Operator estimates for the crushed ice problem

Andrii Khrabustovskyi, Olaf Post

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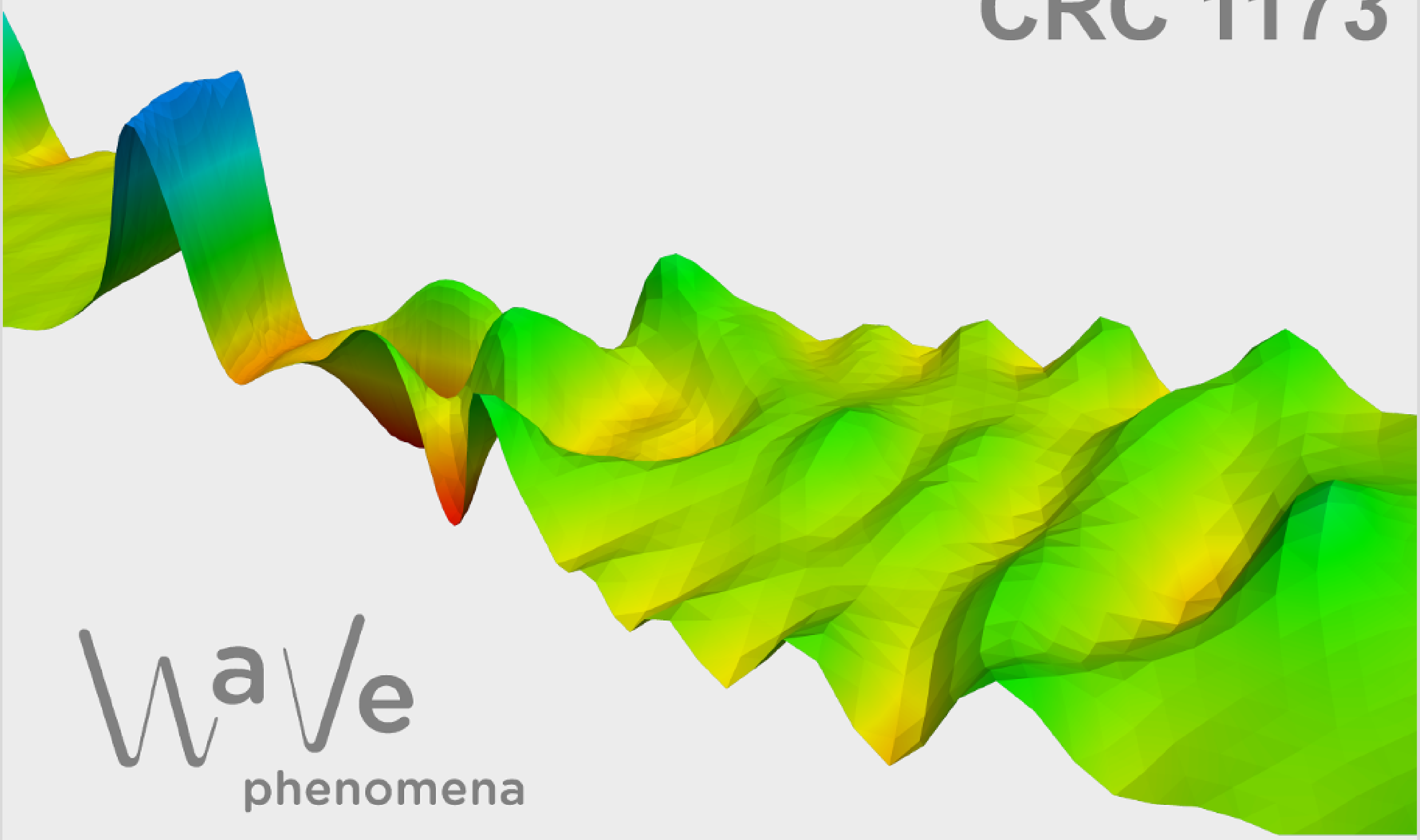

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OPERATOR ESTIMATES FOR THE CRUSHED ICE PROBLEM

ANDRII KHRABUSTOVSKYI AND OLAF POST

ABSTRACT. Let $\Delta_{\Omega_\varepsilon}$ be the Dirichlet Laplacian in the domain $\Omega_\varepsilon := \Omega \setminus (\cup_i D_{i\varepsilon})$. Here $\Omega \subset \mathbb{R}^n$ and $\{D_{i\varepsilon}\}_i$ is a family of tiny identical holes ("ice pieces") distributed periodically in $\mathbb{R}^n$ with period ε. We denote by $\operatorname{cap}(D_{i\varepsilon})$ the capacity of a single hole. It was known for a long time that $-\Delta_{\Omega_\varepsilon}$ converges to the operator $-\Delta_\Omega + q$ in strong resolvent sense provided the limit $q := \lim_{\varepsilon\to 0}\operatorname{cap}(D_{i\varepsilon})\varepsilon^{-n}$ exists and is finite. In the current contribution we improve this result deriving estimates for the rate of convergence in terms of operator norms. As an application, we establish the uniform convergence of the corresponding semi-groups and (for bounded Ω) an estimate for the difference of the k-th eigenvalue of $-\Delta_{\Omega_\varepsilon}$ and $-\Delta_{\Omega_\varepsilon} + q$. Our proofs relies on an abstract scheme for studying the convergence of operators in varying Hilbert spaces developed previously by the second author.

Keywords: crushed ice problem; homogenization; norm resolvent convergence; operator estimates; varying Hilbert spaces

1. INTRODUCTION

In the current work we revisit one of the classical problems in homogenization theory – homogenization of the Dirichlet Laplacian in a domain with a lot of tiny holes. It is also known as *crushed ice problem*. Below, we briefly recall the setting of this problem and the main result.

Let Ω be an open domain in $\mathbb{R}^n$ ($n \geq 2$) and $\{D_{i\varepsilon}\}_i$ be a family of small holes. The holes are identical (up to a rigid motion) and are distributed evenly in Ω along the ε-periodic cubic lattice – see Figure 1. We set

$$\Omega_\varepsilon := \Omega \setminus \Big(\bigcup_i \overline{D_{i\varepsilon}}\Big).$$

The domain Ω_ε is depicted in Figure 1. More precise description of this domain will be given in the next section.

In Ω_ε we study the following problem:

$$-\Delta_{\Omega_\varepsilon} u_\varepsilon + u_\varepsilon = f\restriction_{\Omega_\varepsilon},$$

where $\Delta_{\Omega_\varepsilon}$ is the Dirichlet Laplacian in Ω_ε, $f \in \mathsf{L}_2(\Omega)$ is a given function, $f\restriction_{\Omega_\varepsilon}$ is the restriction of f to Ω_ε. The goal is to describe the behaviour of the solution u_ε to this problem as $\varepsilon \to 0$.

It turns out that the result depends on the limit $q := \lim_{\varepsilon\to 0}\operatorname{cap}(D_{i\varepsilon})\varepsilon^{-n}$ being finite or infinite (here $\operatorname{cap}(D_{i\varepsilon})$ is the capacity of a single hole, see (7) for details). Namely, if

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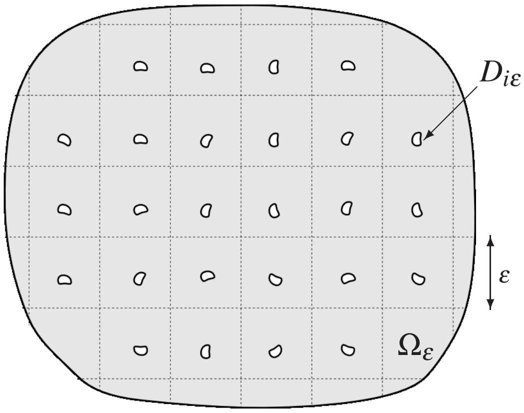

FIGURE 1. The domain Ω_ε obtained from Ω by removing the obstacles $D_{i\varepsilon}$. To avoid technical problems with the boundary of Ω, the obstacles are only placed into cells which lie entirely in Ω.

$q = \infty$ then $\|u_\varepsilon\|_{\mathsf{L}_2(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$. Otherwise, if $q < \infty$, $\|u_\varepsilon - u\|_{\mathsf{L}_2(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$, where u is the solution to the problem

$$-\Delta_\Omega u + qu + u = f.$$

This result was proven independently by V.A. Marchenko, E.Ya. Khruslov [MK64] (the case $q < \infty$), J. Rauch, M. Taylor [RT75] (the cases $q = 0$ and $q = \infty$) and D. Cioranescu, F. Murat [CM82] (all scenario) by using different methods — potential theory, probabilistic methods and a variational approach, respectively. J. Rauch and M. Taylor also treated the case of randomly distributed holes under assumptions resembling the case $q > 0$ in a deterministic case (the pioneer result in this direction was obtained by M. Kac in [Kac74], who investigated the case of uniformly distributed holes). For more details we refer also to [Cha84, CPS07, MK74, MK06, R75, S79].

Note, that this result remains valid if on the external boundary (i.e. on $\partial\Omega \setminus (\bigcup_i \partial D_{i\varepsilon})$) one imposes Neumann, Robin, mixed or any other *ε-independent* boundary conditions (then $-\Delta_\Omega$ is the Laplace operator subject to these conditions on $\partial\Omega$).

Besides the resolvent convergence one can also study the convergence of spectrum or the convergence of the semi-group $\exp(\Delta_{\Omega_\varepsilon} t)$. In the later case the name *crushed ice problem* is indeed reasonable[1].

In what follows, we focus on the case $q < \infty$.

In the language of operator theory one can reformulate the above result as follows: the operator $-\Delta_{\Omega_\varepsilon}$ converges to the operator $-\Delta_\Omega + q$ in strong resolvent sense. Strictly speaking, we are not able to treat the classical resolvent convergence (since the underlying operators act in different Hilbert spaces), but we have its natural analogue for varying domains with $\Omega_\varepsilon \subset \Omega$:

$$\forall f \in \mathsf{L}_2(\Omega): \; \|(-\Delta_{\Omega_\varepsilon} + \mathrm{I})^{-1} J_\varepsilon f - J_\varepsilon(-\Delta_\Omega + q + \mathrm{I})^{-1} f\|_{\mathsf{L}_2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \qquad (1)$$

where $J_\varepsilon f := f\restriction_{\Omega_\varepsilon}$.

[1] Let us assume that Ω is an isolated container occupied by a homogeneous medium, while the sets $D_{i\varepsilon}$ are regarded as a small pieces of ice. Under a certain idealization (the ice pieces do not melt and move) the heat distribution in Ω_ε at time $t > 0$ is described by the function $\exp(\Delta_{\Omega_\varepsilon} t) u_0$, where $\Delta_{\Omega_\varepsilon}$ is the Laplace operator subject to Dirichlet conditions on the boundary of the ice pieces and Neumann conditions on $\partial\Omega$ (since the container is isolated), u_0 is the heat distribution at $t = 0$.

In the recent preprint [DCR17] the authors improved (1) by proving (a kind of) *norm* resolvent convergence, namely

$$\|J'_\varepsilon(-\Delta_{\Omega_\varepsilon}+\mathrm{I})^{-1}-(-\Delta_\Omega+q+\mathrm{I})^{-1}J'_\varepsilon\|_{\mathscr{L}(\mathsf{L}_2(\Omega_\varepsilon),\mathsf{L}_2(\Omega))}\to 0 \text{ as } \varepsilon\to 0, \tag{2}$$

where $J'_\varepsilon\colon \mathsf{L}_2(\Omega_\varepsilon)\to\mathsf{L}_2(\Omega)$ is the operator of extension by zero. The authors assumed that $D_{i\varepsilon}$ are balls, distributed ε-periodically in Ω. For bounded Ω their proof resembles the variational approach developed in [CM82], for unbounded Ω they also utilize a rapid decay of the Green's function of $-\Delta+\mathrm{I}$.

In the current work we extend the result of [DCR17] providing an estimate for the rate of convergence in (2) (see Theorem 2.5 below). We also improve (1) (see Theorem 2.3) deriving the operator estimate

$$\|(-\Delta_{\Omega_\varepsilon}+\mathrm{I})^{-1}J_\varepsilon-J_\varepsilon(-\Delta_\Omega+q+\mathrm{I})^{-1}\|_{\mathscr{L}(\mathsf{L}_2(\Omega),\mathsf{L}_2(\Omega_\varepsilon))}\le 4\delta_\varepsilon,$$

where $\delta_\varepsilon=|q-\lim_{\varepsilon\to 0}\operatorname{cap}(D_\varepsilon)\varepsilon^{-n}|+\gamma_\varepsilon$ with $\gamma_\varepsilon=o(1)$ depending on the dimension n (for the "physical" cases $n=2$ and $n=3$ one has $\gamma_\varepsilon=\mathscr{O}(\varepsilon\ln\varepsilon)$ and $\gamma_\varepsilon=\mathscr{O}(\varepsilon)$, respectively).

As a consequence of our main results, we establish uniform convergence of the corresponding semi-groups and (for bounded Ω) an estimate for the difference between the k-th eigenvalue of $-\Delta_{\Omega_\varepsilon}$ and $-\Delta_\Omega+q$ — see Theorems 2.6–2.7.

Let us stress that in all our results (except Theorem 2.7) we do not assume that the domain Ω is *bounded*.

Our proofs are based on the abstract scheme for studying the convergence of operators in varying Hilbert spaces which was developed by the second author of the present article in [P06] and in more detail in the monograph [P12].

Before proceeding to the main part of the work let us mention several related results:

- Some estimates for the rate of convergence in (1) were obtained in [CPS07, §16]. Namely, assuming that $n=3$, Ω is bounded, $D_{i\varepsilon}$ are balls of radius ε^3 (that is $\operatorname{cap}(D_{i\varepsilon})\varepsilon^{-3}=4\pi=q$) distributed ε-periodically, and the function f belongs to the Hölder class $\mathsf{C}^{0,\alpha}(\overline{\Omega})$, the authors derived the estimates
$$\|(-\Delta_{\Omega_\varepsilon}+\mathrm{I})^{-1}J_\varepsilon f-J_\varepsilon(-\Delta_\Omega+q+\mathrm{I})^{-1}f\|_{\mathsf{L}_2(\Omega_\varepsilon)}\le C\varepsilon\|f\|_{\mathsf{C}^{0,\alpha}(\overline{\Omega})}.$$
$$\|(-\Delta_{\Omega_\varepsilon}+\mathrm{I})^{-1}J_\varepsilon f-J_\varepsilon\varphi_\varepsilon(-\Delta_\Omega+q+\mathrm{I})^{-1}f\|_{\mathsf{H}^1(\Omega_\varepsilon)}\le C\varepsilon\|f\|_{\mathsf{C}^{0,\alpha}(\overline{\Omega})},$$
where φ_ε is the operator of multiplication by a certain cut-off function.
- One can also study a surface distribution of holes, i.e. holes being located near some hypersurface Γ intersecting Ω. This problem was considered in [MK64]; it was proved that the limit operator is $-\Delta_\Omega+q\delta_\Gamma$. Here $q\in\mathsf{L}_\infty(\Gamma)$ is a positive function, and δ_Γ is a delta-distribution supported on Γ. For the case $n=2$, the norm resolvent convergence with estimates on the rate of convergence were obtained in [BCD16]. Note that the method we use in the current work allows to treat the surface distribution of holes as well. Nevertheless, to simplify the presentation, we focus on the bulk distribution of holes only.
- Operator estimates in homogenization theory is a rather young topic. The classical homogenization problem concerning elliptic operators of the form
$$\mathscr{A}_\varepsilon=-\operatorname{div}\left(A\left(\frac{\cdot}{\varepsilon}\right)\nabla\right),\text{where } A(\cdot) \text{ is a } \mathbb{Z}^n\text{-periodic function,}$$

was treated in [BS03, Gr04, Zh05a, ZhP05]. For more results we refer to the paper [BCD16] containing a comprehensive overview on operator estimates in homogenization theory. We also emphasize the paper [Zh05b], where the perturbation is defined by rescaling an abstract periodic measure. The technique developed in [Zh05b] can be applied for deriving operator estimates is the case of periodically perforated domains provided the sizes of holes and distances between them are of the same smallness order (evidently, this does not hold for the problem we study in the current paper).

- In [AP17] we treat (possibly non-compact) manifolds with an increasing (even infinite) number of balls removed (similarly as in [RT75]), and show operator estimates using similar methods as in this article.

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2. Setting of the problem and main results

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain (not necessarily bounded) with C^2-boundary $\partial\Omega$. We denote by $\nu\colon \partial\Omega \to \mathbb{S}^{n-1}$ the unit inward-pointing normal vector field on $\partial\Omega$.

Additionally, we assume that there exists a constant $\theta_\Omega > 0$ such that the map

$$(x,t) \mapsto x + t\nu(x) \tag{3}$$

is injective on $\partial\Omega \times [0,\theta]$ provided $\theta < \theta_\Omega$. We note, that all the results remain valid under less restrictive assumptions on $\partial\Omega$, see Remark 4.8 below.

In what follows we denote by C, C_1 etc. generic constants depending only on the dimension n.

We set $\square := (-1/2, 1/2)^n$.

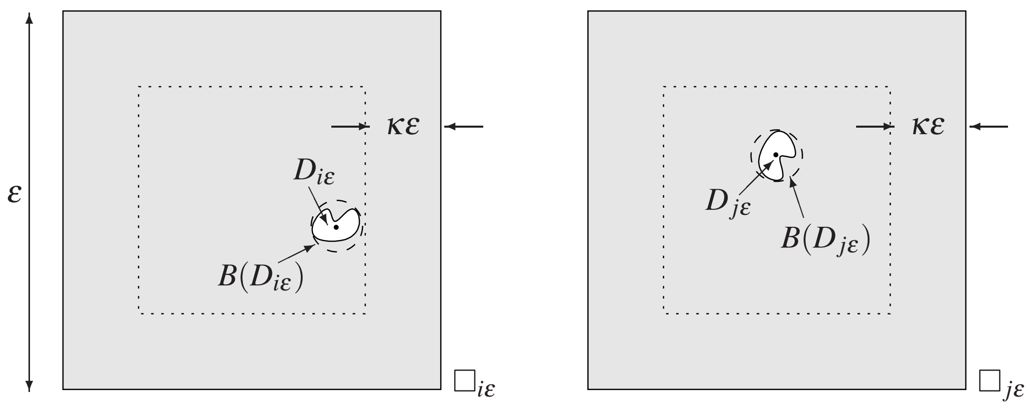

FIGURE 2. Two scaled cells $\square_{i\varepsilon}$ and $\square_{j\varepsilon}$ and possible positions of the obstacles $D_{i\varepsilon}$ and $D_{j\varepsilon}$ (white). The smallest ball $B(D_{i\varepsilon})$ (dashed circle) containing the obstacle $D_{i\varepsilon}$ has security distance $\kappa\varepsilon$ from the boundary of $\square_{i\varepsilon}$, i.e., it should stay inside the dotted cube of side length $(1-2\kappa)\varepsilon$.

Now we describe a family of holes in Ω (see Figure 2). Let D_ε be a Lipschitz domain in $\mathbb{R}^n$ depending on a small parameter $\varepsilon > 0$. We denote by d_ε the radius of the smallest ball containing D_ε. It is assumed that

$$(d_\varepsilon)^{n-2} \le C\varepsilon^n \quad \text{as } n \ge 3, \quad |\ln d_\varepsilon|^{-1} \le C\varepsilon^2 \quad \text{as } n = 2, \quad \text{or} \tag{4}$$

$$d_\varepsilon \le \begin{cases} C_1\varepsilon^{n/(n-2)}, & \text{as } n \ge 3, \\ \exp(-1/(C\varepsilon^2)), & \text{as } n = 2 \end{cases}$$

(hence, in particular, $d_\varepsilon = o(\varepsilon)$). For $i \in \mathbb{Z}^n$, let $D_{i\varepsilon}$ be a set enjoying the following properties:

$$D_{i\varepsilon} \text{ coinsides with } D_\varepsilon \text{ up to a rigid motion,}$$
$$B(D_{i\varepsilon}) \subset \square_{i\varepsilon} := \varepsilon(\square + i),$$
$$\operatorname{dist}\big(B(D_{i\varepsilon}), \partial\square_{i\varepsilon}\big) \ge \kappa\varepsilon \text{ for some } \kappa > 0, \tag{5}$$

where $B(D_{i\varepsilon})$ is the smallest ball containing $D_{i\varepsilon}$ (the radius of this ball is d_ε).

Finally, we set

$$\Omega_\varepsilon := \Omega \setminus \left(\bigcup_{i \in \mathscr{I}_\varepsilon} \overline{D_{i\varepsilon}} \right),$$

where

$$\mathscr{I}_\varepsilon := \{ i \in \mathbb{Z}^n : \ \square_{i\varepsilon} \subset \Omega \},$$

i.e. the set of those indices for which the rescaled unit cell $\square_{i\varepsilon}$ is entirely in Ω (with positive distance to $\partial\Omega$). The domain Ω_ε is depicted in Figure 1.

By $\mathscr{A}_\varepsilon$ we denote the Dirichlet Laplacian on Ω_ε, i.e. the operator acting in the Hilbert space $\mathsf{L}_2(\Omega_\varepsilon)$ associated with the closed densely defined positive sesquilinear form

$$\mathfrak{a}_\varepsilon[u,v] := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \bar{v} \, \mathrm{d}x, \qquad \operatorname{dom}(\mathfrak{a}_\varepsilon) := \mathsf{H}^1_0(\Omega_\varepsilon).$$

Our goal is to describe the behaviour of the resolvent $(\mathscr{A}_\varepsilon + \mathrm{I})^{-1}$ as $\varepsilon \to 0$ under the assumption that the following limit exists and is finite:

$$q = \lim_{\varepsilon \to 0} \frac{\operatorname{cap}(D_\varepsilon)}{\varepsilon^n}, \tag{6}$$

where $\operatorname{cap}(D_\varepsilon)$ is the capacity of the set D_ε. Recall (see, e.g., [T11]), that for $n \ge 3$ the capacity of a set $D \subset \mathbb{R}^n$ is defined via

$$\operatorname{cap}(D) = \int_{\mathbb{R}^n} |\nabla H(x)|^2 \, \mathrm{d}x, \tag{7}$$

where H is a solution to the problem

$$\begin{cases} \Delta H(x) = 0, & x \in \mathbb{R}^n \setminus \overline{D}, \\ H(x) = 1, & x \in \partial D, \\ H(x) \to 0, & |x| \to \infty. \end{cases} \tag{8}$$

One has also the following variational characterization of the capacity, namely

$$\operatorname{cap}(D) = \min \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, \mathrm{d}x, \tag{9}$$

where the minimum is taken over $u \in \mathsf{C}^\infty_0(\mathbb{R}^n)$ being equal to 1 on a neighbourhood of D.

For $n=2$ the right-hand-side of (9) is zero for an arbitrary domain D, hence we need a modified definition. It is as follows:

$$\mathrm{cap}(D)=\int_{B_1}|\nabla H(x)|^2\,\mathrm{d}x, \tag{10}$$

where B_1 is the unit ball concentric with $B(D)$ – the smallest ball containing D (here we assume that the set D is small enough so that $D\subset B(D)\subset B_1$), H solves the problem

$$\begin{cases}\Delta H(x)=0, & x\in B_1\setminus\overline{D},\\ H(x)=1, & x\in\partial D,\\ H(x)=0, & x\in\partial B_1.\end{cases} \tag{11}$$

Further, proving the main results, we will use the following pointwice estimates for the functions H at some positive distance from $B(D)$, see [MK06, Lemma 2.4].

Lemma 2.1. *Let $x\in\mathbb{R}^n\setminus\overline{B(D)}$. We denote by $\rho(x)$ the distance from x to $B(D)$, and by d the radius of $B(D)$. One has:*

$$|H(x)|\le\frac{C\cdot d^{n-2}}{(\rho(x))^{n-2}},\quad |\nabla H(x)|\le\frac{C\cdot d^{n-2}}{(\rho(x))^{n-1}}\quad \text{as } n\ge 3,$$
$$|H(x)|\le\frac{C|\ln d|^{-1}}{|\ln\rho(x)|^{-1}},\quad |\nabla H(x)|\le\frac{C|\ln d|^{-1}}{\rho(x)}\quad \text{as } n=2$$

provided $\rho(x)\ge C_0 d$ as $n\ge 3$ or $\rho(x)\ge\exp\left(-C_0\sqrt{|\ln d|}\right)$ as $n\ge 3$, for some $C_0>0$.

Remark 2.2. Due to (6) one has

$$\mathrm{cap}(D_\varepsilon)\varepsilon^{-n}=\mathscr{O}(1). \tag{12}$$

In fact, this condition also follows directly from (4). Indeed, using the monotonicity of the capacity, we get $\mathrm{cap}(D_\varepsilon)\le\mathrm{cap}(B_\varepsilon)$, where B_ε is ball of radius d_ε containing D_ε. For this ball the function H can be computed explicitly:

$$H(x)=\frac{(d_\varepsilon)^{n-2}}{|x|^{n-2}}\text{ as } n\ge 3,\quad H(x)=\frac{\ln x}{\ln d_\varepsilon}\text{ as } n=2,$$

hence $\mathrm{cap}(B_\varepsilon)=(n-2)|\mathbb{S}^{n-1}|(d_\varepsilon)^{n-2}$ as $n\ge 3$ and $\mathrm{cap}(B_\varepsilon)=2\pi|\ln d_\varepsilon|^{-1}$ as $n=2$, hence, due to (4), we get (12).

Finally, we introduce the limiting operator $\mathscr{A}$. It acts in $\mathsf{L}_2(\Omega)$ and is associated with the form

$$\mathfrak{a}[u,v]:=\int_\Omega(\nabla u\cdot\nabla\bar{v}+qu\bar{v})\,\mathrm{d}x,\qquad \mathrm{dom}(\mathfrak{a}):=\mathsf{H}^1_0(\Omega).$$

Since $\partial\Omega$ is C^2-smooth one has $\mathrm{dom}(\mathscr{A})=\mathsf{H}^2(\Omega)\cap\mathsf{H}^1_0(\Omega)$ and $\mathscr{A}u=-\Delta u+qu$.

The operators $\mathscr{A}_\varepsilon$ and $\mathscr{A}$ act in different Hilbert spaces, namely $\mathscr{H}_\varepsilon:=\mathsf{L}_2(\Omega_\varepsilon)$ and $\mathscr{H}:=\mathsf{L}_2(\Omega)$, respectively. Therefore we are not able to apply the usual notion of resolvent convergence and thus a suitable modification is needed. There are many ways how to do this in a “smart” way. For example (cf. [IOS89, Vai05]), one can treat the behaviour of the operator

$$(\mathscr{A}_\varepsilon+\mathrm{I})^{-1}J_\varepsilon-J_\varepsilon(\mathscr{A}+\mathrm{I})^{-1}\in\mathscr{L}(\mathscr{H},\mathscr{H}_\varepsilon),$$

where $J_\varepsilon\colon\mathscr{H}\to\mathscr{H}_\varepsilon$ is a suitable bounded linear operator satisfying

$$\forall f\in\mathscr{H}:\ \|J_\varepsilon f\|_{\mathscr{H}_\varepsilon}\to\|f\|_{\mathscr{H}}\text{ as }\varepsilon\to 0. \tag{13}$$

It is natural to choose the operator J_ε as the operator of restriction to Ω_ε, i.e.

$$(J_\varepsilon f)(x) = \begin{cases} f(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \bigcup\limits_{i\in\mathscr{I}_\varepsilon} D_{i\varepsilon}. \end{cases} \tag{14}$$

Due to (4) one has for each compact set $K \subset \mathbb{R}^n$

$$\sum_{i:\, D_{i\varepsilon}\subset K} |D_{i\varepsilon}| \le \alpha_\varepsilon |K|,\ \alpha_\varepsilon = o(1) \text{ does not depend on } K, \tag{15}$$

where $|K|$ stands for the Lebesgue measure of K. Hence, evidently, (13) holds. The results of [CM82, MK64, RT75] can be reformulated as follows:

$$\forall f \in \mathscr{H}: \quad \lim_{\varepsilon\to 0} \left\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} J_\varepsilon f - J_\varepsilon(\mathscr{A} + \mathrm{I})^{-1} f\right\|_{\mathscr{H}_\varepsilon} \to 0,$$

i.e. one has a kind of strong resolvent convergence.

Now, we can state our main result.

Theorem 2.3. *One has*

$$\left\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} J_\varepsilon - J_\varepsilon(\mathscr{A} + \mathrm{I})^{-1}\right\|_{\mathscr{L}(\mathscr{H},\mathscr{H}_\varepsilon)} \le 4\delta_\varepsilon,$$

where δ_ε is defined by

$$\delta_\varepsilon = |\operatorname{cap}(D_\varepsilon)\varepsilon^{-n} - q| + C_{\Omega,\kappa,\beta} \cdot \begin{cases} \varepsilon|\ln\varepsilon|, & n = 2, \\ \varepsilon, & n = 3 \\ \varepsilon^{1-\beta},\ \beta > 0, & n = 4, \\ \max\left\{\varepsilon;\, d_\varepsilon \varepsilon^{-1}\right\}, & n \ge 5, \end{cases} \tag{16}$$

and the constant $C_{\Omega,\kappa,\beta}$ depends on the domain Ω, the relative distance κ of the obstacles from the period cell boundary (see (5)*), and, in the case $n = 4$, on β.*

Remark 2.4. Via the same arguments as in Remark 2.2 one gets $(d_\varepsilon)^{n-2}\varepsilon^{-n} \ge C > 0$ provided $q > 0$, hence, using the definition of δ_ε, we obtain

$$q > 0,\ n \ge 5: \quad \delta_\varepsilon = |\operatorname{cap}(D_\varepsilon)\varepsilon^{-n} - q| + C_{\Omega,\kappa}\varepsilon^{2/(n-2)}.$$

Let $J'_\varepsilon\colon \mathscr{H}_\varepsilon \to \mathscr{H}$ be the operator of extension by zero:

$$(J'_\varepsilon u)(x) = \begin{cases} u(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \bigcup\limits_{i\in\mathscr{I}_\varepsilon} D_{i\varepsilon}. \end{cases} \tag{17}$$

Then the main result of [DCR17] is equivalent to

$$\left\|J'_\varepsilon(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - (\mathscr{A} + \mathrm{I})^{-1} J'_\varepsilon\right\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H})} \to 0,\ \varepsilon \to 0.$$

The next theorem gives an improvement of this statement.

Theorem 2.5. *One has*

$$\left\|J'_\varepsilon(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - (\mathscr{A} + \mathrm{I})^{-1} J'_\varepsilon\right\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H})} \le 6\delta_\varepsilon,$$

where δ_ε is defined in (16)*. Moreover,*

$$\left\|J'_\varepsilon(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} J_\varepsilon - (\mathscr{A} + \mathrm{I})^{-1}\right\|_{\mathscr{L}(\mathscr{H},\mathscr{H})} \le 9\delta_\varepsilon,$$

$$\left\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - J_\varepsilon(\mathscr{A} + \mathrm{I})^{-1} J'_\varepsilon\right\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H}_\varepsilon)} \le 13\delta_\varepsilon.$$

One important applications of the norm resolvent convergence is the uniform convergence of semi-groups generated by $\mathscr{A}_\varepsilon$ and $\mathscr{A}$. Namely, we can approximate $\exp(-\mathscr{A}_\varepsilon t)$ in terms of simpler operators $\exp(-\mathscr{A}t)$, J_ε and J'_ε:

Theorem 2.6. *One has for each $t>0$:*

$$\left\|\exp(-\mathscr{A}_\varepsilon t) - J_\varepsilon \exp(-\mathscr{A}t)J'_\varepsilon\right\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H}_\varepsilon)} \le c_t\delta_\varepsilon,$$

where δ_ε is defined in (16)*, and the constant c_t depends only on t.*

Another important application is the Hausdorff convergence of spectra, see [DCR17]. Using Theorem 2.3 we are able to extend this result by obtaining an estimate for the difference between the corresponding eigenvalues. Namely, let the domain Ω be bounded. We denote by $\{\lambda_{k,\varepsilon}\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ the sequences of the eigenvalues of $\mathscr{A}_\varepsilon$ and $\mathscr{A}$, respectively, arranged in the ascending order and repeated according to their multiplicities.

Theorem 2.7. *For each $k\in\mathbb{N}$ one has*

$$\lim_{\varepsilon\to 0}\lambda_{k,\varepsilon} = \lambda_k, \tag{18}$$

moreover

$$|\lambda_{k,\varepsilon}-\lambda_k| \le 4C_\varepsilon(\lambda_{k,\varepsilon}+1)(\lambda_k+1)\delta_\varepsilon, \tag{19}$$

where δ_ε is defined in (16)*, and $|C_\varepsilon|\le C$, $\lim_{\varepsilon\to 0}C_\varepsilon = 1$.*

In the next section we introduce an abstract scheme, which then will be applied for the proof of the above theorems.

3. Abstract framework

In this section we present an abstract scheme for studying the convergence of operators in varying Hilbert spaces. It was developed by the second author of the present article in [P06] and in more detail in the monograph [P12] (see also the later work [MNP13], where non-self-adjoint operators were treated).

Let $\mathscr{H}$ and $\mathscr{H}_\varepsilon$ be two separable Hilbert spaces. Note, that within this section $\mathscr{H}_\varepsilon$ is just a *notation* for some Hilbert space, which (in general) differs from the space $\mathscr{H}$, i.e. the sub-index ε does not mean that this space depends on a small parameter. Of course, further we will use the results of this section for ε-dependent space $\mathscr{H}_\varepsilon = \mathsf{L}_2(\Omega_\varepsilon)$.

Let $\mathfrak{a}$ and $\mathfrak{a}_\varepsilon$ be closed, densely defined, non-negative sesquilinear forms in $\mathscr{H}$ and $\mathscr{H}_\varepsilon$, respectively. We denote by $\mathscr{A}$ and $\mathscr{A}_\varepsilon$ the non-negative, self-adjoint operators associated with $\mathfrak{a}$ and $\mathfrak{a}_\varepsilon$, respectively.

Associated with the operator $\mathscr{A}$, we can introduce a natural *scale of Hilbert spaces $\mathscr{H}^k$* defined via the *abstract Sobolev norm*:

$$\mathscr{H}^k = \operatorname{dom}\mathscr{A}^{k/2}, \quad \|f\|_{\mathscr{H}^k} := \|f\|_k := \|(\mathscr{A}+\mathrm{I})^{k/2}f\|_{\mathscr{H}}.$$

In particular, we have $\mathscr{H}^0=\mathscr{H}$ with $\|f\|_{\mathscr{H}^0}=\|f\|_{\mathscr{H}}$, $\mathscr{H}^1=\operatorname{dom}(\mathfrak{a})$ with $\|f\|_{\mathscr{H}^1} = (\mathfrak{a}[f,f]+\|f\|^2_{\mathscr{H}})^{1/2}$, and $\mathscr{H}^2=\operatorname{dom}(\mathscr{A})$ with $\|f\|_{\mathscr{H}^2}=\|\mathscr{A}f+f\|_{\mathscr{H}}$.

Similarly, we denote by $\mathscr{H}^k_\varepsilon$ the scale of Hilbert spaces associated with $\mathscr{A}_\varepsilon$. The corresponding norms will be denoted by $\|\cdot\|_{\mathscr{H}^k_\varepsilon}$.

We now need pairs of so-called *identification* or *transplantation operators* acting on the Hilbert spaces and later also pairs of identification operators acting on the form domains.

Definition 3.1 ((see [P06, App.] or [P12, Ch. 4]))**.** Let $\delta_\varepsilon \geq 0$ and $k \in \mathbb{N}$. Moreover, let $J_\varepsilon\colon \mathscr{H} \to \mathscr{H}_\varepsilon$ and $J'_\varepsilon\colon \mathscr{H}_\varepsilon \to \mathscr{H}$ be linear bounded operators. In addition, let $J^1_\varepsilon\colon \mathscr{H}^1 \to \mathscr{H}^1_\varepsilon$ and $J^{1\prime}_\varepsilon\colon \mathscr{H}^1_\varepsilon \to \mathscr{H}^1$ be linear bounded operators on the form domains. We say that $(\mathscr{H},\mathfrak{a})$ and $(\mathscr{H}_\varepsilon,\mathfrak{a}_\varepsilon)$ are *δ_ε-close of order k with respect to the operators J_ε, J'_ε, J^1_ε, $J^{1\prime}_\varepsilon$*, if the following conditions hold:

$$\|J_\varepsilon f - J^1_\varepsilon f\|_{\mathscr{H}_\varepsilon} \leq \delta_\varepsilon \|f\|_{\mathscr{H}^1},\ \forall f \in \mathscr{H}^1, \tag{C$_{1a}$}$$

$$\|J'_\varepsilon u - J^{1\prime}_\varepsilon u\|_{\mathscr{H}} \leq \delta_\varepsilon \|u\|_{\mathscr{H}^1_\varepsilon},\ \forall u \in \mathscr{H}^1_\varepsilon, \tag{C$_{1b}$}$$

$$\left|(J_\varepsilon f,u)_{\mathscr{H}_\varepsilon} - (f,J'_\varepsilon u)_{\mathscr{H}}\right| \leq \delta_\varepsilon \|f\|_{\mathscr{H}}\|u\|_{\mathscr{H}_\varepsilon},\ \forall f \in \mathscr{H}, u \in \mathscr{H}_\varepsilon, \tag{C$_2$}$$

$$\|J_\varepsilon f\|_{\mathscr{H}_\varepsilon} \leq (1+\delta_\varepsilon)\|f\|_{\mathscr{H}},\ \forall f \in \mathscr{H}, \tag{C$_{3a}$}$$

$$\|J'_\varepsilon u\|_{\mathscr{H}} \leq (1+\delta_\varepsilon)\|u\|_{\mathscr{H}_\varepsilon},\ \forall u \in \mathscr{H}_\varepsilon, \tag{C$_{3b}$}$$

$$\|f - J'_\varepsilon J_\varepsilon f\|_{\mathscr{H}} \leq \delta_\varepsilon \|f\|_{\mathscr{H}^1},\ \forall f \in \mathscr{H}^1, \tag{C$_{4a}$}$$

$$\|u - J_\varepsilon J'_\varepsilon u\|_{\mathscr{H}_\varepsilon} \leq \delta_\varepsilon \|u\|_{\mathscr{H}^1_\varepsilon},\ \forall u \in \mathscr{H}^1_\varepsilon, \tag{C$_{4b}$}$$

$$\left|\mathfrak{a}_\varepsilon(J^1_\varepsilon f,u) - \mathfrak{a}(f,J^{1\prime}_\varepsilon u)\right| \leq \delta_\varepsilon \|f\|_{\mathscr{H}^k}\|u\|_{\mathscr{H}^1_\varepsilon}, \forall f \in \mathscr{H}^k, u \in \mathscr{H}^1_\varepsilon. \tag{C$_5$}$$

Remark 3.2. For $\delta_\varepsilon = 0$ the definition above implies that the operators $\mathscr{A}$ and $\mathscr{A}_\varepsilon$ are unitary equivalent. Indeed, (C$_2$)–(C$_{4b}$) assure that the operator J_ε is unitary with the inverse J'_ε; due to (C$_{1a}$)–(C$_{1b}$) J^1_ε and $J^{1\prime}_\varepsilon$ are the restrictions of J_ε and J^1_ε onto $\mathrm{dom}(\mathfrak{a})$ and $\mathrm{dom}(\mathfrak{a}_\varepsilon)$, respectively. Hence, in view of (C$_5$), J_ε realises the unitary equivalence of $\mathscr{A}$ and $\mathscr{A}_\varepsilon$.

Now, we present the main implications of the definition of δ_ε-closeness.

Theorem 3.3 ([P06, Th. A.5])**.** *One has*

$$\left\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1}J_\varepsilon - J_\varepsilon(\mathscr{A} + \mathrm{I})^{-1}\right\|_{\mathscr{L}(\mathscr{H},\mathscr{H}_\varepsilon)} \leq 4\delta_\varepsilon,$$

provided conditions (C$_{1a}$)*,* (C$_{1b}$)*,* (C$_2$)*, and* (C$_5$) *hold with* $k \leq 2$*.*

Remark 3.4. Let $\mathscr{A}_\varepsilon$ ($\varepsilon > 0$), $\mathscr{A}$ be non-negative self-adjoint operators in the *same* Hilbert space $\mathscr{H}$, and let $\mathfrak{a}_\varepsilon$ and $\mathfrak{a}$ be the corresponding sesquilinear forms. We assume that $\mathrm{dom}(\mathfrak{a}_\varepsilon) = \mathrm{dom}(\mathfrak{a})$ and

$$|\mathfrak{a}_\varepsilon(f,u) - \mathfrak{a}(f,u)| \leq \delta_\varepsilon \sqrt{\mathfrak{a}[f,f] + \|f\|_{\mathscr{H}}}\sqrt{\mathfrak{a}_\varepsilon[u,u] + \|u\|_{\mathscr{H}}},\ \forall f,u \in \mathrm{dom}(\mathfrak{a}), \tag{20}$$

where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Due to (20) $(\mathscr{H},\mathfrak{a})$ and $(\mathscr{H},\mathfrak{a}_\varepsilon)$ are δ_ε-close of order 1 with respect to the *identity* maps $J_\varepsilon, J'_\varepsilon$ (on $\mathscr{H}$) and $J^1_\varepsilon, J^{1\prime}_\varepsilon$ (on $\mathrm{dom}(\mathfrak{a})$). Then by Theorem 3.3

$$\left\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - (\mathscr{A} + \mathrm{I})^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \to 0. \tag{21}$$

In fact, it would suffice for (21) if (20) is satisfied whenever $f = u$, see Theorem VI.3.6 in T. Kato's monograph [Kat66]. In this sense, Theorem 3.3 can be regarded as a generalization of this classical result to the setting of varying spaces.

Theorem 3.5 ([P06, Th. A.8])**.** *Let $U \subset \overline{\mathbb{R}_+}$ be an open set containing either $\sigma(\mathscr{A})$ or $\sigma(\mathscr{A}_\varepsilon)$. Let $\psi\colon \overline{\mathbb{R}_+} \to \mathbb{C}$ be a bounded measurable function, continuous on U and such that the limit* $\lim_{\lambda\to\infty}\psi(\lambda)$ *exists.*

Then there exists $\eta_\psi(\delta) > 0$ with $\eta_\psi(\delta) \to 0$ as $\delta \to 0$ such that

$$\|\psi(\mathscr{A}_\varepsilon)J_\varepsilon - J_\varepsilon\psi(\mathscr{A})\|_{\mathscr{L}(\mathscr{H},\mathscr{H}_\varepsilon)} \le \eta_\psi(\delta_\varepsilon) \tag{22}$$

for all pairs $(\mathscr{H},\mathfrak{a})$ and $(\mathscr{H}_\varepsilon,\mathfrak{a}_\varepsilon)$, which are δ_ε-close of order $k \le 2$.

Remark 3.6. The important example of the function ψ satisfying the requirements of the above theorem is $\psi(\lambda) = \exp(-\lambda t)$, $t > 0$ is a parameter. Another important example is the function $\psi = \mathbf{1}_{(\alpha,\beta)}$ – the characteristic function of the interval (α,β) with $\alpha,\beta \notin \sigma(\mathscr{A})$ or $\alpha,\beta \notin \sigma(\mathscr{A}_\varepsilon)$. In this case Theorem 3.5 gives the closeness of the spectral projections.

Theorem 3.7 ([P06, Th. A.10]). *Let for some function ψ the estimate* (22) *be valid. Then*

$$\begin{aligned}
\big\|J'_\varepsilon\psi(\mathscr{A}_\varepsilon) - \psi(\mathscr{A})J'_\varepsilon\big\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H})} &\le \eta_\psi(\delta_\varepsilon) + 2\|\psi\|_\infty\delta_\varepsilon,\\
\big\|J'_\varepsilon\psi(\mathscr{A}_\varepsilon)J_\varepsilon - \psi(\mathscr{A})\big\|_{\mathscr{L}(\mathscr{H},\mathscr{H})} &\le C_\psi\delta_\varepsilon + 2\eta_\psi(\delta_\varepsilon),\\
\big\|\psi(\mathscr{A}_\varepsilon) - J_\varepsilon\psi(\mathscr{A})J'_\varepsilon\big\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H}_\varepsilon)} &\le 5C_\psi\delta_\varepsilon + 2\eta_\psi(\delta_\varepsilon)
\end{aligned}$$

provided (C_2)–(C_{4b}) *hold true. Here $\eta_\psi(\delta_\varepsilon)$ comes from* (22)*, $\|\cdot\|_\infty$ stands for the L_∞-norm, and C_ψ is a constant satisfying $|\psi(\lambda)| \le C_\psi(1+\lambda)^{-\frac{1}{2}}$ for all $\lambda \ge 0$.*

For $\psi(\lambda) = (1+\lambda)^{-1}$ one has $\eta_\psi(\delta_\varepsilon) = 4\delta_\varepsilon$ (see Theorem 3.3), $C_\psi = 1$, and hence we immediately get the following corollary from Theorem 3.7.

Corollary 3.8. *One has*

$$\begin{aligned}
\big\|J'_\varepsilon(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - (\mathscr{A} + \mathrm{I})^{-1}J'_\varepsilon\big\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H})} &\le 6\delta_\varepsilon,\\
\big\|J'_\varepsilon(\mathscr{A}_\varepsilon + \mathrm{I})^{-1}J_\varepsilon - (\mathscr{A} + \mathrm{I})^{-1}\big\|_{\mathscr{L}(\mathscr{H},\mathscr{H})} &\le 9\delta_\varepsilon,\\
\big\|(\mathscr{A}_\varepsilon + \mathrm{I})^{-1} - J_\varepsilon(\mathscr{A} + \mathrm{I})^{-1}J'_\varepsilon\big\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H}_\varepsilon)} &\le 13\delta_\varepsilon
\end{aligned}$$

provided $(\mathscr{H},\mathfrak{a})$ and $(\mathscr{H}_\varepsilon,\mathfrak{a}_\varepsilon)$ are δ_ε-close of order $k \le 2$.

For “good enough” functions the last statement of Theorem 3.7 can be improved. Evidently the function $\psi(\lambda) = \exp(-\lambda t)$ $(t > 0)$ satisfies the requirements of the theorem below.

Theorem 3.9 ([MNP13, Th. 3.7][2]). *Let $\Sigma_\Theta := \{z \in \mathbb{C} : |\arg(z+1)| < \Theta\}$ with $\Theta \in (0,\pi)$ and $\psi : \Sigma_\Theta \to \mathbb{C}$ be a holomorphic function satisfying $\psi(z) = \mathscr{O}(|z|^{-\mu})$ for some $\mu > \frac{1}{2}$. Let $(\mathscr{H},\mathfrak{a})$ and $(\mathscr{H}_\varepsilon,\mathfrak{a}_\varepsilon)$ be δ_ε-close of order $k \le 2$. Then*

$$\big\|\psi(\mathscr{A}_\varepsilon) - J_\varepsilon\psi(\mathscr{A})J'_\varepsilon\big\|_{\mathscr{L}(\mathscr{H}_\varepsilon,\mathscr{H}_\varepsilon)} \le c_\psi\delta_\varepsilon, \tag{23}$$

where c_ψ is a constant depending on ψ.

Remark 3.10. In fact, (23) is valid even for less regular functions. For instance, it holds for $\psi = \mathbf{1}_{(\alpha,\beta)}$ as in Remark 3.6, see [P12, Sec. 4.5, Cor. 4.5.15].

The last result concerns the convergence of spectra in general. For two compact sets $X, Y \subset \mathbb{R}$ we denote by $\mathrm{dist_H}(X,Y)$ the *Hausdorff distance* between these sets, i.e.

$$\mathrm{dist_H}(X,Y) = \max\Big\{\sup_{x\in X}\mathrm{dist}(x,Y);\ \sup_{y\in Y}\mathrm{dist}(y,X)\Big\},$$

[2] Actually, (23) is proven in [MNP13, Th. 3.7] only for the case $k = 1$. For $k = 2$ the proof is repeated word-by-word since it relies only on the last estimate in Corollary 3.8.

where $\operatorname{dist}(x,Y) = \inf_{y\in Y}|x-y|$.

Remark 3.11. Let $\{X_\varepsilon \subset \mathbb{R}\}_\varepsilon$ be a family of compact domains and

$$\operatorname{dist}_{\mathrm{H}}(X_\varepsilon, X) \to 0 \text{ as } \varepsilon \to 0 \tag{24}$$

for some compact domain $X \subset \mathbb{R}$. It is easy to prove (see, e.g., [P12, Proposition A.1.6]) that (24) holds iff the following two conditions are fulfilled:

(i) Let $\lambda_0 \in \mathbb{R}\setminus X$. Then there exists $d > 0$ such that $X_\varepsilon \cap \{\lambda : \ |\lambda-\lambda_0| < d\} = \varnothing$.
(ii) Let $\lambda_0 \in X$. Then there exists a family $\{\lambda_\varepsilon\}_\varepsilon$ with $\lambda_\varepsilon \in X_\varepsilon$ such that $\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda_0$.

Theorem 3.12 ([P06, Th. A.13]). *There exists $\eta(\delta) > 0$ with $\eta(\delta) \to 0$ as $\delta \to 0$ such that*

$$\operatorname{dist}_{\mathrm{H}}\left(\frac{1}{1+\sigma(\mathscr{A})}, \frac{1}{1+\sigma(\mathscr{A}_\varepsilon)}\right) \le \eta(\delta_\varepsilon)$$

for all pairs $(\mathscr{H}, \mathfrak{a})$ and $(\mathscr{H}_\varepsilon, \mathfrak{a}_\varepsilon)$ which are δ_ε-close of some order $k \in \mathbb{N}$.

4. Proof of the main results

For an open subset $M \subset \mathbb{R}^n$ ($M \neq \emptyset$) we denote by $\langle f\rangle_M$ the *mean value* of f over M, i.e.

$$\langle f\rangle_M := \frac{1}{|M|}\int_M f(x)\,\mathrm{d}x.$$

Recall that $\mathscr{H}_\varepsilon$ and $\mathscr{H}$ stand for the spaces $\mathsf{L}_2(\Omega_\varepsilon)$ and $\mathsf{L}_2(\Omega)$, respectively; $\mathfrak{a}_\varepsilon$ and $\mathfrak{a}$ are the sesqulilinear forms associated with the operators $\mathscr{A}_\varepsilon$ and $\mathscr{A}$. Also, recall that $\mathscr{H}^1_\varepsilon$ (respectively, $\mathscr{H}^1$) is a Hilbert space of functions from $\operatorname{dom}(\mathfrak{a}_\varepsilon)$ (respectively, $\operatorname{dom}(\mathfrak{a})$) equipped with the scalar product $(u,v)_{\mathscr{H}^1_\varepsilon} = \mathfrak{a}_\varepsilon[u,v] + (u,v)_{\mathscr{H}_\varepsilon}$ (respectively, $(u,v)_{\mathscr{H}^1} = \mathfrak{a}[u,v] + (u,v)_{\mathscr{H}}$).

Our goal is to show that $(\mathscr{H}, \mathfrak{a})$ and $(\mathscr{H}_\varepsilon, \mathfrak{a}_\varepsilon)$ are δ_ε-close of order $k = 2$ with respect to the operators $J_\varepsilon\colon \mathscr{H} \to \mathscr{H}_\varepsilon$ defined in (14), $J'_\varepsilon\colon \mathscr{H}_\varepsilon \to \mathscr{H}$ defined in (17) and suitable operators $J^1_\varepsilon\colon \mathscr{H}^1 \to \mathscr{H}^1_\varepsilon$, $J'_\varepsilon\colon \mathscr{H}^1_\varepsilon \to \mathscr{H}^1$. Then Theorem 2.3 follows immediately from Theorem 3.3, Theorem 2.5 follows from Corollary 3.8, and Theorem 2.6 follows from Theorem 3.9. The proof of Theorem 2.7 needs an additional step. For convenience, we postpone it to the end of this section.

We define the operator $J^{1\prime}_\varepsilon$ being equal to J'_ε on $\mathscr{H}^1_\varepsilon$. Thus the only non-obvious definition is the one of J^1_ε as we have to assure that $J^1_\varepsilon f\restriction_{\cup_{i\in\mathscr{I}_\varepsilon} D_{i\varepsilon}} = 0$.

We define

$$J^1_\varepsilon f := f - \sum_{i\in\mathscr{I}_\varepsilon} P_{i\varepsilon} f - \sum_{i\in\mathscr{I}_\varepsilon} Q_{i\varepsilon} f$$

with

$$(P_{i\varepsilon} f)(x) := \big(f(x) - f_{i\varepsilon}\big)\chi_{i\varepsilon}(x) \qquad \text{and} \qquad (Q_\varepsilon f)(x) := f_{i\varepsilon} H_{i\varepsilon}(x)\widehat{\chi}_{i\varepsilon}(x)$$

Here (see also Figure 3)

- $f_{i\varepsilon} := \langle f\rangle_{\square_{i\varepsilon}}$,

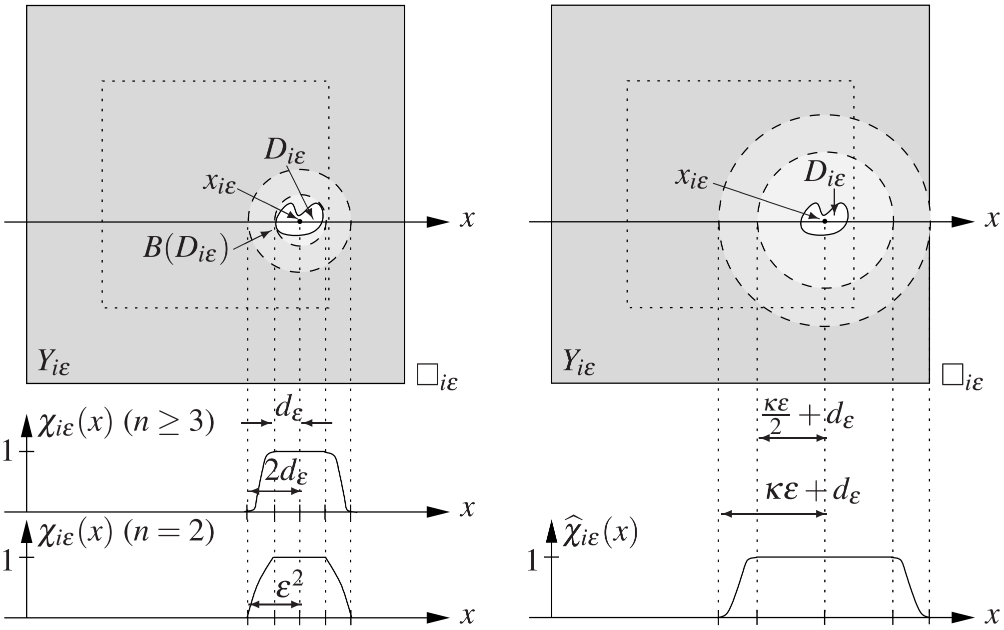

FIGURE 3. The two cut-off functions $\chi_{i\varepsilon}$ and $\widehat{\chi}_{i\varepsilon}$ with decay on the scale d_ε and ε, respectively. On the left, there is the cut-off function $\chi_{i\varepsilon}$, which is 1 inside the small ball $B(D_{i\varepsilon})$ (light gray) with radius d_ε, and 0 outside the larger ball around $x_{i\varepsilon}$ with radius $2d_\varepsilon$ ($n \geq 3$) resp. ε^2 ($n = 2$).
On the right, there is the cut-off function $\widehat{\chi}_{i\varepsilon}$, which is 1 inside the light gray ball of radius $\kappa\varepsilon/2 + d_\varepsilon$, and 0 on the dark gray area outside the larger ball of radius $\kappa\varepsilon + d_\varepsilon$. Both cut-off functions have support in $\square_{i\varepsilon}$.

- $x_{i\varepsilon}$ denotes the center of the smallest ball $B(D_{i\varepsilon})$ containing the set $D_{i\varepsilon}$ (recall that this ball has radius d_ε),
- for $n \geq 3$:
$$\chi_{i\varepsilon}(x) = \chi\left(\frac{|x - x_{i\varepsilon}|}{d_\varepsilon}\right),$$
where $\chi \in \mathsf{C}^\infty(\mathbb{R})$ is a smooth cut-off function such that $|\chi(t)| \leq 1$ and
$$\chi(t) = 1 \text{ as } t < 1 \text{ and } \chi(t) = 0 \text{ as } t > 2,$$
- for $n = 2$:
$$\chi_{i\varepsilon}(x) = \begin{cases} 1 & \text{as } |x - x_{i\varepsilon}| \leq d_\varepsilon, \\ \dfrac{\ln|x - x_{i\varepsilon}| - 2\ln\varepsilon}{\ln d_\varepsilon - 2\ln\varepsilon} & \text{as } |x - x_{i\varepsilon}| \in (d_\varepsilon, \varepsilon^2), \\ 0 & \text{as } |x - x_{i\varepsilon}| \geq \varepsilon^2, \end{cases}$$
- $\widehat{\chi}_{i\varepsilon}(x) := \chi\left(\dfrac{(2/\kappa) \cdot (|x - x_{i\varepsilon}| - d_\varepsilon)}{\varepsilon}\right)$,
- for $n \geq 3$: $H_{i\varepsilon}$ is the solution to the problem
$$\begin{cases} \Delta H_{i\varepsilon}(x) = 0, & x \in \mathbb{R}^n \setminus \overline{D_{i\varepsilon}}, \\ H_{i\varepsilon}(x) = 1, & x \in \partial D_{i\varepsilon}, \\ H_{i\varepsilon}(x) \to 0, & |x| \to \infty \end{cases}$$

- for $n=2$: $H_{i\varepsilon}$ is the solution to the problem

$$\begin{cases}\Delta H_{i\varepsilon}(x)=0, & x\in B_1(x_{i\varepsilon})\setminus\overline{D_{i\varepsilon}},\\ H_{i\varepsilon}(x)=1, & x\in\partial D_{i\varepsilon},\\ H_{i\varepsilon}(x)=0, & x\in\partial B_1(x_{i\varepsilon}),\end{cases}$$

$B_1(x_{i\varepsilon})$ is the unit ball centered at $x_{i\varepsilon}$,

extended by 0 to $\mathbb{R}^n\setminus\overline{B_1(x_{i\varepsilon})}$.

Note, that the function $H_{i\varepsilon}$ is defined on $\mathbb{R}^n\setminus D_{i\varepsilon}$ (resp. $B_1(x_{i\varepsilon})\setminus D_{i\varepsilon}$ if $n=2$). We extend it onto $D_{i\varepsilon}$ by 1 (and onto $\mathbb{R}^n\setminus B_1(x_{i\varepsilon})$ by 0 if $n=2$), keeping the same notation $H_{i\varepsilon}$. Note also that $\|\nabla H_{i\varepsilon}\|^2_{\mathsf{L}_2(\mathbb{R}^n)}=\mathrm{cap}(D_{i\varepsilon})$ by the definition of capacity in (7) and (10).

We set $Y_{i\varepsilon}:=\square_{i\varepsilon}\setminus\overline{D_{i\varepsilon}}$. It is easy to see that

$$P_{i\varepsilon}f\restriction_{D_{i\varepsilon}}=f\restriction_{D_{i\varepsilon}}-f_{i\varepsilon},\quad Q_{i\varepsilon}f\restriction_{D_{i\varepsilon}}=f_{i\varepsilon},\quad \mathrm{supp}(P_{i\varepsilon}f)\subset\square_{i\varepsilon},\quad \mathrm{supp}(Q_{i\varepsilon}f)\subset\square_{i\varepsilon}$$

(the inclusions are valid for $d_\varepsilon\le\kappa\varepsilon$, which holds true for small enough ε in view of (4)). Consequently $J^1_\varepsilon f\in\mathsf{H}^1_0(\Omega_\varepsilon)$.

Now, we are in position to start the proof of (C_{1a})–(C_5).

At first, we note that conditions (C_{1b}), (C_2), (C_{4b}) hold with $\delta_\varepsilon=0$, following from the definitions of the operators J_ε, J'_ε and $J^{1\prime}_\varepsilon$. Also, obviously, for each $f\in\mathsf{L}_2(\Omega)$ and $u\in\mathsf{L}_2(\Omega_\varepsilon)$ we have

$$\|J_\varepsilon f\|_{\mathsf{L}_2(\Omega_\varepsilon)}\le\|f\|_{\mathsf{L}_2(\Omega)},\quad \|J'_\varepsilon u\|_{\mathsf{L}_2(\Omega)}=\|u\|_{\mathsf{L}_2(\Omega_\varepsilon)},$$

and therefore conditions (C_{3a})–(C_{3b}) are valid as well with $\delta_\varepsilon=0$. Thus, it remains to check the non-trivial conditions (C_{1a}), (C_{4a}) and (C_5).

The following Friedrichs- and Poincare-type inequalities will be frequently used further.

Lemma 4.1. *One has*

$$\forall v\in\mathsf{H}^1_0(\square_{i\varepsilon}):\quad \|v\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\le C\varepsilon^2\|\nabla v\|^2_{\mathsf{L}_2(\square_{i\varepsilon})},\tag{25}$$

$$\forall v\in\mathsf{H}^1(\square_{i\varepsilon}):\quad \|v-\langle v\rangle_{\square_{i\varepsilon}}\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\le C\varepsilon^2\|\nabla v\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}.\tag{26}$$

Proof. By the min-max principle

$$\Lambda^{\mathrm{D}}_\varepsilon=\min\left\{\frac{\|\nabla u\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}}{\|u\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}}:\ u\in\mathsf{H}^1_0(\square_{i\varepsilon})\setminus\{0\}\right\},$$

$$\Lambda^{\mathrm{N}}_\varepsilon=\min\left\{\frac{\|\nabla u\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}}{\|u\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}}:\ u\in\mathsf{H}^1(\square_{i\varepsilon})\setminus\{0\},\ \int_{\square_{i\varepsilon}}u(x)\,\mathrm{d}x=0\right\},$$

where $\Lambda^{\mathrm{D}}_\varepsilon$ (respectively, $\Lambda^{\mathrm{N}}_\varepsilon$) is the first (respectively, the second) eigenvalue of the Dirichlet (respectively, the Neumann) Laplacian on $\square_{i\varepsilon}$. Straightforward calculations gives

$$\lambda^{\mathrm{D}}_\varepsilon=n\left(\frac{\pi}{\varepsilon}\right)^2,\qquad \lambda^{\mathrm{N}}_\varepsilon=\left(\frac{\pi}{\varepsilon}\right)^2,$$

hence we easily get the required inequalities (25)–(26). ☐

4.1. **Proof of** (C_{1a})**.** Let $f \in \mathsf{L}_2(\Omega)$. We have

$$\|J_\varepsilon f - J^1_\varepsilon f\|^2_{\mathscr{H}_\varepsilon} \le 2 \sum_{i\in\mathscr{I}_\varepsilon} \left(\|P_{i\varepsilon} f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} + \|Q_{i\varepsilon} f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\right). \tag{27}$$

Using (26) and taking into account that $|\chi_{i\varepsilon}(x)| \le 1$, we obtain:

$$\sum_{i\in\mathscr{I}_\varepsilon} \|P_{i\varepsilon} f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} \le \sum_{i\in\mathscr{I}_\varepsilon} \|f - f_{i\varepsilon}\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} \le C\varepsilon^2 \sum_{i\in\mathscr{I}_\varepsilon} \|\nabla f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} \le C\varepsilon^2 \|f\|^2_{\mathscr{H}^1}. \tag{28}$$

Using (25) and taking into account that $|\widehat{\chi}_{i\varepsilon}| \le 1$, $|\widehat{\chi}'_{i\varepsilon}| \le C\kappa^{-1}\varepsilon^{-1}$ and $\widehat{\chi}_{i\varepsilon}\restriction_{D_{i\varepsilon}} = H_{i\varepsilon}\restriction_{D_{i\varepsilon}} = 1$ we obtain:

$$\begin{aligned}\sum_{i\in\mathscr{I}_\varepsilon} \|Q_{i\varepsilon} f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} &\le \sum_{i\in\mathscr{I}_\varepsilon} |f_{i\varepsilon}|^2 \|\widehat{\chi}_{i\varepsilon} H_{i\varepsilon}\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} \le C\varepsilon^2 \sum_{i\in\mathscr{I}_\varepsilon} |f_{i\varepsilon}|^2 \|\nabla(\widehat{\chi}_{i\varepsilon} H_{i\varepsilon})\|^2_{\mathsf{L}_2(Y_{i\varepsilon})} \\ &\le C_1\varepsilon^2 \sum_{i\in\mathscr{I}_\varepsilon} |f_{i\varepsilon}|^2 \left(\|\nabla H_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})} + \kappa^{-2}\varepsilon^{-2} \|H_{i\varepsilon}\|^2_{\mathsf{L}_2(\mathrm{supp}(\nabla\widehat{\chi}_{i\varepsilon}))}\right).\end{aligned} \tag{29}$$

From (6) and the definition of $H_{i\varepsilon}$ we obtain the estimate

$$\|\nabla H_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})} \le \|\nabla H_{i\varepsilon}\|^2_{\mathsf{L}_2(\mathbb{R}^n)} = \mathrm{cap}(D_{i\varepsilon}) \le C\varepsilon^n. \tag{30}$$

Using Lemma 2.1, we obtain

$$|H_{i\varepsilon}(x)| \le C\tau_n(d_\varepsilon)/\tau_n(\varepsilon), \quad \text{where} \quad \tau_n(r) := \begin{cases} r^{n-2}, & n \ge 3, \\ |\ln r|^{-1}, & n = 2, \end{cases}$$

for $x \in \mathrm{supp}(\nabla\widehat{\chi}_{i\varepsilon}) \subset \{x \in \mathbb{R}^n :\ d_\varepsilon + \kappa\varepsilon/2 \le |x - x_{i\varepsilon}| \le d_\varepsilon + \kappa\varepsilon\}$. Hence, taking into account that $\tau_n(d_\varepsilon) \le C\varepsilon^n$ (see (4)), we deduce the asymptotics

$$\|H_{i\varepsilon}\|^2_{\mathsf{L}_2(\mathrm{supp}(\nabla\chi_{i\varepsilon}))} = o(\varepsilon^{n+2}). \tag{31}$$

Finally, applying the Cauchy-Schwarz inequality, one gets

$$|f_{i\varepsilon}|^2 \le \varepsilon^{-n} \|f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}. \tag{32}$$

Combining (29)–(32) we arrive at

$$\sum_{i\in\mathscr{I}_\varepsilon} \|Q_{i\varepsilon} f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})} \le C_\kappa \varepsilon^2 \|f\|^2_{\mathscr{H}}. \tag{33}$$

Here and in what follows by C_κ we denote a generic constant depending on κ and n.

From (27), (28), (33) we obtain

$$\|J_\varepsilon f - J^1_\varepsilon f\|_{\mathscr{H}_\varepsilon} \le C_\kappa \varepsilon \|f\|_{\mathscr{H}^1} \le \delta_\varepsilon \|f\|_{\mathscr{H}^1},$$

where δ_ε is defined in (16). Therefore, we have checked Condition (C_{1a}).

4.2. **Proof of** (C_{4a})**.** We need the following lemma, which was proven in [MK06, Lem. 4.9 and Rem. 4.2)].

Lemma 4.2. *Let $D \subset \mathbb{R}^n$ be a bounded convex domain, and let $D_1, D_2 \subset D$ be measurable subsets with $|D_2| \ne 0$. Then*

$$\|v\|^2_{\mathsf{L}_2(D_1)} \le \frac{2|D_1|}{|D_2|} \|v\|^2_{\mathsf{L}_2(D_2)} + C\frac{(\mathrm{diam}(D))^{n+1}|D_1|^{1/n}}{|D_2|} \|\nabla v\|^2_{\mathsf{L}_2(D)}$$

for all $v \in \mathsf{H}^1(D)$, where C depends only on the dimension n.

Let $f\in \mathsf{L}_2(\Omega)$. Applying Lemma 4.2 with $v:=f$ and $D:=\square_{i\varepsilon}$, $D_1:=D_{i\varepsilon}$, $D_2:=\square_{i\varepsilon}$ we obtain

$$\|f-J_\varepsilon' J_\varepsilon f\|^2_{\mathscr{H}}=\sum_{i\in\mathscr{I}_\varepsilon}\|f\|^2_{\mathsf{L}_2(D_{i\varepsilon})}\le C\sum_{i\in\mathscr{I}_\varepsilon}\left(\left(\frac{d_\varepsilon}{\varepsilon}\right)^n\|f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}+\frac{\varepsilon^{n+1}d_\varepsilon}{\varepsilon^n}\|\nabla f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\right)$$
$$\le C\left(\left(\frac{d_\varepsilon}{\varepsilon}\right)^n+\varepsilon d_\varepsilon\right)\|f\|^2_{\mathscr{H}^1}.$$

It is straightforward to show, using (4) and the definition of δ_ε in (16), that

$$\left(\frac{d_\varepsilon}{\varepsilon}\right)^n+\varepsilon d_\varepsilon\le C\varepsilon^2\le C(\delta_\varepsilon)^2$$

and thus condition (C_{4a}) is also valid.

4.3. **Proof of the form estimate** (C_5).

We will show that (C_5) holds with $k=2$.

Recall that $\mathscr{H}^2$ is a Hilbert space consisting of functions from $\operatorname{dom}(\mathscr{A})=\mathsf{H}^2(\Omega)\cap\mathsf{H}^1_0(\Omega)$ with scalar product $(f,g)_{\mathscr{H}^2}:=(\mathscr{A}f+f,\mathscr{A}g+g)_{\mathscr{H}}$.

Since $\partial\Omega$ is C^2-smooth, we can apply standard elliptic regularity theory (see, e.g., [GT77]): namely, the $\mathscr{H}^2$-norm is equivalent to the $\mathsf{H}^2(\Omega)$-norm, i.e. there is $C_\Omega>0$ such that for each $f\in\operatorname{dom}(\mathscr{A})$ we have

$$\|f\|_{\mathsf{H}^2(\Omega)}\le C_\Omega\|f\|_{\mathscr{H}^2}. \tag{34}$$

Note, that this is the only estimate in our proof in which the constant depends on the domain Ω. This results to Ω-dependence of the constant standing in the definition of δ_ε.

Let $f\in\operatorname{dom}(\mathscr{A})$, $u\in\mathsf{H}^1_0(\Omega_\varepsilon)$. One has:

$$\left|\mathfrak{a}_\varepsilon[u,J^1_\varepsilon f]-\mathfrak{a}[J^{1\prime}_\varepsilon u,f]\right|$$
$$\le\left|\sum_{i\in\mathscr{I}_\varepsilon}(\nabla u,\nabla P_{i\varepsilon}f)_{\mathsf{L}_2(Y_{i\varepsilon})}\right|+\left|\sum_{i\in\mathscr{I}_\varepsilon}(\nabla u,\nabla Q_{i\varepsilon}f)_{\mathsf{L}_2(Y_{i\varepsilon})}+q(u,f)_{\mathsf{L}_2(\Omega_\varepsilon)}\right|=:\mathscr{J}_{\varepsilon,1}+\mathscr{J}_{\varepsilon,2}.$$

4.3.1. *Estimates for* $\mathscr{J}_{\varepsilon,1}$.

One has, taking into account that $|\chi_{i\varepsilon}(x)|\le 1$:

$$\mathscr{J}_{\varepsilon,1}\le\left|\sum_{i\in\mathscr{I}_\varepsilon}(\nabla u,\chi_{i\varepsilon}\nabla f)_{\mathsf{L}_2(\operatorname{supp}(\chi_{i\varepsilon}))}\right|+\left|\sum_{i\in\mathscr{I}_\varepsilon}(\nabla u,(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon})_{\mathsf{L}_2(\operatorname{supp}(\chi_{i\varepsilon}))}\right|$$
$$\le\|\nabla u\|_{\mathsf{L}_2(\Omega_\varepsilon)}\left(\left(\sum_{i\in\mathscr{I}_\varepsilon}\|\nabla f\|^2_{\mathsf{L}_2(\operatorname{supp}(\chi_{i\varepsilon}))}\right)^{\frac12}+\left(\sum_{i\in\mathscr{I}_\varepsilon}\|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})}\right)^{\frac12}\right) \tag{35}$$

Applying Lemma 4.2 with $v:=\partial_j f$, $D:=\square_{i\varepsilon}$, $D_1:=\operatorname{supp}(\chi_{i\varepsilon})$ and $D_2:=\square_{i\varepsilon}$, and taking into account (4) and (34) we obtain the estimates:

$$\begin{aligned}
n\ge 2:&\quad\sum_{i\in\mathscr{I}_\varepsilon}\|\nabla f\|^2_{\mathsf{L}_2(\operatorname{supp}(\chi_{i\varepsilon}))}\le C\left(\left(\frac{d_\varepsilon}{\varepsilon}\right)^n+\varepsilon\cdot d_\varepsilon\right)\|f\|^2_{\mathsf{H}^2(\Omega)}\le C\varepsilon^2\|f\|^2_{\mathscr{H}^2},\\
n=2:&\quad\sum_{i\in\mathscr{I}_\varepsilon}\|\nabla f\|^2_{\mathsf{L}_2(\operatorname{supp}(\chi_{i\varepsilon}))}\le C\left(\left(\frac{\varepsilon^2}{\varepsilon}\right)^2+\varepsilon\cdot\varepsilon^2\right)\|f\|^2_{\mathsf{H}^2(\Omega)}\le C\varepsilon^2\|f\|^2_{\mathscr{H}^2}.
\end{aligned} \tag{36}$$

Now, we estimate the second term in (35). One has the following Hölder-type inequality:

$$\forall p,q\in[2,\infty],\ \frac{1}{p}+\frac{1}{q}=\frac{1}{2}:\ \|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|_{\mathsf{L}_2(Y_{i\varepsilon})}\le\|f-f_{i\varepsilon}\|_{\mathsf{L}_p(Y_{i\varepsilon})}\|\nabla\chi_{i\varepsilon}\|_{\mathsf{L}_q(Y_{i\varepsilon})} \quad (37)$$

(for $p=\infty$ we use a convention $p^{-1}=0$). Indeed, the classical Hölder inequality states that $\|FG\|_{\mathsf{L}_1(Y_{i\varepsilon})}\le\|F\|_{\mathsf{L}_{\mathbf{p}}(Y_{i\varepsilon})}\|G\|_{\mathsf{L}_{\mathbf{q}}(Y_{i\varepsilon})}$ provided $\mathbf{p},\mathbf{q}\in[1,\infty]$, $\mathbf{p}^{-1}+\mathbf{q}^{-1}=1$. Setting $\mathbf{p}:=p/2$, $\mathbf{q}:=q/2$, $F:=|f-f_{i\varepsilon}|^2$, $G:=|\nabla\chi_{i\varepsilon}|^2$ we easily arrive at (37).

One needs the following re-scaled Sobolev inequality.

Lemma 4.3. *For each* $f\in\mathsf{H}^2(\square_{i\varepsilon})$

$$\|f-f_{i\varepsilon}\|_{\mathsf{L}_p(\square_{i\varepsilon})}\le C_p\varepsilon^{n/p+(2-n)/2}\|f\|_{\mathsf{H}^2(\square_{i\varepsilon})} \quad (38)$$

provided p *satisfies*

$$2\le p\le\frac{2n}{n-4}\ \text{ as } n\ge5,\quad 2\le p<\infty\ \text{ as } n=4,\quad p=\infty\ \text{ as } n=2,3. \quad (39)$$

The constant C_p *depends only on* p.

Proof. Recall that $\square=(-1/2,1/2)^n$. Since $\mathsf{H}^2(\square)\hookrightarrow\mathsf{L}_p(\square)$ provided (39) holds (see, e.g., [Ad75, Theorem 5.4]) one has for each $g\in\mathsf{H}^2(\square)$:

$$\|g\|_{\mathsf{L}_p(\square)}\le C_p\|g\|_{\mathsf{H}^2(\square)}. \quad (40)$$

Now, making the change of variables $\square\ni x=y\varepsilon^{-1}-i$ with $y\in\square_{i\varepsilon}$ in (40), we infer from (40): for each $g\in\mathsf{H}^2(\square_{i\varepsilon})$

$$\varepsilon^{-n/p}\|g\|_{\mathsf{L}_p(\square)}\le C_p\Big(\varepsilon^{-n}\|g\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}+\varepsilon^{2-n}\|\nabla g\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}+\varepsilon^{4-n}\sum_{k,l=1}^{n}\|\partial^2_{kl}g\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\Big)^{1/2}. \quad (41)$$

Finally, we set $g:=f-f_{i\varepsilon}$. Then, due to (26), the estimate (41) becomes

$$\begin{aligned}\|f-f_{i\varepsilon}\|_{\mathsf{L}_p(\square)}&\le C_p\varepsilon^{n/p}\Big(\varepsilon^{2-n}\|\nabla f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}+\varepsilon^{4-n}\sum_{k,l=1}^{n}\|\partial^2_{kl}f\|^2_{\mathsf{L}_2(\square_{i\varepsilon})}\Big)^{1/2}\\&\le C_p\varepsilon^{n/p+(2-n)/2}\|f\|_{\mathsf{H}^2(\square_{i\varepsilon})}.\end{aligned}$$

□

We also need the estimate for $\chi_{i\varepsilon}$, which is proved via a straightforward calculations.

Lemma 4.4. *One has*

$$\begin{aligned}&\|\nabla\chi_{i\varepsilon}\|_{\mathsf{L}_q(Y_{i\varepsilon})}\le C(d_\varepsilon)^{n/q-1},\quad n\ge3,\ q\in[2,\infty],\\&\|\nabla\chi_{i\varepsilon}\|_{\mathsf{L}_2(Y_{i\varepsilon})}\le C|\ln d_\varepsilon|^{-1/2},\quad n=2.\end{aligned} \quad (42)$$

Now, we choose the largest p for which (39) holds:

$$p:=\frac{2n}{n-4}\ \text{as } n\ge5,\quad p:=4\beta^{-1}\text{ with }\beta\in(0,2)\ \text{as } n=4,\quad p=\infty\ \text{as } n=2,3. \quad (43)$$

As before

$$q=\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}. \quad (44)$$

Plugging the estimates (38) and (42) into (37) and taking into account (4), (43)–(44) we arrive easily at

$$\begin{aligned}
n\ge 5: &\quad \Big(\sum_{i\in\mathscr{I}_\varepsilon}\|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})}\Big)^{1/2}\le C\cdot\frac{d_\varepsilon}{\varepsilon}\|f\|_{\mathsf{H}^2(\Omega)},\\
n=4: &\quad \Big(\sum_{i\in\mathscr{I}_\varepsilon}\|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})}\Big)^{1/2}\le C_\beta\cdot\Big(\frac{d_\varepsilon}{\varepsilon}\Big)^{1-\beta}\|f\|_{\mathsf{H}^2(\Omega)}\le C_\beta\varepsilon^{1-\beta}\|f\|_{\mathsf{H}^2(\Omega)},\\
n=3: &\quad \Big(\sum_{i\in\mathscr{I}_\varepsilon}\|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})}\Big)^{1/2}\le C\cdot\Big(\frac{d_\varepsilon}{\varepsilon}\Big)^{1/2}\|f\|_{\mathsf{H}^2(\Omega)}\le C\varepsilon\|f\|_{\mathsf{H}^2(\Omega)},\\
n=2: &\quad \Big(\sum_{i\in\mathscr{I}_\varepsilon}\|(f-f_{i\varepsilon})\nabla\chi_{i\varepsilon}\|^2_{\mathsf{L}_2(Y_{i\varepsilon})}\Big)^{1/2}\le C|\ln d_\varepsilon|^{-\frac12}\|f\|_{\mathsf{H}^2(\Omega)}\le C\varepsilon\|f\|_{\mathsf{H}^2(\Omega)}.
\end{aligned}\tag{45}$$

Combining (36) and (45) and taking into account (34) and the definition of δ_ε, we get the estimate

$$\mathscr{J}_{\varepsilon,1}\le\delta_\varepsilon\|u\|_{\mathscr{H}^1}\|f\|_{\mathscr{H}^2}.\tag{46}$$

4.3.2. *Estimates for $\mathscr{J}_{\varepsilon,2}$.* One has

$$\mathscr{J}_{\varepsilon,2}=\left|\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}(\nabla u,\nabla(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))_{\mathsf{L}_2(Y_{i\varepsilon})}+q(u,f)_{\mathsf{L}_2(\Omega_\varepsilon)}\right|.\tag{47}$$

Besides (7) (or (10) for $n=2$) there is another equivalent characterization of the capacity.

Lemma 4.5. *Let $D\subset\mathbb{R}^n$, and let H be the solution of either* (8) *if $n\ge3$ or* (11) *if $n=2$. Then*

$$\operatorname{cap}(D)=-\int_{\partial D}\frac{\partial H}{\partial\nu}\,\mathrm{d}s,\tag{48}$$

where ν is the outward-pointing unit normal to ∂D, $\mathrm{d}s$ *is the area measure on ∂D.*

Proof. For $n=2$ the result follows from

$$\int_{\partial D}\frac{\partial H}{\partial\nu}\,\mathrm{d}s=\int_{\partial D}\frac{\partial H}{\partial\nu}H\,\mathrm{d}s=-\int_{B_1\setminus\overline{D}}|\nabla H|^2\,\mathrm{d}x.$$

Here the first equality is due to $H\restriction_{\partial D}=1$, while the second one is the Green formula, in which the surface integral over ∂B_1 vanishes since $H\restriction_{\partial B_1}=0$, and the second volume integral vanishes since $\Delta H=0$.

For $n\ge3$ we proceed as follows. Let B_R be the ball of radius $R>1$ being concentric with the smallest ball containing D. One has:

$$\int_{\partial D}\frac{\partial H}{\partial\nu}\,\mathrm{d}s=\int_{\partial D}\frac{\partial H}{\partial\nu}H\,\mathrm{d}s=-\int_{B_R\setminus\overline{D}}|\nabla H|^2\,\mathrm{d}x-\int_{\partial B_R}\frac{\partial H}{\partial\nu}H\,\mathrm{d}s\tag{49}$$

(in the last integral ν is the inward-pointing unit normal to ∂B_R). Lemma 2.1 implies the estimate

$$\left|\int\limits_{\partial B_R}\frac{\partial H}{\partial\nu}H\,\mathrm{d}s\right|\leq\frac{C}{R^{n-2}}. \tag{50}$$

Passing to the limit $R\to\infty$ in (49) and taking into account (7) and (50) we arrive at the required equality (48). The lemma is proved. □

Denote $u_{i\varepsilon}:=\langle J'_\varepsilon u\rangle_{\square_{i\varepsilon}}$. Integrating by parts and using (48) we get

$$\begin{aligned}\sum_{i\in\mathscr{I}_\varepsilon}&f_{i\varepsilon}(\nabla u,\nabla(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))_{\mathsf{L}_2(Y_{i\varepsilon})}\\ &=-\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}(u,\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))_{\mathsf{L}_2(Y_{i\varepsilon})}=-\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}u_{i\varepsilon}(1,\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))_{\mathsf{L}_2(Y_{i\varepsilon})}+\zeta_\varepsilon\\ &=\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}u_{i\varepsilon}\int\limits_{\partial D_{i\varepsilon}}\frac{\partial H_{i\varepsilon}}{\partial\nu}\mathrm{d}s+\zeta_\varepsilon=-\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}u_{i\varepsilon}\operatorname{cap}(D_\varepsilon)+\zeta_\varepsilon,\end{aligned} \tag{51}$$

where $\zeta_\varepsilon=\sum_{i\in\mathscr{I}_\varepsilon}f_{i\varepsilon}(u_{i\varepsilon}-u,\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))_{\mathsf{L}_2(Y_{i\varepsilon})}$. Here we have used the facts that $\widehat{\chi}_{i\varepsilon}$ vanishes on $\{x\in Y_{i\varepsilon}:\ |x-x_{i\varepsilon}|\geq d_\varepsilon+\varepsilon\kappa\}$ with all its derivatives, $u\restriction_{\partial D_{i\varepsilon}}=0$, and $\widehat{\chi}_{i\varepsilon}(x)=1$ in a neighbourhood of $D_{i\varepsilon}$.

The remainder term ζ_ε is small; namely the following estimate holds:

Lemma 4.6. *One has:*

$$|\zeta_\varepsilon|\leq C_\kappa\cdot\begin{cases}\varepsilon\|\nabla u\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|f\|_{\mathsf{L}_2(\Omega)}, & n\geq 3,\\ \varepsilon|\ln\varepsilon|\|\nabla u\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|f\|_{\mathsf{L}_2(\Omega)}, & n=2.\end{cases} \tag{52}$$

Proof. At first we consider the case $n\geq 3$. Since $\Delta H_{i\varepsilon}=0$ we have

$$\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon})=2\nabla H_{i\varepsilon}\cdot\nabla\widehat{\chi}_{i\varepsilon}+H_{i\varepsilon}\Delta\widehat{\chi}_{i\varepsilon},$$
$$\operatorname{supp}(\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))\subset\{x\in Y_{i\varepsilon}:\ d_\varepsilon+\kappa\varepsilon/2\leq|x-x_{i\varepsilon}|\leq d_\varepsilon+\kappa\varepsilon\},$$

hence, due to Lemma 2.1, (4) and $|\nabla\widehat{\chi}_{i\varepsilon}(x)|\leq C\kappa^{-1}\varepsilon^{-1}$, $|\Delta\widehat{\chi}_{i\varepsilon}(x)|\leq C\kappa^{-2}\varepsilon^{-2}$, we get

$$|(\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))(x)|\leq C\left(\frac{(d_\varepsilon)^{n-2}}{\varepsilon^{n-1}}\cdot\frac{1}{\kappa\varepsilon}+\frac{(d_\varepsilon)^{n-2}}{\varepsilon^{n-2}}\cdot\frac{1}{\kappa^2\varepsilon^2}\right)\leq C_\kappa, \tag{53}$$

Then, using the Cauchy-Schwarz inequality, (26) and (53), we obtain

$$|\zeta_\varepsilon|\leq C_\kappa\left(\sum_{i\in\mathscr{I}_\varepsilon}\|J'_\varepsilon u-u_{i\varepsilon}\|_{\mathsf{L}_2(\square_{i\varepsilon})}\right)^{1/2}\left(\sum_{i\in\mathscr{I}_\varepsilon}|f_{i\varepsilon}|^2\varepsilon^n\right)^{1/2}\leq C_\kappa\varepsilon\|\nabla u\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|f\|_{\mathsf{L}_2(\Omega)}.$$

In the case $n=2$ Lemma 2.1 gives

$$|(\Delta(H_{i\varepsilon}\widehat{\chi}_{i\varepsilon}))(x)|\leq C_\kappa|\ln\varepsilon|,$$

and, via the same arguments as in the case $n\geq 3$, we obtain

$$|\zeta_\varepsilon|\leq C_\kappa\varepsilon|\ln\varepsilon|\|\nabla u\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|f\|_{\mathsf{L}_2(\Omega)}.$$

The lemma is proved. □

We need also the estimate for $\mathsf{H}^1_0(\Omega)$-functions in a neighbourhood of $\partial\Omega$.

Lemma 4.7. *Let $T_\varepsilon = \Omega_\varepsilon \setminus \overline{\cup_{i\in\mathscr{I}_\varepsilon} Y_{i\varepsilon}}$. One has for each $f \in \mathsf{H}^1_0(\Omega)$:*

$$\|f\|_{\mathsf{L}_2(T_\varepsilon)} \le C\varepsilon \|\nabla f\|_{\mathsf{L}_2(\Omega)}. \tag{54}$$

Proof. We denote

$$\widehat{\Omega}_\varepsilon = \{y \in \mathbb{R}^n:\ y = x + t\nu(x),\ x \in \partial\Omega,\ t \in [0, \sqrt{n}\varepsilon)\}$$

(recall that $\nu\colon \partial\Omega \to \mathbb{S}^{n-1}$ is a unit inward-pointing normal vector field on $\partial\Omega$). Note, that $\sqrt{n}$ is the length of the diagonal of the cube $\square$. Taking this into account one can easily deduce from the definition of the set $\mathscr{I}_\varepsilon$ that $T_\varepsilon \subset \widehat{\Omega}_\varepsilon$.

Let $\Delta_{\widehat{\Omega}_\varepsilon}$ be the Laplace operator on $\widehat{\Omega}_\varepsilon$ subject to the Dirichlet conditions on $\partial\Omega$ and the Neumann conditions on $\partial\widehat{\Omega}_\varepsilon \setminus \partial\Omega$. One has the following asymptotic equality (see [Kr14]):

$$\inf\sigma(-\Delta_{\widehat{\Omega}_\varepsilon}) = \left(\frac{\pi}{2\sqrt{n}\varepsilon}\right)^2 + \frac{C_\Omega}{\varepsilon} + \mathscr{O}(1),$$

where the constant C_Ω depends on the principal curvatures of $\partial\Omega$. Note, that the result of [Kr14] is obtained under the assumption that the map (3) is injective on $\partial\Omega \times [0, \sqrt{n}\varepsilon]$, that indeed holds true provided ε is small enough, namely $\varepsilon < \theta_\Omega/\sqrt{n}$.

Hence, using the minimax principle, we get the inequality

$$\|f\|^2_{\mathsf{L}_2(\widehat{\Omega}_\varepsilon)} \le C\varepsilon^2 \|\nabla f\|^2_{\mathsf{L}_2(\widehat{\Omega}_\varepsilon)}, \tag{55}$$

which holds for each $f \in \mathsf{H}^1(\widehat{\Omega}_\varepsilon)$ with $f\restriction_{\partial\Omega} = 0$. Obviously, (54) follows from (55). $\square$

Using (26), (51), (54) we obtain from (47):

$$\begin{aligned}
\mathscr{J}_{\varepsilon,2} &\le |\zeta_\varepsilon| + \left|\sum_{i\in\mathscr{I}_\varepsilon} f_{i\varepsilon}u_{i\varepsilon}\operatorname{cap}(D_\varepsilon) - q\sum_{i\in\mathscr{I}_\varepsilon}(f,u)_{\mathsf{L}_2(Y_{i\varepsilon})}\right| + q\left|(f,u)_{\mathsf{L}_2(T_\varepsilon)}\right| \\
&\le |\zeta_\varepsilon| + \left|(\operatorname{cap}(D_\varepsilon)\varepsilon^{-n} - q)\sum_{i\in\mathscr{I}_\varepsilon}\varepsilon^n f_{i\varepsilon}u_{i\varepsilon}\right| + q\left|\sum_{i\in\mathscr{I}_\varepsilon}(f_{i\varepsilon} - f, u)_{\mathsf{L}_2(Y_{i\varepsilon})}\right| + q\left|(f,u)_{\mathsf{L}_2(T_\varepsilon)}\right| \\
&\le |\zeta_\varepsilon| + |\operatorname{cap}(D_\varepsilon)\varepsilon^{-n} - q|\|f\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|u\|_{\mathsf{L}_2(\Omega_\varepsilon)} + q\left(\sum_{i\in\mathscr{I}_\varepsilon}\|f - f_{i\varepsilon}\|_{\mathsf{L}_2(\square_{i\varepsilon})}\right)^{1/2}\|u\|_{\mathsf{L}_2(\Omega_\varepsilon)} \\
&\quad + q\|f\|_{\mathsf{L}_2(T_\varepsilon)}\|u\|_{\mathsf{L}_2(T_\varepsilon)} \le |\zeta_\varepsilon| + |\operatorname{cap}(D_\varepsilon)\varepsilon^{-n} - q|\|f\|_{\mathsf{L}_2(\Omega_\varepsilon)}\|u\|_{\mathsf{L}_2(\Omega_\varepsilon)} \\
&\quad + C\varepsilon\|\nabla f\|_{\mathsf{L}_2(\Omega)}\|u\|_{\mathsf{L}_2(\Omega_\varepsilon)} + q\|f\|_{\mathsf{L}_2(T_\varepsilon)}\|u\|_{\mathsf{L}_2(T_\varepsilon)},
\end{aligned}$$

hence, taking into account (52), we get

$$\mathscr{J}_{\varepsilon,2} \le \delta_\varepsilon \|u\|_{\mathscr{H}^1}\|f\|_{\mathscr{H}^1}. \tag{56}$$

Combining estimates (46) and (56) we obtain (C_5) with $k = 2$.

Thus, we have checked the fulfilment of conditions (C_{1a})–(C_5), hence we immediately get Theorems 2.3–2.6.

Remark 4.8. It is evident from the proof that the assumptions on $\partial\Omega$ can be weakened. We use them twice: to guarantee the fulfilment of (34) (elliptic regularity) and to prove estimate (55), where we utilize the result from [Kr14]. It is well-known, that the elliptic regularity is still valid under less restrictive assumptions, for example, if $\partial\Omega$ belongs to $C^{1,1}$ class or Ω is a convex domain with Lipschitz boundary (see, e.g., [Gv85, Theorems 2.2.2.3 and 3.2.1.2]). Apparently, inequality (55) can be proved for Lipschitz domains under additional restrictions on principal curvatures.

4.4. **Proof of Theorem 2.7.** We will use the results of [IOS89]. Let $\mathscr{H}_\varepsilon$ and $\mathscr{H}$ be separable Hilbert spaces, and $B_\varepsilon\colon \mathscr{H}_\varepsilon \to \mathscr{H}_\varepsilon$, $B\colon \mathscr{H} \to \mathscr{H}$ be linear compact self-adjoint positive operators. We denote by $\{\mu_{k,\varepsilon}\}_{k\in\mathbb{N}}$ and $\{\mu_k\}_{k\in\mathbb{N}}$ the eigenvalues of the operators B_ε and B, respectively, being renumbered in the descending order and with account of their multiplicity.

Theorem 4.9 ([IOS89]). *Assume that the following conditions $A_1 - A_4$ hold:*

A_1. *The linear bounded operator $J_\varepsilon\colon \mathscr{H} \to \mathscr{H}_\varepsilon$ exists such that for each $f \in \mathscr{H}$*

$$\|J_\varepsilon f\|_{\mathscr{H}_\varepsilon} \to \|f\|_{\mathscr{H}} \text{ as } \varepsilon \to 0.$$

A_2. *The norms $\|B_\varepsilon\|_{\mathscr{L}(\mathscr{H}_\varepsilon)}$ are bounded uniformly in ε.*

A_3. *For any $f \in \mathscr{H}$: $\|B_\varepsilon J_\varepsilon f - J_\varepsilon B f\|_{\mathscr{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$.*

A_4. *For any family $\{f_\varepsilon \in \mathscr{H}_\varepsilon\}_\varepsilon$ with $\sup_\varepsilon \|f_\varepsilon\|_{\mathscr{H}_\varepsilon} < \infty$ there exist a sequence $(\varepsilon_m)_m$ and $w \in \mathscr{H}$ such that $\|B_{\varepsilon_m} f_{\varepsilon_m} - J_{\varepsilon_m} w\|_{\mathscr{H}_{\varepsilon_m}} \to 0$ and $\varepsilon_m \to 0$ as $m \to \infty$.*

Then for any $k \in \mathbb{N}$ we have

$$|\mu_{k,\varepsilon} - \mu_k| \le C_\varepsilon \sup_f \|B_\varepsilon J_\varepsilon f - J_\varepsilon B f\|_{\mathscr{H}_\varepsilon},$$

where $|C_\varepsilon| \le C$, $\lim_{\varepsilon\to 0} C_\varepsilon = 1$, the supremum is taken over all $f \in \mathscr{H}$ belonging to the eigenspace associated with μ_k and satisfying $\|f\|_{\mathscr{H}} = 1$.

We apply this theorem with $B_\varepsilon = (\mathscr{A}_\varepsilon + I)^{-1}$, $B = (\mathscr{A} + I)^{-1}$. These operators are positive, self-adjoint and compact (recall that Ω is a bounded domain here), moreover $\|B_\varepsilon\|_{\mathscr{L}(\mathscr{H}_\varepsilon)} \le 1$. Thus condition A_2 is fulfilled. We choose the operator J_ε by (14); due to (15) condition A_1 is valid. By Theorem 2.3 condition A_3 holds as well. Finally, since $\|B_\varepsilon\|_{\mathscr{L}(\mathscr{H}_\varepsilon)} \le 1$, the set $\{\|B_\varepsilon f_\varepsilon\|_{\mathsf{H}^1(\Omega_\varepsilon)}\}_\varepsilon$ is also bounded. Then the sequence $\{J'_\varepsilon B_\varepsilon f_\varepsilon\}_\varepsilon$ is bounded in $\mathsf{H}^1(\Omega)$ (recall that the operator J'_ε is defined in (17)), and by Rellich's embedding theorem it is compact in $\mathsf{L}_2(\Omega)$ provided Ω is bounded. Thus there exist $w \in \mathsf{L}_2(\Omega)$ and a sequence $(\varepsilon_m)_m$ such that $\|J'_{\varepsilon_m} B_{\varepsilon_m} f_{\varepsilon_m} - w\|_{\mathsf{L}_2(\Omega)} \to 0$ and $\varepsilon_m \to 0$ as $m \to \infty$, hence we immediately obtain Condition A_4.

Combining Theorems 2.3 and 4.9 we arrive at the estimate

$$\left|\mu_{k,\varepsilon} - \mu_k\right| \le 4 C_\varepsilon \delta_\varepsilon, \tag{57}$$

where $|C_\varepsilon| \le C$, $\lim_{\varepsilon\to 0} C_\varepsilon = 1$ and δ_ε is given in (16). Since $\mu_{k,\varepsilon} = (\lambda_{k,\varepsilon} + 1)^{-1}$, and $\mu_k = (\lambda_k + 1)^{-1}$, (57) is equivalent to (19).

Finally, we observe that for each fixed $k \in \mathbb{N}$

$$\lambda_{k,\varepsilon} \le C_k \tag{58}$$

that follows from Theorem 3.12 and Remark 3.11 (otherwise, we will easily obtain a contradiction with Condition (ii) from this remark). (19), (58) imply (18). Theorem 2.7 is proved.

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INSTITUT FÜR ANALYSIS, KARLSRUHE INSTITUTE OF TECHNOLOGY, KARLSRUHE, GERMANY

E-mail address: andrii.khrabustovskyi@kit.edu

FACHBEREICH 4 – MATHEMATIK, UNIVERSITÄT TRIER, 54286 TRIER, GERMANY

E-mail address: olaf.post@uni-trier.de