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Rainer Mandel

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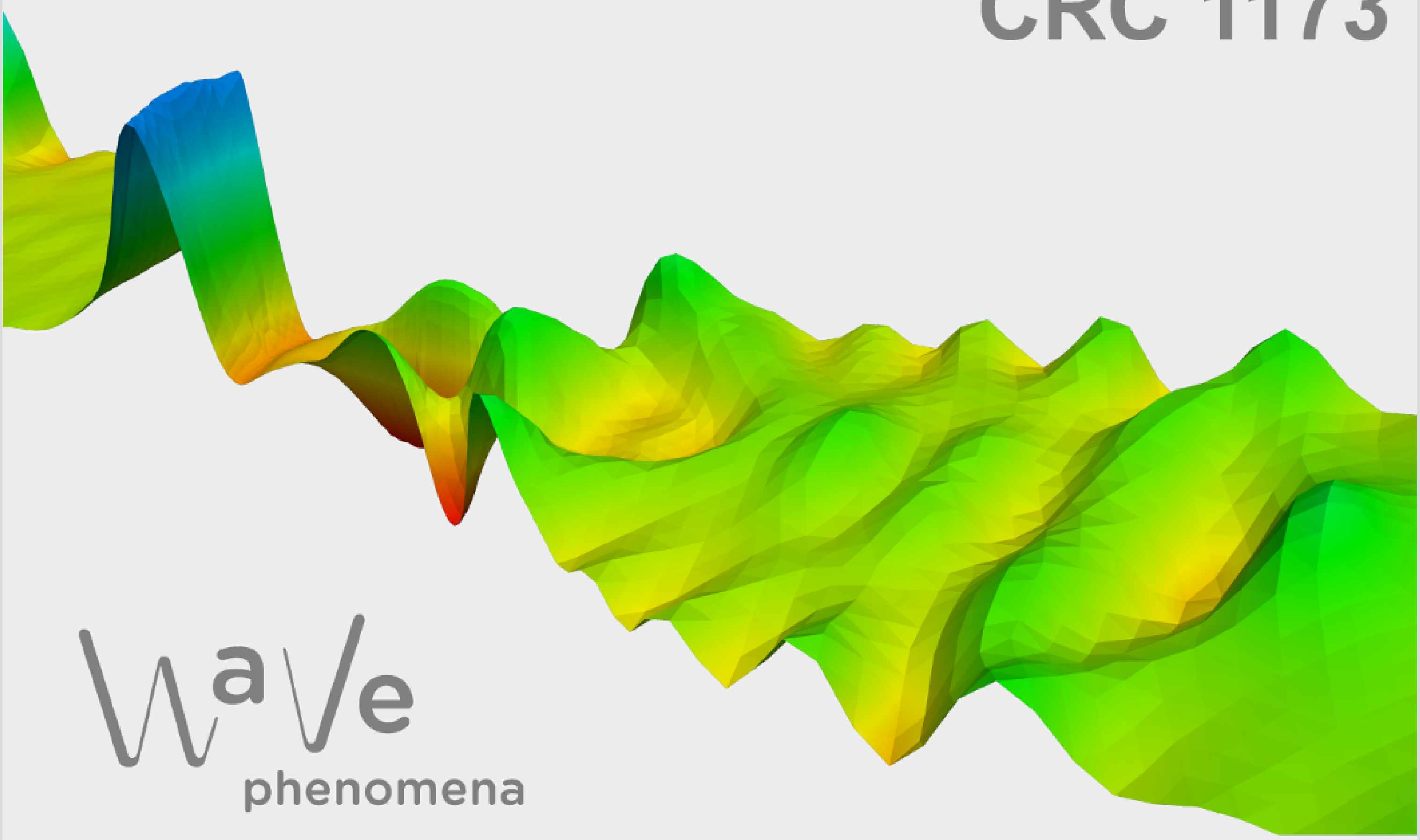

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THE LIMITING ABSORPTION PRINCIPLE FOR PERIODIC DIFFERENTIAL OPERATORS AND APPLICATIONS TO NONLINEAR HELMHOLTZ EQUATIONS

RAINER MANDEL

Abstract. We prove an L^p-version of the limiting absoprtion principle for a class of periodic elliptic differential operators of second order. The result is applied to the construction of nontrivial solutions of nonlinear Helmholtz equations with periodic coefficient functions.

1. Introduction

In this paper we study elliptic partial differential equations of the form

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d \tag{1}$$

where $L\psi := -\operatorname{div}(A(\cdot)\nabla\psi) + V(\cdot)\psi$ is a Schrödinger-type operator with periodic coefficient functions that are sufficiently regular. For λ outside the spectrum of the selfadjoint operator $L : L^2(\mathbb{R}^d) \supset H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ this equation is invertible, i.e. a unique solution $u \in H^2(\mathbb{R}^d)$ of (1) exists. What about λ inside the spectrum of L? This issue is much more delicate and a general answer for large classes of operators is missing. There is, however, a general strategy called "limiting absoprtion principle" how to find nontrivial solutions of (1) for such λ. On an abstract level, any such limiting absoprtion principle is characterized by a class of coefficient functions A, V and real function spaces X, Y such that for all $f \in Y$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$ there is a unique solution $u^\varepsilon \in X + iX$ of the perturbed equation

$$Lu - (\lambda + i\varepsilon)u = f \quad \text{in } \mathbb{R}^d \tag{2}$$

such that u^ε converges as $\varepsilon \to 0^\pm$ to a solution $u^\pm \in X + iX$ of (1) in a suitable topology. Let us give some examples for Schrödinger-type operators of the form $L = -\Delta + V(x)$ in $\mathbb{R}^3$.

One of the first results on limiting absorption principles for such operators is due to Odeh [23] who proved uniform convergence of the u^ε for square integrable[1] right hand sides f with compact support provided the potential V decays sufficiently fast at infinity in an averaged sense. Another famous result is due to Agmon (Theorem 4.1 in [1]) who used differently weighted L^2−spaces X and Y and so-called short range potentials satisfying $V(x) = O(|x|^{-1-\delta})$ as $|x| \to \infty$ for some $\delta > 0$. A generalization to Helmholtz equations in unbounded and asymptotically conic manifolds was recently proved by Rodnianski and Tao [29]. Further versions of the limiting absorption principle in Morrey-Campanato-spaces, again for evanescent potentials, can be found in [5] or [25]. Goldberg and Schlag [12] proved an L^p-version of the limiting

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[1]Odeh requires the right hand side to be "integrable", but probably "square integrable" is meant in view of the fact that he speaks of a unique L^2-solution of (2).

absorption principle ($X = L^4(\mathbb{R}^3), Y = L^{4/3}(\mathbb{R}^3)$) for potentials $V \in L^r(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^d)$ for some $r > \frac{3}{2}$. Each of the preceding results relies on the decay of the potential V, which ensures that the resolvent of $-\Delta + V(x) - \lambda - i\varepsilon$ resembles the one of $-\Delta - \lambda - i\varepsilon$ as far as the asymptotic properties at infinity are concerned. We stress that a control of the global regularity and integrability of the functions u^ε represents the main difficulty since convergence on compact sets can be proved under very mild assumptions on V. For instance, in 1962 Eidus [8] proved a convergence result in $H^2_{loc}(\mathbb{R}^3)$ whenever V is bounded from below and locally bounded from above. Being interested in global regularity for solutions of periodic problems, we need to take a different approach. The main tool of our analysis is Floquet-Bloch theory, which allows to give a qualitative description of the spectrum of elliptic periodic differential operators. As we will see, combining this approach with suitable assumptions on the so-called band structure of L leads to a new limiting absorption principle. In our analysis we mainly take advantage of the papers by Gutiérrez [13] and Radosz [27]. The first-mentioned paper provides an L^p-version of the limiting absorption principle for the Helmholtz operator $-\Delta - \lambda$, while the second paper contains the main ideas how Floquet-Bloch analysis may be used in order to establish a limiting absorption principle for periodic problems. Our contribution is to combine the methods from both papers in order to prove an L^p-version for the limiting absorption principle in the periodic setting. Accordingly, both papers are of fundamental importance for this paper, so we provide some details.

In [13] Theorem 6 Gutiérrez shows that for all $\lambda > 0$ the family of resolvent operators $(-\Delta - \lambda - i\varepsilon)^{-1} : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is equibounded with respect to $\varepsilon \in \mathbb{R} \setminus \{0\}$ provided $d \geq 3$ and p, q are chosen suitably, see (13). The task is to analyze the functions

$$u^\varepsilon := (-\Delta - \lambda - i\varepsilon)^{-1} f = \mathcal{F}^{-1}\Big(\frac{\hat{f}(\cdot)}{|\cdot|^2 - \lambda - i\varepsilon}\Big).$$

Gutiérrez' a priori estimates allow to pass to a weak limit of the u^ε in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0^\pm$ and the limit functions $u^+, u^- \in L^p(\mathbb{R}^d)$ are given by

$$\begin{aligned} u^\pm(x) &= \int_{\mathbb{R}^d} \frac{i}{4} (2\pi|x-y|)^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(x-y) f(y)\,dy \\ &= (2\pi)^{-\frac{d}{2}} \Big(p.v. \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{|\xi|^2 - \lambda} e^{i\langle x,\xi\rangle}\,d\xi + i\pi \int_{\{|\xi|^2=\lambda\}} \frac{\hat{f}(\xi)}{2\sqrt{\lambda}} e^{i\langle x,\xi\rangle}\,d\mathcal{H}^{d-1}(\xi) \Big), \end{aligned} \tag{3}$$

where $H^{(1)}_{(d-2)/2} : \mathbb{R} \to \mathbb{C}$ denotes the Hankel function of the first kind, see (11) in [10]. The formula from the second line follows from Lemma 5.1 in [30]. It shows some similarities with the formula obtained by Radosz in the case of a periodic Schrödinger operator $L = -\Delta + V(x)$, see Theorem 2.13 in [27]. Using Floquet-Bloch theory [3,11] Radosz analyzed the convergence of the functions $u^\varepsilon(\lambda, \cdot) := (L - \lambda + i\varepsilon)^{-1} f$ as $\varepsilon \to 0^\pm$ and determined complex-valued functions u^+, u^- satisfying

$$\int_{\mathbb{R}^d \times I} u^\pm(\lambda, x)\big((L-\lambda)\phi(\lambda,\cdot)\big)(x)\,d(x,\lambda) = \int_{\mathbb{R}^d \times I} f(x)\phi(\lambda,x)\,d(x,\lambda) \quad \text{for all } \phi \in C_0^\infty(I \times \mathbb{R}^d),$$

where $I \subset \mathbb{R}$ is a sufficiently small interval containing a "regular frequency" $\lambda \in \sigma(L)$, cf. Definition 1.1 in [27]. More precisely, she shows in Theorem 1.2 that the functions u^ε converge to some $u^\pm$ as $\varepsilon \to 0^\pm$ in the space $L^2(I, Z)$ where Z is a suitably weighted L^2–space. Nonetheless Radosz' results are weaker than one may hope for in view of Gutiérrez' results for constant potentials. First of all, it is expected that a convergence result holds true for every fixed regular frequency λ in the spectrum of L, which cannot be deduced from convergence in $L^2(I, Z)$. Furthermore, the topology of the weighted L^2–space Z is rather coarse given that the weight function is assumed to have some decay at infinity, see p.255-256 and Definition 2.7 in [27]. As a consequence, Radosz' techniques do not allow to control the decay of the functions $u^\varepsilon(\lambda, \cdot)$ and $u^\pm(\lambda, \cdot)$ at infinity. These shortcomings were our motivation to look for a limiting absorption principle that may substitute Gutiérrez' results [13] when the differential operator L has periodic instead of constant coefficient functions. Our Theorem 1 provides such a new result for a class of differential operators L and regular frequencies λ satisfying the assumptions (A1),(A2),(A3) that we are going to introduce and discuss next.

Our first assumption says that we deal with uniformly elliptic partial differential equations of second order in divergence form with $\mathbb{Z}^d$-periodic coefficient functions so that Floquet-Bloch theory is applicable. Clearly, by a change of coordinates, other periodicities can be dealt with, too. So we require the following:

(A1) $L\psi = -\operatorname{div}(A(\cdot)\nabla\psi) + V(\cdot)\psi$ for $\mathbb{Z}^d$-periodic coefficient functions $A \in C^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $V \in L^\infty(\mathbb{R}^d)$ such that $A(x)$ is symmetric and $\langle \xi, A(x)\xi\rangle \geq c|\xi|^2$ holds for some $c > 0$ and all $x, \xi \in \mathbb{R}^d$.

Under this assumption the operator L is selfadjoint on $\mathbb{R}^d$ with domain $H^2(\mathbb{R}^d)$ and its spectrum has a so-called band structure. This means that the spectrum of L is the union of infinitely many bands $\lambda_s(B)$ where the band functions λ_s are continuous and $B = [-\pi, \pi]^d$ is the so-called Brilluoin zone, named after Léon Brillouin in honor of his contributions to the study of wave propagation in periodic materials [4]. The relation between the band functions λ_s and the operator L is given by the following k-dependent selfadjoint quasiperiodic eigenvalue problems on the periodicity cell $\Omega := [0, 1]^d$:

$$\begin{aligned} L\psi &= \lambda\psi && \text{in } \Omega, \\ \psi(x + n) &= e^{i\langle k,n\rangle}\psi(x) && \text{on } \partial\Omega \text{ for all } n \in \mathbb{Z}^d. \end{aligned} \tag{4}$$

For every $k \in B$ there is an orthonormal basis $\{\psi_s(\cdot, k) : s \in \mathbb{Z}^d\}$ in $L^2(\Omega; \mathbb{C})$ consisting of eigenfunctions of (4) with associated eigenvalues $\{\lambda_s(k) : s \in \mathbb{Z}^d\}$ so that the band structure takes the form

$$\sigma(L) = \bigcup_{s\in\mathbb{Z}^d} \lambda_s(B) = \bigcup_{s\in\mathbb{Z}^d} \bigcup_{k\in B} \{\lambda_s(k)\}. \tag{5}$$

A proof of (5) may be found in Lemma 4 and Lemma 5 in the paper bei Odeh and Keller [24]. Notice that their result is formulated for continuous and $\mathbb{Z}^d$–periodic potentials V, but extends to bounded ones as in (A1). We will use that the functions $k \mapsto \psi_s(\cdot, k) \in L^2(\Omega; \mathbb{C})$ can be chosen to be measurable, see Lemma 5.3 b) in [2]. Moreover, we may extend the ψ_s continuously to $\mathbb{R}^d \times B$ by quasiperiodicity, i.e. by defining $\psi_s(x + n, k) = e^{i\langle k,n\rangle}\psi_s(x, k)$ for $x \in \Omega, n \in \mathbb{Z}^d$, see (4). A very subtle point concerns the labeling of the eigenpairs

$(\psi_s(\cdot,k),\lambda_s(k))$. A common way to do this is to use $\mathbb{N}_0$ instead of $\mathbb{Z}^d$ as an index set and to order the eigenvalues by requiring $\lambda_j(k)\le\lambda_{j+1}(k)$ for all $j\in\mathbb{N}_0$. This approach is used for instance in [24] or in Eastham's book, see Chapter 6 in [7]. The advantage of this numbering is two-fold: Firstly, it is intuitive and secondly, the $\mathbb{Z}^d$–periodicity and Lipschitz continuity of the band functions immediately follow from the min-max-characterization of eigenvalues. In this paper, however, we do not use this labeling. The reason is that for this labeling Lipschitz continuity is the best regularity one may in general hope for. Indeed, it is possible that eigenvalue surfaces $\lambda_s(B)$ intersect each other "transversally" so that the crossings destroy every kind of differentiability property of the band functions λ_s and the corresponding surfaces $\lambda_s(B)$, but not their Lipschitz continuity. This phenomenon is illustrated schematically in Figure XIII.15 in [28] in the one-dimensional setting. A numerical example for $d=2$ and $L=-\Delta+V(x)$ with a concrete potential V may be found on p.863 in [6]. We choose the index set $\mathbb{Z}^d$ for the numbering of the orthonormal basis, which is motivated by the explicit example of a constant potential where the Floquet-Bloch eigenpairs $(\psi_s(\cdot,k),\lambda_s(k))$ are given by

$$\psi_s(x,k)=e^{i\langle k+2\pi s,x\rangle},\quad \lambda_s(k)=|k+2\pi s|^2\qquad \text{for } k\in B, s\in\mathbb{Z}^d, x\in\Omega, \tag{6}$$

see (6.8.1),(6.8.2) in [7]. So one finds that ψ_s,λ_s are smooth with $\psi_s(x,k+2\pi n)=\psi_{s+n}(x,k)$, $\lambda_s(k+2\pi n)=\lambda_{s+n}(k)$ for all $n\in\mathbb{Z}^d$. We conclude that with our choice of the index set smoothness may be gained at the expense of $\mathbb{Z}^d$-periodicity with respect to the quasimomenta k. We will say more on regularity issues below.

The band functions λ_s satisfy the estimates

$$c|s|^2-C\le\lambda_s(k)\le C|s|^2+C\qquad (s\in\mathbb{Z}^d,\ k\in B) \tag{7}$$

for some $c,C>0$ independent of k. Notice that $|s|^2$ has to be replaced by $|s|^{2/d}$ when $\mathbb{N}_0$ or $\mathbb{Z}$ is used as an index set. Let us quickly recall why this is true. In the case $L=-\Delta$ Theorem 6.3.1 in [7] shows that the jth largest eigenvalue among the $\lambda_s(k)$ can be enclosed between the j-th Neumann and the j-th Dirichlet eigenvalue. Since the asymptotics for both eigenvalue sequences are given by Weyl's law, (7) follows for this special case. For differential operators L as in $(A1)$ one has $c\cdot(-\Delta)-C\le L\le C\cdot(-\Delta+1)$ for some $c,C>0$ in the sense of symmetric operators so that (7) results from Courant's min-max characterization for the eigenvalues of self-adjoint compact operators and the corresponding result for $-\Delta$ mentioned above. For more information about the qualitative properties of the eigenpairs $(\psi_s(\cdot,k),\lambda_s(k))$ in a one-dimensional setting we refer to Theorem XIII.89 and Theorem XIII.90 in [28] or Chapter 2.8 in [2]. Important tools from Floquet-Bloch analysis are the Floquet-Bloch transformation U and its inverse U^{-1} that allow to transfer problems from $\mathbb{R}^d$ to k-dependent problems on the periodicity cell $\Omega=[0,1]^d$ and vice versa. It is given by

$$\begin{aligned} U&:L^2(\mathbb{R}^d;\mathbb{C})\to L^2(\Omega\times B;\mathbb{C}),\quad f\mapsto\Big[(x,k)\mapsto\sum_{n\in\mathbb{Z}^d}f(x-n)e^{ink}\Big],\\ U^{-1}&:L^2(\Omega\times B;\mathbb{C})\to L^2(\mathbb{R}^d;\mathbb{C}),\quad g\mapsto\Big[x\mapsto\frac{1}{\sqrt{|B|}}\int_B g(x,k)\,dk\Big] \end{aligned} \tag{8}$$

where g is to be understood quasiperiodically extended via the formula $g(x+n,k) = e^{i\langle k,n\rangle}g(x,k)$ ($n \in \mathbb{Z}^d$) from $\Omega \times B$ to $\mathbb{R}^d \times B$, see Lemma 2 and Lemma 3 in [24]. The Floquet-Bloch transformation is an isometry, see Theorem 2.2.5 in [19] or Corollary 2 in [24].

With the above preparations we may now introduce and discuss the precise regularity assumptions that we have to impose on the Floquet-Bloch eigenpairs $(\psi_s(\cdot,k),\lambda_s(k))$ from above. In the case of the trivial potential, see (6), there are real analytic functions $\Lambda : \mathbb{R}^d \to \mathbb{R}, \Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that $\lambda_s(k) = \Lambda(k+2\pi s), \psi_s(x,k) = \Psi(x,k+2\pi s)$ and the so-called Fermi surfaces

$$F_\tau := \{k \in \mathbb{R}^d : \Lambda(k) = \tau\} \tag{9}$$

are spheres of radius $\sqrt{\tau}$ for all positive τ, i.e. for all τ in the interior the spectrum $[0,\infty)$. In the general case, our assumption (A2) on the Fermi surfaces of the operator L will ensure that for τ close to a given frequency $\lambda \in \sigma(L)$ the associated Fermi surfaces F_τ show a somewhat similar behaviour. More precisely, we will require them to be compact, sufficiently smooth and to have positive Gaussian curvature in each point of the surface. From the physical point of view it is reasonable to assume that at least small periodic perturbations of constant potentials have this property. At this point we want to stress that in most of the textbooks and papers the term "Fermi surface" is used differently. There it is the uniquely defined subset of the Brillouin zone $B = [-\pi,\pi]^d$ that contains a $2\pi\mathbb{Z}^d$-translate of a point from F_τ. In other words, it is given as follows:

$$\mathcal{F}_\tau = \{k \in B : \lambda_s(k) = \tau \text{ for some } s \in \mathbb{Z}^d\}. \tag{10}$$

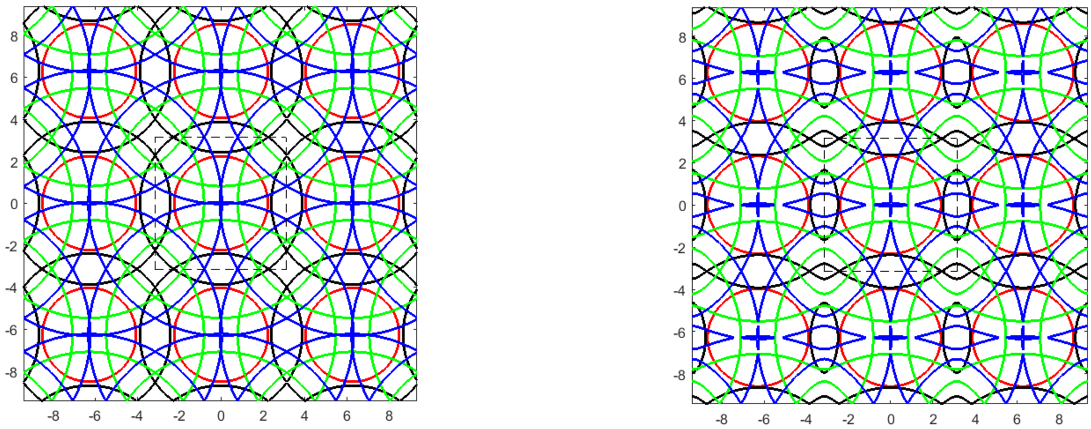
The following statement about the $\mathcal{F}_\tau$ is taken literally from Sólyom's book [31], page 89:

> "... However, the presence of a periodic potential can drastically distort the spherical shape of the Fermi surface – and, as we shall see, it can even disappear. For a relatively small number of electrons only the states at the bottom of the lowest-lying band are occupied. The Fermi surface is then a simply connected continuous surface that deviates little from the spherical shape. When the number of electrons is increased, the surface may cease to be simply connected ... In such cases more than one band can be partially filled. The Fermi surface separating occupied and unoccupied states must then be given for each of these – hence the Fermi surface is made up of several pieces."

Trasferred to our situation this means that for small τ one typically observes that $\mathcal{F}_\tau = F_\tau$ has a spherical shape. For larger τ, however, $\mathcal{F}_\tau \neq F_\tau$ is possible and $\mathcal{F}_\tau$ may be disconnected. Indeed, this phenomenon can be easily verified for the constant potential $V \equiv 0$, which is again based on (6). For $\tau > \sqrt{\pi}$ the sphere $F_\tau = \{k \in \mathbb{R}^d : \Lambda(k) = |k|^2 = \tau\}$ does not fit into the Brillouin zone $B = [-\pi,\pi]^d$ and thus $\mathcal{F}_\tau$ becomes disconnected. It is however remarkable that the Fermi surfaces F_τ according to our definition from (9) keep their shape regardless of the precise value of $\tau > 0$. This makes us believe that, firstly, the sets F_τ are actually more meaningful and "physical" than the $\mathcal{F}_\tau$. Notice that the fact of the $\mathcal{F}_\tau$ becoming disconnected for $\tau > \sqrt{\pi}$ does not produce any physical effects; the Helmholtz equation $-\Delta u - \tau u = f$ for τ bigger or smaller than $\sqrt{\pi}$ may be transformed into each other by a simple rescaling so that the qualitative description of the solutions does not change. Secondly, assuming a spherical shape in terms of positive Gaussian curvature also makes sense from a physical point of view. Our assumptions for the Fermi surfaces concern their shape as well as their regularity.

(A2) There is a $\rho > 0$, an open set $U \subset \mathbb{R}^d$ and $\Lambda : U \to \mathbb{R}, \Psi : \mathbb{R}^d \times U \to \mathbb{C}$ such that $\lambda_s(k) = \Lambda(k + 2\pi s)$, $\psi_s(x,k) = \Psi(x, k + 2\pi s)$ whenever $\lambda_s(B) \cap [\lambda - \rho, \lambda + \rho] \neq \emptyset$ and the following holds:
   (a) $\sup_{x\in\Omega} \|\Psi(x,\cdot)\|_{C^N(U)} < \infty$ and $\Lambda \in C^{N+1}(U,\mathbb{R})$ for some $N \geq 2, N > \frac{d-1}{2}$.
   (b) The Fermi surface $F_\lambda := \{k \in U : \Lambda(k) = \lambda\}$ is a closed compact hypersurface with positive Gaussian curvature.
   (c) $\nabla\Lambda \neq 0$ on F_λ.

We stress that we require the functions Λ, Ψ to have these properties on a sufficiently small neighbourhood U of the Fermi surface F_λ and only for finitely many indices $s \in \mathbb{Z}^d$ as follows from (7). At first sight this seems to be a technical point, but in fact it is known for $d = 2$ that an entire function $\Lambda : \mathbb{C}^d \to \mathbb{C}$ with $\lambda_s(k) = \Lambda(k+2\pi s)$ can only exist for constant potentials V, see Theorem 4.4.6 in [18]. We are not aware of any global regularity results for the band functions λ_s or $\psi_s(x,\cdot)$ that would allow to deduce (A2) from whatever property of V, A, so we have to require them. As mentioned above, eigenvalue surfaces associated with different indices may intersect so that differentiability properties across these intersection points are not as easy to get. Away from these intersections the dependence on k is analytic, see for instance Theorem 2 and Remark (iii) in [24]. In [34] Wilcox proves that for all $s \in \mathbb{Z}^d$ the mappings $k \mapsto \psi_s(\cdot,k) \in C(\overline{\Omega})$ are holomorphic on $B \setminus Z_s$ where Z_s is a closed null set, but this regularity result is not sufficient for our purposes. As far as the band functions λ_s are concerned, there are global regularity results that allow to continue the λ_s analytically in a certain sense (see Chapter 3.5.4 in [2] and in particular Theorem 5.2) but we did not see how such tools can be used for our purposes. In Figure 1 the Fermi surfaces (and translates of it) are plotted numerically for an almost constant potential (left) and for a strongly oscillating one (right). The figure on the left suggest that assumption (A2)(b) is satisfied for all depicted frequencies τ. The Fermi surfaces on the right hand side are more complicated and for some τ more than one connected components of the Fermi surface can be found as well as parts with negative Gaussian curvature. So in this case the geometry of the Fermi surfaces does not seem to be covered by our assumption (A2)(b). The author thanks T.Dohnal (University of Dortmund) for providing these pictures. Part (c) of assumption (A2) was introduced in [27]. Frequencies $\lambda \in \sigma(L)$ with this property are called regular. As mentioned in Remark 2.2 of [27] almost all frequencies are regular under this assumption. Indeed, Sard's Lemma and $\Lambda \in C^1(U,\mathbb{R})$ imply that the set of irregular frequencies $\Lambda(\{k \in U : \nabla\Lambda(k) = 0\})$ is a null set. For the constant potential all frequencies $\tau > 0$ are regular, whereas $\tau = 0$ is irregular. The assumption (A2) will allow us to analyze the properties of certain integrals over the Fermi surfaces that may be interpreted as a generalized version of Herglotz waves, which are known to play a fundamental role in the study of homogeneous Helmholtz equations, see [30] for more details in this direction. Let us mention that Herglotz waves also appear in Gutiérrez' proof of the limiting absoprtion principle for the Helmholtz operator [13] so that it may not surprise that such integrals are involved in our analysis.



(a) Fermi surfaces F_τ for the potential $V(x,y) = 0.2\sin(2\pi x)^2\cos(2\pi y)$ and $\tau = 5, 15, 30, 40$ (red/black/green/blue)

(b) Fermi surfaces F_τ for the potential $V(x,y) = 10\sin(2\pi x)^2\cos(2\pi y)$ and $\tau = 5, 15, 30, 40$ (red/black/green/blue)

Figure 1. Fermi surfaces

Our last assumption concerns the eigenfunctions $\psi_s(\cdot,k)$ introduced above.

(A3) There is a $C > 0$ such that $\|\psi_s(\cdot,k)\|_{L^\infty(\Omega;\mathbb{C})} \le C$ for all $s \in \mathbb{Z}^d, k \in B$.

Again we deduce from (6) that this assumption holds for the constant potential. Using ODE methods we can show that this assumption even holds for $A = \mathrm{id}_{d\times d}$ and so-called separable potentials $V(x) = V_1(x) + \ldots + V_d(x_d)$ with 1-periodic potentials $V_1,\ldots,V_d \in L^\infty(\mathbb{R})$. This fact is proved in the Appendix. The general case, however, seems to be completely open.

We finally come to our main result, which is the limiting absoprtion principle for periodic differential operators satisfying the assumptions (A1),(A2),(A3). It is formulated in terms of the resolvent operators

$$\mathcal{R}^\varepsilon(\lambda) := \mathcal{R}(\lambda + i\varepsilon) := (L - \lambda - i\varepsilon)^{-1} \qquad \text{for } \varepsilon \in \mathbb{R}\setminus\{0\} \tag{11}$$

that we will consider as bounded linear operators from $L^q(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d;\mathbb{C})$ for p,q according to the following inequalities:

$$\begin{aligned} &1 \le q < \frac{2(d+1)}{d+3}, && \frac{2dq}{2+q(d-3)} < p \begin{cases} < \frac{qd}{d-2q} & ,q \le \frac{d}{2} \\ \le \infty & ,q > \frac{d}{2} \end{cases} \qquad \text{or} \\ &\frac{2(d+1)}{d+3} \le q < \frac{2d}{d+1}, && \frac{2q}{2d - q(d+1)} < p \begin{cases} < \frac{qd}{d-2q} & ,q \le \frac{d}{2} \\ \le \infty & ,q > \frac{d}{2} \end{cases}. \end{aligned} \tag{12}$$

Our limiting absorption principle for periodic differential operators reads as follows.

Theorem 1. *Let $d \in \mathbb{N}, d \ge 2, p, q$ satisfy* (12) *and let the assumptions (A1),(A2),(A3) hold for some $\lambda \in \sigma(L)$. Then the family of resolvent operators $\mathcal{R}^\varepsilon(\lambda) : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{C})$ from* (11) *is equibounded and there exist bounded linear operators* $\mathcal{R}^{\pm}(\lambda) : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{C})$ *such that*

$$\mathcal{R}^{\varepsilon}(\lambda) \to \mathcal{R}^{\pm}(\lambda) \quad \text{as } \varepsilon \to 0^{\pm}$$

in the operator norm. For all $f \in L^q(\mathbb{R}^d)$ *the functions* $\mathcal{R}^{\pm}(\lambda)f \in W^{2,p}(\mathbb{R}^d;\mathbb{C}) + W^{2,q}(\mathbb{R}^d;\mathbb{C})$ *define strong solutions of* $Lu - \lambda u = f$ *in* $\mathbb{R}^d$.

Additionally, the functions $\mathcal{R}^{\pm}(\lambda)f$ are expected to satisfy a generalized form of Sommerfeld's radiation condition at infinity. Similarly, the farfield expansions of these functions are of interest and generalized versions of the corresponding results for constant potentials (see for instance Proposition 2.7 and Proposition 2.8 in [10]) are expected to hold. Let us compare the ranges for p, q from (12) with the ones from [13]. Gutiérrez' limiting absorption principle holds for exponents $p, q \in [1,\infty]$ satisfying the inequalities

$$\frac{1}{q} > \frac{d+1}{2d}, \qquad \frac{1}{p} < \frac{d-1}{2d}, \qquad \frac{2}{d+1} \le \frac{1}{q} - \frac{1}{p} \le \frac{2}{d}. \tag{13}$$

In the case $1 \le q < \frac{2(d+1)}{d+3}$ we have $\frac{q(d+1)}{d+1-2q} < \frac{2dq}{2+q(d-3)}$ so that Gutiérrez' bounds allow for more (i.e. smaller) exponents p. In the case $\frac{2(d+1)}{d+3} < q < \frac{2d}{d+1}$, however, we find $\frac{q(d+1)}{d+1-2q} > \frac{2q}{2d-q(d+1)}$ so that our range for p covers smaller values than those of Gutiérrez. In particular, given that the assumptions (A1)-(A3) hold for $L = -\Delta$, we see that Theorem 1 partly improves the limiting absorption principle from [13]. The reason for this comes from a different interpolation procedure for the "resonant part" of the resolvent operators $\mathcal{R}^{\varepsilon}(\lambda)$, as we will see later. It is an open and interesting question what the optimal ranges are.

Finally, we discuss an application of the limiting absorption principle from Theorem 1. We study real-valued solutions of the nonlinear Helmholtz equations

$$Lu - \lambda u = \pm\Gamma(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^d \tag{14}$$

where L, λ satisfy the assumptions of the theorem and $\Gamma \in L^\infty(\mathbb{R}^d)$ is a positive $\mathbb{Z}^d$–periodic function. In the case $L = -\Delta$ and $\lambda > 0$ Evequoz and Weth [10] showed that (14) admits a dual variational formulation in $L^{p'}(\mathbb{R}^d)$ for $\frac{2(d+1)}{d-1} \le p \le \frac{2d}{d-2}$ that relies on the selfdual estimates for the associated resolvent operators $\mathcal{R}^{\pm}(\lambda) : L^{p'}(\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{C})$. For p in the interior of this interval they proved the existence of a mountain pass critical point in $L^{p'}(\mathbb{R}^d)$ of the dual functional and thus the existence of a so-called dual ground state of the equation belonging to $L^p(\mathbb{R}^d)$. This solution even lies in $W^{2,r}(\mathbb{R}^d) \cap C^{1,\alpha}(\mathbb{R}^d)$ for all $r \ge p$ and $\alpha \in (0,1)$. One of the major limitations in their approach is the specific form of the linear operator L, which is due to the fact that only in this case the mapping properties of the resolvent-type operators $\mathcal{R}^{\pm}(\lambda)$ are known (thanks to Gutiérrez' results we mentioned above). We refer to the beginning of section 2 in [10] for the details. Given that the selfdual estimates $\mathcal{R}^{\pm}(\lambda) : L^{p'}(\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{C})$ from Theorem 1 hold for $\frac{2(d+1)}{d-1} < p < \frac{2d}{d-2}$, we may apply the same variational techniques provided the linear operator satisfies (A1),(A2),(A3).

Corollary 1. *Let* $d \in \mathbb{N}, d \ge 2, \frac{2(d+1)}{d-1} < p < \frac{2d}{d-2}$ *and let the assumptions (A1),(A2),(A3) hold for some* $\lambda \in \sigma(L)$, *let* $\Gamma \in L^\infty(\mathbb{R}^d)$ *be positive and* $\mathbb{Z}^d$*–periodic. Then the nonlinear Helmholtz equation* (14) *has a (nontrivial) dual ground state solution* $u \in L^p(\mathbb{R}^d)$ *with* $u \in W^{2,p}(\mathbb{R}^d) + W^{2,p'}(\mathbb{R}^d)$.

As in [9,10] the existence of infinitely many nontrivial solutions may be shown by invoking the Symmetric Mountain Pass Theorem under the assumption that Γ is evanescent at infinity so that the associated dual functional satisfies the Palais-Smale condition, cf. Lemma 5.2 in [10] for the case $d \geq 3$ and p.10 in [9] for the case $d = 2$. We mention that our statement concerning the global regularity of the solution is weaker than the corresponding claims in [9, 10] because a replacement for the bootstrap procedure from Theorem 4.4 in [10] has not been established yet.

The paper is organized as follows: In section 2 we analyze the mapping properties of the resolvent operators $\mathcal{R}^\varepsilon(\lambda)$ for $\varepsilon \in \mathbb{R} \setminus \{0\}$ and identify the limit operators $\mathcal{R}^\pm(\lambda)$. This will be done by splitting $\mathcal{R}^\varepsilon(\lambda)$ into a nonresonant and a resonant part the analysis of which is substantially different. We mention that this splitting already appears in the work of Radosz [26, 27] and it seems to be indispensable. The estimates from section 2 will then be used in section 3 where Theorem 1 and Corollary 1 are proved. Two results from section 2 with long and technical proofs will be discussed in section 4 and section 5. Finally, in the Appendix we verify assumption (A3) when A is the identity matrix and V is separable. Throughout the paper $c, C > 0$ will denote positive numbers that may change from line to line.

2. Estimates

Throughout this section we make use of the assumptions of Theorem 1. Following the strategy outlined above we intend to split up the resolvent operators according to $\mathcal{R}^\varepsilon(\lambda) = \mathcal{R}_1^\varepsilon(\lambda) + \mathcal{R}_2^\varepsilon(\lambda)$ where $\mathcal{R}_1^\varepsilon(\lambda), \mathcal{R}_2^\varepsilon(\lambda)$ define linear and bounded operators between appropriate Lebesgue spaces that converge as $\varepsilon \to 0^\pm$. In order to prove this assertion we first provide a representation formula for the resolvent using the eigenfunction expansion for the eigenvalue problems (4) on the periodicity cell $\Omega = [0,1]^d$. With the aid of the Floquet-Bloch transformation and the notation from the first section we get the following result.

Proposition 1. *For all $f \in C_0^\infty(\mathbb{R}^d)$ we have*

$$(\mathcal{R}^\varepsilon(\lambda) f)(x) = \int_{\mathbb{R}^d} K^\varepsilon(x,y) f(y)\, dy$$

where the kernel function $K^\varepsilon \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$ is given by

$$K^\varepsilon(x,y) = \fint_B \sum_{s \in \mathbb{Z}^d} \frac{\psi_s(x,k)\overline{\psi_s(y,k)}}{\lambda_s(k) - \lambda - i\varepsilon}\, dk. \tag{15}$$

Proof. For $\varepsilon \in \mathbb{R} \setminus \{0\}$ we set $u^\varepsilon := \mathcal{R}^\varepsilon(\lambda) f$. Then $u^\varepsilon \in H^2(\mathbb{R}^d; \mathbb{C})$ satisfies

$$Lu^\varepsilon - (\lambda + i\varepsilon) u^\varepsilon = f \quad \text{in } \mathbb{R}^d$$

in the strong sense. Now we apply the Floquet-Bloch transformation which commutes with the differential operator L thanks to periodicity assumption (A1). So we have $Uu^\varepsilon(\cdot,k) \in H^2(\Omega;\mathbb{C})$ for all $k \in B$ as well as

$$L(Uu^\varepsilon)(\cdot,k) - (\lambda + i\varepsilon)(Uu^\varepsilon)(\cdot,k) = (Uf)(\cdot,k) \quad \text{in } \Omega.$$

Since $(\psi_s(\cdot,k))_{s\in\mathbb{Z}^d}$ is an orthonormal basis in $L^2(\Omega;\mathbb{C})$ consisting of eigenfunctions associated with (4) and eigenvalues $\lambda_s(k)$, we get for all $k\in B$ and almost all $x\in\Omega$

$$(Uu^\varepsilon)(x,k)=\sum_{s\in\mathbb{Z}^d}\frac{\langle (Uf)(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}}{\lambda_s(k)-\lambda-i\varepsilon}\psi_s(x,k). \tag{16}$$

Notice that for every given $k\in B$ and $\varepsilon\in\mathbb{R}\smallsetminus\{0\}$ this series converges in $L^2(\Omega;\mathbb{C})$ thanks to (7). Since f has compact support, we get

$$\begin{aligned}\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})} &= \int_\Omega Uf(y,k)\overline{\psi_s(y,k)}\,dy\\ &= |B|^{-1/2}\int_\Omega\sum_{n\in\mathbb{Z}^d} f(y+n)e^{-ink}\overline{\psi_s(y,k)}\,dy\\ &= |B|^{-1/2}\sum_{n\in\mathbb{Z}^d}\int_\Omega f(y+n)\overline{\psi_s(y+n,k)}\,dy\\ &= |B|^{-1/2}\int_{\mathbb{R}^d} f(y)\overline{\psi_s(y,k)}\,dy,\end{aligned} \tag{17}$$

and thus

$$\begin{aligned}(Uu^\varepsilon)(x,k) &= \sum_{s\in\mathbb{Z}^d}\frac{|B|^{-1/2}\int_{\mathbb{R}^d} f(y)\overline{\psi_s(y,k)}\,dy}{\lambda_s(k)-\lambda-i\varepsilon}\psi_s(x,k)\\ &= |B|^{-1/2}\int_{\mathbb{R}^d}\sum_{s\in\mathbb{Z}^d}\frac{\psi_s(x,k)\overline{\psi_s(y,k)}}{\lambda_s(k)-\lambda-i\varepsilon}f(y)\,dy\end{aligned}$$

for all $k\in B$ and almost all $x\in\Omega$. Finally, we apply the inverse Floquet-Bloch transformation given by (8) and get from Fubini's Theorem

$$\begin{aligned}u^\varepsilon(x) &= |B|^{-1/2}\int_B Uu^\varepsilon(x,k)\,dk\\ &= \int_{\mathbb{R}^d}\Big(⨍_B\sum_{s\in\mathbb{Z}^d}\frac{\psi_s(x,k)\overline{\psi_s(y,k)}}{\lambda_s(k)-\lambda-i\varepsilon}\,dk\Big)f(y)\,dy\\ &= \int_{\mathbb{R}^d}K^\varepsilon(x,y)f(y)\,dy,\end{aligned}$$

which is all we had to show. □

We note that an explicit formula for K^ε does not seem to be available except for the special case of the Helmholtz operator $L-\lambda=-\Delta-\lambda$ for $\lambda>0$, see (3). The representation formula from Proposition 1 in fact holds for more general functions f. Based on estimates involving K^ε we will see that the integral representation for $\mathcal{R}^\varepsilon(\lambda)f$ also makes sense for $f\in L^q(\mathbb{R}^d)$ when q is chosen suitably. To see this, we split the sum and the integration into one part where $\lambda_s(k)-\lambda$ is bounded away from zero and a second part where $\lambda_s(k)-\lambda$ is close to zero. We will call the associated operators the nonresonant part (indexed by 1) or the resonant part (indexed by 2) of the resolvent, respectively. With ρ as in assumption (A2) we choose a cutoff function $\chi\in C_0^\infty(\mathbb{R}^d)$ with the properties

$$0\le\chi\le 1,\qquad \operatorname{supp}(\chi)\subset B_\rho(0),\qquad \chi\equiv 1 \text{ on } B_{\rho/2}(0). \tag{18}$$

Then the splitting

$$\mathcal{R}^\varepsilon(\lambda)f = \mathcal{R}_1^\varepsilon(\lambda)f + \mathcal{R}_2^\varepsilon(\lambda)f$$

holds for $f \in C_0^\infty(\mathbb{R}^d)$ where the operators on the right hand side are defined via

$$\begin{aligned}(\mathcal{R}_1^\varepsilon(\lambda)f)(x) &:= \int_{\mathbb{R}^d} K_1^\varepsilon(x,y)f(y)\,dy,\\ (\mathcal{R}_2^\varepsilon(\lambda)f)(x) &:= \int_{\mathbb{R}^d} K_2^\varepsilon(x,y)f(y)\,dy\end{aligned} \tag{19}$$

and $K_1^\varepsilon, K_2^\varepsilon \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{C})$ are given by

$$\begin{aligned}K_1^\varepsilon(x,y) &:= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{1-\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon}\psi_s(x,k)\overline{\psi_s(y,k)}\,dk,\\ K_2^\varepsilon(x,y) &:= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon}\psi_s(x,k)\overline{\psi_s(y,k)}\,dk.\end{aligned} \tag{20}$$

In view of (7) we find that $K_2^\varepsilon(x,y)$ should be seen as a finite sum of singular terms (as $\varepsilon \to 0$) whereas $K_1^\varepsilon(x,y)$ is an infinite series of regular terms. Moreover, we observe

$$K_j^\varepsilon(x,y) = \overline{K_j^{-\varepsilon}(y,x)}, \quad K_j^\varepsilon(x+m,y) = K_j^\varepsilon(x,y-m) \quad \text{for } \varepsilon \in \mathbb{R}\setminus\{0\}, x,y \in \mathbb{R}^d, m \in \mathbb{Z}^d \ \ (j=1,2).$$

Using the assumptions (A1),(A2),(A3) we will show that, roughly speaking, the resonant part is responsible for low decay rates at infinity because it maps into Lebesgue spaces $L^p(\mathbb{R}^d)$ with certain exponents $p > 2$. On the contrary the nonresonant part will give the upper bound for p from (12). In the following we study the mapping properties of $\mathcal{R}_1^\varepsilon(\lambda), \mathcal{R}_2^\varepsilon(\lambda)$ for small $|\varepsilon|$ that will be used in section 3 when we prove Theorem 1 and Corollary 1.

2.1. **Estimates for the nonresonant part.** Using the equiboundedness of the eigenfunctions ψ_s from assumption (A3) we first prove an estimate for the family of sequences $(\langle Uh(\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})})$ where k ranges over the Brillouin zone B and $h \in L^{r'}(\mathbb{R}^d)$ for some $r \in [2,\infty]$. For notational convenience we suppress k as well as the index $s \in \mathbb{Z}^d$ of these sequences. These estimates involve the Banach spaces $L^r(B\times\mathbb{Z}^d)$ for $r \in [2,\infty]$ which we define to be the Lebesgue space with exponent r induced by the product of the Lebesgue measure on $B \subset \mathbb{R}^d$ and the counting measure on $\mathbb{Z}^d$. The corresponding norm is given by

$$\|(\langle Uh,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|_{L^r(B\times\mathbb{Z}^d;\mathbb{C})} := \Big(\int_B \sum_{s\in\mathbb{Z}^d} |\langle Uh(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}|^r\,dk\Big)^{1/r}$$

for $2 \le r < \infty$ and

$$\|(\langle Uh,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|_{L^\infty(B\times\mathbb{Z}^d;\mathbb{C})} := \sup_{(k,s)\in B\times\mathbb{Z}^d} |\langle Uh(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}|.$$

Here, sup stands for the essential supremum. In view of (17) the following result resembles the Hausdorff-Young inequality for Fourier series.

Proposition 2. *There is a $C > 0$ such that for all $k \in B$ and $2 \le r \le \infty$ and $h \in L^{r'}(\mathbb{R}^d)$*

$$\|(\langle Uh,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|_{L^r(B\times\mathbb{Z}^d;\mathbb{C})} \le C\|h\|_{L^{r'}(\mathbb{R}^d)}.$$

Proof. For $r=2$ we have the identity

$$\begin{aligned}\|(\langle Uh,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|^2_{L^2(B\times\mathbb{Z}^d;\mathbb{C})} &= \int_B \sum_{s\in\mathbb{Z}^d} |\langle Uh(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}|^2\,dk \\ &= \int_B \|Uh(\cdot,k)\|^2_{L^2(\Omega;\mathbb{C})}\,dk \\ &= \|Uh\|^2_{L^2(\Omega\times B;\mathbb{C})} \\ &= \|h\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Here we used that the functions $\psi_s(\cdot,k)$ form an orthonormal basis in $L^2(\Omega;\mathbb{C})$ and that the Floquet-Bloch transformation $U: L^2(\mathbb{R}^d;\mathbb{C})\to L^2(\Omega\times B;\mathbb{C})$ is an isometry. In the proof of the inequality for $r=\infty$ we use (A3), so let $C>0$ be given with $|\psi_s(x,k)|\le C$ for all $x\in\Omega, k\in B, s\in\mathbb{Z}^d$. Then we get from (8)

$$\begin{aligned}\|(\langle Uh,\psi_s\rangle)\|_{L^\infty(B\times\mathbb{Z}^d;\mathbb{C})} &= \sup_{(k,s)\in B\times\mathbb{Z}^d} |\langle Uh(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}| \\ &\le C \sup_{(k,s)\in B\times\mathbb{Z}^d} \int_\Omega |Uh(x,k)|\,dx \\ &\le C\int_\Omega \sum_{n\in\mathbb{Z}^d} |h(x-n)|\,dx \\ &\le C\|h\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Interpolating both estimates yields the result. □

Next we use the estimates from Proposition 2 to prove some mapping properties of the nonresonant part of the resolvent operator.

Lemma 1. *Let p,q satisfy*

$$d\ge 2 \quad \text{and} \quad 0\le \frac{1}{q}-\frac{1}{p}<\frac{2}{d}, \quad 1\le q\le 2\le p\le\infty. \tag{21}$$

Then there is a $C>0$ such that for all $\varepsilon\in\mathbb{R}$ and $f\in C_0^\infty(\mathbb{R}^d)$ the following estimates hold

$$\|\mathcal{R}_1^\varepsilon(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})}\le C\|f\|_{L^q(\mathbb{R}^d)},$$
$$\|\mathcal{R}_1^\varepsilon(\lambda)f-\mathcal{R}_1^0(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})}\le C\varepsilon\|f\|_{L^q(\mathbb{R}^d)}.$$

Proof. Applying (17) to $f,g\in C_0^\infty(\mathbb{R}^d)$ we get

$$\begin{aligned}&\int_{\mathbb{R}^d} g(x)(\mathcal{R}_1^\varepsilon(\lambda)f)(x)\,dx \\ &= \int_{\mathbb{R}^d} g(x)\Big(\int_{\mathbb{R}^d} K_1^\varepsilon(x,y)f(y)\,dy\Big)\,dx \\ &= \int_B \sum_{s\in\mathbb{Z}^d}\alpha_s^\varepsilon(k)\Big(\int_{\mathbb{R}^d} f(y)\overline{\psi_s(y,k)}\,dy\Big)\Big(\int_{\mathbb{R}^d} g(x)\psi_s(x,k)\,dx\Big)\,dk \\ &= \int_B \sum_{s\in\mathbb{Z}^d}\alpha_s^\varepsilon(k)\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}\overline{\langle Ug(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}}\,dk\end{aligned}$$

where

$$\alpha_s^\varepsilon(k) := \frac{1-\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon}.$$

Now let $r > \frac{d}{2}$ be given by $\frac{1}{q}-\frac{1}{p}=\frac{1}{r}$. Due to (7) we find a $C>0$ such that for all $\varepsilon\in\mathbb{R}$ we have $\|(\alpha_s^\varepsilon)\|_{L^r(B\times\mathbb{Z}^d;\mathbb{C})}\le C$. So Hölder's inequality and Proposition 2 (we have $q',p\ge 2$) yield

$$\begin{aligned}&\int_B\sum_{s\in\mathbb{Z}^d}\left|\alpha_s^\varepsilon(k)\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}\overline{\langle Ug(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}}\right|dk\\&\le\|(\alpha_s^\varepsilon)\|_{L^r(B\times\mathbb{Z}^d;\mathbb{C})}\|(\langle Uf,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|_{L^{q'}(B\times\mathbb{Z}^d;\mathbb{C})}\|(\langle Ug,\psi_s\rangle_{L^2(\Omega;\mathbb{C})})\|_{L^p(B\times\mathbb{Z}^d;\mathbb{C})}\\&\le C\|g\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

This entails

$$\int_{\mathbb{R}^d}g(x)(\mathcal{R}_1^\varepsilon(\lambda)f)(x)\,dx\le C\|g\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^q(\mathbb{R}^d)}$$

for all $f,g\in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^{p'}(\mathbb{R}^d)$ and the dual of $L^{p'}(\mathbb{R}^d)$ is $L^p(\mathbb{R}^d)$ due to $p\ge 2$ we get the first asserted estimate. The same way we get the second estimate from $\|(\alpha_s^\varepsilon-\alpha_s^0)\|_{L^r(B\times\mathbb{Z}^d;\mathbb{C})}\le C\varepsilon$. □

2.2. **Estimates for the resonant part.** Now we discuss the mapping properties of the integral operator

$$(\mathcal{R}_2^\varepsilon(\lambda)f)(x)=\int_{\mathbb{R}^d}K_2^\varepsilon(x,y)f(y)\,dy$$

where K_2^ε was defined in (20). Our first result is a pointwise estimate for the kernel function, which is the most challenging result in this paper from the technical point of view. Its proof is based on a refinement of the method of (non-)stationary phase and its application to decay estimates for oscillatory integrals over nicely curved hypersurfaces in $\mathbb{R}^d$. Exploiting the regularity assumptions for the Fermi surfaces of L from (A2) we get the following:

Proposition 3. *There are measurable functions $K_2^\pm:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ and a $C>0$ such that for all $\varepsilon\in\mathbb{R}\setminus\{0\},x,y\in\mathbb{R}^d,m\in\mathbb{Z}^d$ we have $K_2^\pm(x,y)=\overline{K_2^\mp(y,x)}$, $K^\pm(x+m,y)=K^\pm(x,y-m)$ as well as*

$$\begin{aligned}&|K_2^\varepsilon(x,y)|\le C(1+|x-y|)^{\frac{1-d}{2}},\\&|K_2^\varepsilon(x,y)-K_2^\pm(x,y)|=o(1)(1+|x-y|)^{\frac{1-d}{2}}\quad\text{as }\varepsilon\to0^\pm.\end{aligned}\tag{22}$$

Here the factor $o(1)$ indicates that the convergence is uniform with respect to $x,y\in\mathbb{R}^d$. The proof of Proposition 3 is very long, so we prefer to present it in the appendix. The estimate (22) already yields some mapping properties of $\mathcal{R}_2^\varepsilon(\lambda)$ between certain Lebesgue spaces, but those are not strong enough to prove Theorem 1. As in the proof of Theorem 2.2 in [16] or Theorem 6 in [13] an estimate based on "spectral properties" has to be added in order to improve them via interpolation, i.e. with the aid of the Riesz-Thorin Interpolation Theorem. In [13, 16] this strategy applies in the context of elliptic differential operators with constant coefficients. For instance in the case $L=-\Delta-1$ one finds that the kernel function associated with the operator $L-i\varepsilon$ is given by $K^\varepsilon(x,y)=\Phi^\varepsilon(x-y)$ with $\mathcal{F}(\Phi^\varepsilon)(\xi)=(|\xi|^2-1-i\varepsilon)^{-1}$. The estimates for the resonant part $(||\xi|^2-1|\ge c>0)$ of the associated integral operator are based on Bessel potential estimates – their counterpart in the periodic setting was presented in the previous section. The resonant part ($||\xi|^2-1|\le c$) is estimated differently. The resonant part of the kernel function K_2^ξ is split into infinitely many pieces $K_2^{\varepsilon,j}$ that only depend on the behaviour of K_2^ξ in the annuli $2^{j-1}\le|x-y|<2^j$ for $j\in\mathbb{N}$. The j-dependent mapping properties of these infinitely many integral operators result from the pointwise decay of K_2^ξ and from estimates based on the Stein-Tomas theorem, see for instance (36) in [13] for the decomposition into annular regions and Lemma 1 in [13] for the resulting j-dependent estimates on these regions. For the Stein-Tomas theorem we refer to [33] or p.375,p.414 for $d\ge 3, d=2$ in [32].

In the case of general periodic elliptic differential operators the Fourier transformation is not suitable and a replacement for the above-mentioned estimates based on the "spectral properties" has to be found. In our situation it turns out that estimates for the Floquet-Bloch transforms (similar to the ones in the paper [15]) of $K_2^\xi(x,y)$ for (x,y) in the jth dyadic shell are helpful. These dyadic shells should be seen as the the analogues of the annuli used in the constant coefficient case. More precisely, we define the grid points $R_0:=\{0\}, R_j:=\{m\in\mathbb{Z}^d: 2^{j-1}\le|m_i|<2^j \text{ for } i=1,\dots,d\}$ and then, for each $j\in\mathbb{N}_0$ and $\varepsilon\in\mathbb{R}\setminus\{0\}$,

$$(23)\qquad \begin{aligned}(\mathcal{R}_2^{\varepsilon,j}(\lambda)f)(x) &:= \int_{\mathbb{R}^d} K_2^{\varepsilon,j}(x,y)f(y)\,dy \qquad \text{where}\\ K_2^{\varepsilon,j}(x,y) &:= K_2^\xi(x,y)1_{R_j}([x]-[y]).\end{aligned}$$

Here, $[x]:=([x_1],\dots,[x_d])\in\mathbb{Z}^d$ denotes the componentwise floor function. This definition guarantees $K_2^{\varepsilon,j}(x+m,y)=K_2^{\varepsilon,j}(x,y-m)$ for all $x,y\in\mathbb{R}^d, m\in\mathbb{Z}^d$ so that $K_2^{\varepsilon,j}$ inherits this important symmetry property from K_2^ξ. Analogously, we define

$$(24)\qquad \begin{aligned}(\mathcal{R}_2^{\pm,j}(\lambda)f)(x) &:= \int_{\mathbb{R}^d} K_2^{\pm,j}(x,y)f(y)\,dy \qquad \text{where}\\ K_2^{\pm,j}(x,y) &:= K_2^\pm(x,y)1_{R_j}([x]-[y]).\end{aligned}$$

First we provide the estimates based on the pointwise bounds from Proposition 3.

Proposition 4. *There is a $C>0$ such that we have for all $\varepsilon\in\mathbb{R}\setminus\{0\}$ and $p,q,r\in[1,\infty]$ satisfying $1+\frac1p=\frac1r+\frac1q$ and all $f\in L^q(\mathbb{R}^d)$*

$$\begin{aligned}\|\mathcal{R}_2^{\varepsilon,j}(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})} &\le C2^{j(\frac{1-d}{2}+\frac{d}{r})}\|f\|_{L^q(\mathbb{R}^d)} && \text{for all } j\in\mathbb{N}_0 \text{ and}\\ \|\mathcal{R}_2^{\varepsilon,j}(\lambda)f-\mathcal{R}_2^{\pm,j}(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})} &= o(1)2^{j(\frac{1-d}{2}+\frac{d}{r})}\|f\|_{L^q(\mathbb{R}^d)} && \text{for all } j\in\mathbb{N}_0 \text{ as } \varepsilon\to 0^\pm.\end{aligned}$$

Proof. We only show the first estimate, the proof of the second being similar. For $x,y\in\mathbb{R}^d$ such that $[x]-[y]\in R_j$ we have the inequality $c\cdot 2^j\le|x-y|\le C\cdot 2^j$ for some positive c,C. In particular Proposition 3 gives

$$|K_2^{\varepsilon,j}(x,y)|\le C(1+|x-y|)^{\frac{1-d}{2}}1_{|x-y|\le C2^j}\le C2^{\frac{j(1-d)}{2}}1_{|x-y|\le C2^j}\qquad \text{for all } j\in\mathbb{N}_0.$$

Hence, Young's convolution inequality yields the desired estimate. □

As outlined above we go on with an L^2-L^2–estimate for $\mathcal{R}_2^{\varepsilon,j}(\lambda)$ based on a pointwise estimate of the Floquet-Bloch transform of the kernel function $K_2^{\varepsilon,j}(\cdot,y)$ which relies on the regularity assumptions for the Fermi surfaces from assumption (A2). Since it is quite long, we will give the proof in the appendix.

Proposition 5. *For all $\delta > 0$ there is a $C_\delta > 0$ such that for all $\varepsilon \in \mathbb{R} \setminus \{0\}$ we have*

$$\sup_{x,y\in\Omega, l\in B} \left| U(K_2^{\varepsilon,j}(\cdot,y))(x,l) \right| \le C_\delta 2^{j(1+\delta)} \qquad \text{for all } j \in \mathbb{N}_0 \text{ and}$$

$$\sup_{x,y\in\Omega, l\in B} \left| U(K_2^{\varepsilon,j}(\cdot,y) - K_2^{\pm,j}(\cdot,y))(x,l) \right| = o(1) 2^{j(1+\delta)} \qquad \text{for all } j \in \mathbb{N}_0 \text{ as } \varepsilon \to 0^\pm.$$

In this result and in the following ones $o(1)$ indicates that the estimated quantities converge to zero uniformly with respect to j.

Proposition 6. *For all $\delta > 0$ there is a $C_\delta > 0$ such that for all $\varepsilon \in \mathbb{R} \setminus \{0\}, f \in C_0^\infty(\mathbb{R}^d)$ we have*

$$\|\mathcal{R}_2^{\varepsilon,j}(\lambda) f\|_{L^2(\mathbb{R}^d;\mathbb{C})} \le C_\delta 2^{j(1+\delta)} \|f\|_{L^2(\mathbb{R}^d)} \qquad \text{for all } j \in \mathbb{N}_0 \text{ and}$$

$$\|\mathcal{R}_2^{\varepsilon,j}(\lambda) f - \mathcal{R}_2^{\pm,j}(\lambda) f\|_{L^2(\mathbb{R}^d;\mathbb{C})} = o(1) 2^{j(1+\delta)} \|f\|_{L^2(\mathbb{R}^d)} \qquad \text{for all } j \in \mathbb{N}_0 \text{ as } \varepsilon \to 0^\pm.$$

Proof. Again, we only prove the first estimate since it relies on the first inequality from Proposition 5 in the same way as the second estimate relies on the second inequality from Proposition 5. First we recall the convolution formula for the Floquet-Bloch transformation. By the quasiperiodicity of the eigenfunctions we have $K_2^{\varepsilon,j}(x+n,y) = K_2^{\varepsilon,j}(x,y-n)$ for all $x, y \in \mathbb{R}^d, n \in \mathbb{Z}^d$, see (23). This yields the following formula for $x \in \Omega, l \in B$ and $f \in C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
U(\mathcal{R}_2^{\varepsilon,j}(\lambda) f)(x,l) &= U\Big(\int_{\mathbb{R}^d} K_2^{\varepsilon,j}(\cdot,y) f(y)\, dy \Big)(x,l) \\
&= |B|^{-1/2} \sum_{m\in\mathbb{Z}^d} e^{iml} \int_{\mathbb{R}^d} K_2^{\varepsilon,j}(x-m,y) f(y)\, dy \\
&= |B|^{-1/2} \sum_{m,n\in\mathbb{Z}^d} e^{i(m-n)l} e^{inl} \int_\Omega K_2^{\varepsilon,j}(x-m,y-n) f(y-n)\, dy \\
&= |B|^{-1/2} \sum_{m,n\in\mathbb{Z}^d} \int_\Omega e^{i(m-n)l} K_2^{\varepsilon,j}(x-(m-n),y) f(y-n) e^{inl}\, dy \\
&= |B|^{1/2} \int_\Omega U(K_2^{\varepsilon,j}(\cdot,y))(x,l) Uf(y,l)\, dy
\end{aligned}$$

and hence by Proposition 5

$$\left| U(\mathcal{R}_2^{\varepsilon,j}(\lambda) f)(x,l) \right| \le |B|^{1/2} C_\delta 2^{j(1+\delta)} \int_\Omega |Uf(y,l)|\, dy.$$

Taking the L^2−norm over $\Omega \times B$ and using the isometry property of the Floquet transformation as well as Hölder's inequality we arrive at

$$\begin{aligned}
\|\mathcal{R}_2^{\varepsilon,j}(\lambda) f\|_{L^2(\mathbb{R}^d;\mathbb{C})} &= \|U(\mathcal{R}_2^{\varepsilon,j}(\lambda) f)\|_{L^2(\Omega\times B;\mathbb{C})} \\
&\le (|B||\Omega|)^{1/2} C_\delta 2^{j(1+\delta)} \Big(\int_B \Big(\int_\Omega |Uf(y,l)|\, dy \Big)^2 dl \Big)^{1/2} \\
&\le (2\pi)^{d/2} C_\delta 2^{j(1+\delta)} \|Uf\|_{L^2(\Omega\times B;\mathbb{C})} \\
&= (2\pi)^{d/2} C_\delta 2^{j(1+\delta)} \|f\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

By interpolation we deduce the following estimates.

Lemma 2. *Assume that*

$$
\begin{aligned}
&1 \le q < \frac{2(d+1)}{d+3}, \qquad && \frac{2dq}{2+q(d-3)} < p \le \infty \qquad \text{or} \\
&\frac{2(d+1)}{d+3} \le q < \frac{2d}{d+1}, && \frac{2q}{2d-q(d+1)} < p \le \infty.
\end{aligned}
\tag{25}
$$

Then there are $C > 0 > \gamma$ *such that we have for all* $\varepsilon \in \mathbb{R} \setminus \{0\}$ *and* $f \in C_0^\infty(\mathbb{R}^d)$

$$
\begin{aligned}
\|\mathcal{R}_2^{\varepsilon,j}(\lambda) f\|_{L^p(\mathbb{R}^d;\mathbb{C})} &\le C 2^{\gamma j} \|f\|_{L^q(\mathbb{R}^d)} \qquad && \text{for all } j \in \mathbb{N}_0 \text{ and} \\
\|\mathcal{R}_2^{\varepsilon,j}(\lambda) f - \mathcal{R}_2^{\pm,j}(\lambda) f\|_{L^p(\mathbb{R}^d\, C)} &= o(1) 2^{\gamma j} \|f\|_{L^q(\mathbb{R}^d)} && \text{for all } j \in \mathbb{N}_0 \text{ as } \varepsilon \to 0^\pm.
\end{aligned}
$$

Proof. We interpolate the estimates from Proposition 4 and Proposition 6; let $p, q \in [1, \infty]$. If we choose $\theta \in [0,1]$ and $\tilde{q}, \tilde{p}, r \in [1, \infty]$ such that

$$
\frac{1}{q} = \frac{\theta}{\tilde{q}} + \frac{1-\theta}{2}, \qquad \frac{1}{p} = \frac{\theta}{\tilde{p}} + \frac{1-\theta}{2}, \qquad 1 + \frac{1}{\tilde{p}} = \frac{1}{r} + \frac{1}{\tilde{q}},
\tag{26}
$$

then we have $\mathcal{R}_2^{\varepsilon,j}(\lambda) : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ with operator norm bounded from above by

$$
\big(C 2^{j(\frac{1-d}{2} + \frac{d}{r})}\big)^\theta \big(C_\delta 2^{j(1+\delta)}\big)^{1-\theta} \le C'_\delta 2^{j\gamma_\delta} \quad \text{where } \gamma_\delta := \Big(\frac{1-d}{2} + \frac{d}{r}\Big)\theta + (1-\theta)(1+\delta).
$$

Since $\delta > 0$ can be chosen arbitrarily small, we have to show that for p, q chosen according to (25) we have $\gamma_0 < 0$ and thus $\gamma_\delta < 0$ for small positive δ. We will even show that such a choice is possible if and only if p, q satisfy (25).

We start with discussing the admissible ranges for p, q under the additional assumption $q < 2$. There are $\tilde{q}, \tilde{p}, r \in [1, \infty], \theta \in [0,1]$ satisfying (26) and $\gamma_0 < 0$ if and only if there exist $\tilde{q} \in [1, \infty], r \in [1, \frac{\tilde{q}}{\tilde{q}-1}], \theta \in [0,1]$ with

$$
\frac{1}{q} = \frac{\theta}{\tilde{q}} + \frac{1-\theta}{2}, \qquad \frac{1}{p} = \frac{1}{2} + \theta\Big(\frac{1}{r} + \frac{1}{\tilde{q}} - \frac{3}{2}\Big), \qquad r((d+1)\theta - 2) > 2d\theta.
$$

Here, we solved the last equation in (26) for $\tilde{p}$ and the last inequality is equivalent to $\gamma_0 > 0$. The first of these equations may be reduced to $\theta = \frac{\tilde{q}(2-q)}{q(2-\tilde{q})}$ so that $\theta \in [0,1]$ leads to $\tilde{q} \le q$ because of $q < 2$. Hence, the problem is equivalent to finding $\tilde{q} \in [1, q], r \in [1, \frac{\tilde{q}}{\tilde{q}-1}]$ such that

$$
\frac{1}{p} = \frac{1}{2} + \frac{\tilde{q}(2-q)}{q(2-\tilde{q})}\Big(\frac{1}{r} + \frac{1}{\tilde{q}} - \frac{3}{2}\Big), \qquad r > \frac{2d\tilde{q}(2-q)}{(\tilde{q}(2(d+1) - q(d-1)) - 4q)_+}.
\tag{27}
$$

The inequality has to interpreted as being impossible if the denominator on the right hand side equals zero. Using $\tilde{q} \le q < 2$ we find that this lower bound for r is always larger than 1 and

$$
\frac{2d\tilde{q}(2-q)}{(\tilde{q}(2(d+1) - q(d-1)) - 4q)_+} < \frac{\tilde{q}}{\tilde{q}-1} \quad \Leftrightarrow \quad \tilde{q} > \frac{2(q(d+2) - 2d)}{(q(d+1) - 2(d-1))_+}.
$$

This lower bound for $\tilde{q}$ is bigger than 1 for $\frac{2(d+1)}{d+3} < q < 2$ and smaller or equal to 1 for $1 \le q \le \frac{2(d+1)}{d+3}$. So it remains to find $\tilde{q}, r \in [1,\infty]$ such that the equation for p from (27) holds as well as

$$1 \le q < \frac{2(d+1)}{d+3}, \qquad 1 \le \tilde{q} \le q, \qquad \frac{2d\tilde{q}}{\tilde{q}(2(d+1)-q(d-1))-4q} < r \le \frac{\tilde{q}}{\tilde{q}-1} \qquad \text{or}$$

$$\frac{2(d+1)}{d+3} \le q < 2, \ \frac{2(q(d+2)-2d)}{q(d+1)-2(d-1)} < \tilde{q} \le q, \ \frac{2d\tilde{q}}{\tilde{q}(2(d+1)-q(d-1))-4q} < r \le \frac{\tilde{q}}{\tilde{q}-1}.$$

Pluggin these bounds for r into (27) we find that such a choice for r is possible if and only if the conditions from the following two lines hold:

$$1 \le q < \frac{2(d+1)}{d+3}, 1 \le \tilde{q} \le q \quad \text{or} \quad \frac{2(d+1)}{d+3} \le q < 2, \ \frac{2(q(d+2)-2d)}{q(d+1)-2(d-1)} < \tilde{q} \le q,$$

$$\frac{q-\tilde{q}}{2q(2-\tilde{q})} \le \frac{1}{p} < \frac{\tilde{q}(q(d+1)+2-4d)+4d-4q}{2dq(2-\tilde{q})}.$$

Finally, such a choice for $\tilde{q}$ is possible if and only if (25) holds.

Now we show that for $q \ge 2$ such a choice for $\tilde{p}, \tilde{q}, r$ is not possible. In this case $\theta \in [0,1]$ leads to $\tilde{q} \ge q$ so that (27) has to hold for $\tilde{q} \in [q,\infty], r \in [1, \frac{\tilde{q}}{\tilde{q}-1}]$. The lower bound for r is smaller than the upper bound if and only if there is $\tilde{q}$ such that

$$q \le \tilde{q} < \frac{2(q(d+2)-2d)}{q(d+1)-2(d-1)}, \qquad \frac{-\tilde{q}(q(d+1)+2-4d)-4d+4q}{2dq(\tilde{q}-2)} < \frac{1}{p} \le \frac{\tilde{q}-q}{2q(\tilde{q}-2)}.$$

From the first inequality we deduce $(q-2)(q(d+1)-2d) < 0$, which is impossible. □

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1: The first step of the proof is the definition of the operators $\mathcal{R}^{\pm}(\lambda)$. In view of the results of the previous chapter it is reasonable to define for $f \in C_0^\infty(\mathbb{R}^d)$

$$\mathcal{R}^{\pm}(\lambda)f := \mathcal{R}_1^0(\lambda)f + \sum_{j=0}^{\infty} \mathcal{R}_2^{\pm,j}(\lambda)f,$$

see (19) and (24). For p, q as in (12) these mappings satisfy an estimate of the form

$$\|\mathcal{R}^{\pm}(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})} \le C\|f\|_{L^q(\mathbb{R}^d)}$$

for a positive number C independent of f, see Lemma 1 and Lemma 2. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$, $\mathcal{R}^{\pm}(\lambda)$ extend to bounded linear operators (denoted with the same symbol) from $L^q(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d;\mathbb{C})$. The same lemmas provide the equiboundedness of the bounded linear operators $\mathcal{R}^{\varepsilon}(\lambda) : L^q(\mathbb{R}^d) \to L^p(\mathbb{R}^d;\mathbb{C})$ as well as

$$\|\mathcal{R}^{\varepsilon}(\lambda)f - \mathcal{R}^{\pm}(\lambda)f\|_{L^p(\mathbb{R}^d;\mathbb{C})} = o(1)\|f\|_{L^q(\mathbb{R}^d)} \qquad \text{as } \varepsilon \to 0^{\pm}.$$

From this we deduce $\mathcal{R}^{\varepsilon} \to \mathcal{R}^{\pm}$ as $\varepsilon \to 0^{\pm}$ in the operator norm.

We now show that $\mathcal{R}^{\pm}(\lambda)$ defines a formal resolvent operator for $L - \lambda$. For $f \in L^q(\mathbb{R}^d)$ we set $u^{\varepsilon} := \mathcal{R}^{\varepsilon}(\lambda)f$ so that u^{ε} is a strong solution of $Lu - (\lambda + i\varepsilon)u = f$. The first part of the proof implies $u^\varepsilon \to u^\pm := \mathcal{R}^\pm(\lambda)f$ as $\varepsilon \to 0^\pm$ in $L^p(\mathbb{R}^d;\mathbb{C})$ and hence we obtain for all test functions $g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)g(x)\,dx &= \int_{\mathbb{R}^d} (L-\lambda-i\varepsilon)u^\varepsilon(x)g(x)\,dx \\ &= \int_{\mathbb{R}^d} u^\varepsilon(x)(L-\lambda-i\varepsilon)g(x)\,dx \\ &\to \int_{\mathbb{R}^d} u^\pm(x)(L-\lambda)g(x)\,dx \qquad \text{as } \varepsilon \to 0^\pm.\end{aligned}$$

As a consequence, $u^\pm$ is a distributional solution of the linear elliptic PDE $(L-\lambda)u = f$ on $\mathbb{R}^d$ and therefore (see for instance Theorem 2 in [20]) it satisfies this PDE in the strong sense as an element of $W^{2,p}_{loc}(\mathbb{R}^d;\mathbb{C}) + W^{2,q}_{loc}(\mathbb{R}^d;\mathbb{C})$.

It is left to prove that $\mathcal{R}^\pm(\lambda)f$ lies in $W^{2,p}(\mathbb{R}^d;\mathbb{C}) + W^{2,q}(\mathbb{R}^d;\mathbb{C})$. To this end set $L_0\psi := -\operatorname{div}(A\nabla\psi)$ and write $u^\varepsilon = v^\varepsilon + w$ where

$$v^\varepsilon := (L_0+1)^{-1}((1-V-\lambda-i\varepsilon)u^\varepsilon), \qquad w := (L_0+1)^{-1}f.$$

These functions are well-defined because of $u^\varepsilon \in L^p(\mathbb{R}^d;\mathbb{C}), f \in L^q(\mathbb{R}^d)$. Next we use the $W^{2,p}$-estimates for the operator L_0+1 from Proposition 3.2 in [22] (for $U \equiv 1, F \equiv 0$). The boundedness of the linear operators $(L_0+1)^{-1} : L^r(\mathbb{R}^d) \to W^{2,r}(\mathbb{R}^d)$ for all $r \in (1,\infty)$ and $V \in L^\infty(\mathbb{R}^d)$ imply

$$\|v^\varepsilon\|_{W^{2,p}(\mathbb{R}^d;\mathbb{C})} + \|w\|_{W^{2,q}(\mathbb{R}^d)} \le C(\|u^\varepsilon\|_{L^p(\mathbb{R}^d;\mathbb{C})} + \|f\|_{L^q(\mathbb{R}^d)}) \le C\|f\|_{L^q(\mathbb{R}^d)},$$

where the last inequality follows from the L^p-estimates we proved above. Hence, we may pass to a subsequence again denoted by v^ε such that v^ε converges in $L^p(\mathbb{R}^d;\mathbb{C})$ (see the first part of the proof), weakly in $W^{2,p}(\mathbb{R}^d;\mathbb{C})$ and pointwise almost everywhere. Hence, we get

$$\mathcal{R}^\pm(\lambda)f = \lim_{\varepsilon\to 0^\pm} u^\varepsilon = \lim_{\varepsilon\to 0^\pm} v^\varepsilon + w \in W^{2,p}(\mathbb{R}^d) + W^{2,q}(\mathbb{R}^d),$$

since the pointwise limit, the limit in $L^p(\mathbb{R}^d)$ and the weak limit in $W^{2,p}(\mathbb{R}^d)$ coincide. □

We notice that the operators $\mathcal{R}^\pm(\lambda)$ are defined as integral operators with a kernel function

$$K^\pm(x,y) := K_1^0(x,y) + K_2^\pm(x,y) := K_1^0(x,y) + \sum_{j=1}^\infty K_2^{\pm,j}(x,y) \tag{28}$$

where the integral has to be understood in the sense of an oscillatory integral, i.e.

$$(\mathcal{R}^\pm(\lambda)f)(x) = \int_{\mathbb{R}^d} K_1^0(x,y)f(y)\,dy + \sum_{j=0}^\infty \int_{\mathbb{R}^d} K_2^{\pm,j}(x,y)f(y)\,dy.$$

We will use $K^\pm(x,y) = \overline{K^\mp(y,x)}$ for all $x,y \in \mathbb{R}^d$ as well as $K^\pm(x+m,y) = K^\pm(x,y-m)$ for all $x,y \in \mathbb{R}^d$ and $m \in \mathbb{Z}^d$, which follows from the corresponding properties of each of the summands in (28).

Proof of Corollary 1: As pointed out in the introduction the idea for the proof of this result is completely due to Evequoz and Weth [10]. We quickly review in which way our construction of the resolvent from Theorem 1 makes it possible to use their methods. Following their notation we set

$$\mathbf{R}f := \frac{1}{2}\mathcal{R}^+(\lambda)f + \frac{1}{2}\overline{\mathcal{R}^-(\lambda)f} = \int_{\mathbb{R}^d} K^*(x,y))f(y)\,dy, \qquad \text{where}$$
$$K^*(x,y) := \frac{1}{2}(\mathrm{Re}(K^+(x,y) + K^-(x,y))) = \frac{1}{2}(\mathrm{Re}(K^+(x,y) + K^+(y,x))).$$

This formula defines a bounded linear operator from $L^{p'}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, cf. (45) in [10] and $\mathbf{R}f$ is a real-valued strong solution of $Lu - \lambda u = f$. By construction, we have

$$K^*(x,y) = K^*(y,x), \quad K^*(x+m,y) = K^*(x,y-m) \qquad \text{for all} m \in \mathbb{Z}^d,\ x,y \in \mathbb{R}^d. \tag{29}$$

The nonlinear Helmholtz equation $Lu - \lambda u = \pm\Gamma(x)|u|^{p-2}u$ for $u \in L^p(\mathbb{R}^d)$ is then equivalent to finding $v \in L^{p'}(\mathbb{R}^d)$ such that

$$|v|^{p'-2}v = \pm\Gamma^{1/p}\mathbf{R}(\Gamma^{1/p}v), \tag{30}$$

see (47) in [10]. Exploiting the first equation in (29) we conclude that the equation (30) is variational and its Euler functional $J : L^{p'}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$J(v) = \frac{1}{p'}\|v\|_{L^{p'}(\mathbb{R}^d)}^{p'} \mp \frac{1}{2}\int_{\mathbb{R}^d} \Gamma^{1/p} v \mathbf{R}(\Gamma^{1/p}v)\,dx,$$

cf. (48) in [10]. This functional is continuously differentiable. Moreover, J has the mountain pass geometry, see Lemma 4.2 in [10]. The only point in the verification of this lemma that is not so obvious, is the existence of a nontrivial functions $z_+, z_- \in L^{p'}(\mathbb{R}^d)$ such that

$$\mp \int_{\mathbb{R}^d} \Gamma^{1/p} z_\pm \mathbf{R}(\Gamma^{1/p} z_\pm)\,dx < 0. \tag{31}$$

In order to find such a function we adapt the idea from Lemma 3.1 in [21]. To this end let $s \in \mathbb{Z}^d, k \in B$ be given with $\lambda_s(k) = \lambda$, see (5). By assumption (A2)(c) we have $\nabla\lambda_s(k) \neq 0$ so that the subsets $B_\pm := \{k \in B : \pm(\lambda_s(k) - \lambda) > \delta\}$ have positive measure provided $\delta > 0$ is chosen sufficiently small. Then we define $z_\pm$ via

$$z_\pm := 1_{B_R(0)} y_\pm, \qquad U(\Gamma^{1/p} y_\pm)(x,k) := 1_{B_\pm}(k)\psi_s(x,k), \qquad R \text{ large.} \tag{32}$$

We get from (16) and

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\Big[\mp \int_{\mathbb{R}^d} \Gamma^{1/p} y_\pm \mathcal{R}^\varepsilon(\Gamma^{1/p} y_\pm)\,dx\Big] \\
&= \lim_{\varepsilon\to 0^+}\Big[\mp \int_{\mathbb{R}^d} \Gamma^{1/p} y_\pm \mathcal{R}^\varepsilon(\lambda)(\Gamma^{1/p} y_\pm)\,dx\Big] \\
&= \lim_{\varepsilon\to 0^+}\Big[\mp \int_\Omega \int_B \overline{U(\Gamma^{1/p} y_\pm)} U\big(\mathcal{R}^\varepsilon(\lambda)(\Gamma^{1/p} y_\pm)\big)\,dk\,dx\Big) \\
&= \lim_{\varepsilon\to 0^+}\Big[\mp \int_\Omega \int_B 1_{B_\pm}(k)\overline{\psi_s(x,k)} \cdot \sum_{t\in\mathbb{Z}^d} \frac{\langle U(\Gamma^{1/p} y_\pm)(\cdot,k), \psi_t(\cdot,k)\rangle_{L^2(\Omega;\mathbb{C})}}{\lambda_t(k) - \lambda - i\varepsilon}\psi_t(x,k)\,dk\,dx\Big] \\
&= \lim_{\varepsilon\to 0^+}\Big[\mp \int_\Omega \int_{B_\pm} \frac{|\psi_s(x,k)|^2}{\lambda_s(k) - \lambda - i\varepsilon}\,dk\,dx\Big]
\end{aligned}$$

$$= \mp \int_\Omega \int_{B_\pm} \frac{|\psi_s(x,k)|^2}{\lambda_s(k)-\lambda}\,dk\,dx$$
$$< 0.$$

Here we used that $\{\psi_t(\cdot,k) : t \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\Omega;\mathbb{C})$. The same calculations for the limit $\varepsilon \to 0^-$ yield (after taking the real parts)

$$\mp \int_{\mathbb{R}^d} \Gamma^{1/p} y_\pm \mathbf{R}(\Gamma^{1/p} y_\pm)\,dx < 0.$$

Choosing now R large enough in (32) we get (31) as well as $z_\pm \in L^{p'}(\mathbb{R}^d)$ by the explicit formula for U^{-1} from (8). So the Mountain Pass Theorem provides a Palais-Smale sequence for J at its mountain pass level $c > 0$, which is defined as in section 6 of [10]. This sequence is bounded and using the periodicity of Γ as well as (29) we get from the "nonvanishing property" (see Theorem 3.1 in [10]) that, up to translation, the Palais-Smale sequence converges weakly to a nontrivial solution $v \in L^{p'}(\mathbb{R}^d)$ of (30) which has the right energy level c. As in [10] this provides an $L^p(\mathbb{R}^d)$-solution of (14). As in the above theorem we deduce $u \in W^{2,p}(\mathbb{R}^d) + W^{2,p'}(\mathbb{R}^d)$, which finishes the proof. □

4. Proof of Proposition 3

The proof of Proposition 3 uses the method of stationary phase (p.348ff. [32]) in order to derive the pointwise bounds for $K_2^\varepsilon(x,y)$. The crucial observation is that in the definition of this kernel function from (15) the integration takes place over those regions in the Brillouin zone which, by assumption (A2), correspond to a foliation by Fermi surfaces (F_τ) for $\tau \in (\lambda-\rho,\lambda+\rho)$. Possibly after shrinking $\rho > 0$ these hypersurfaces have positive Gaussian curvature by (A2)(b) so that we may prove decay estimates for integrals of the form

$$\int_{\mathbb{R}} \frac{\chi(\lambda-\tau)}{\lambda-\tau-i\varepsilon}\Big(\int_{F_\tau} h(k)e^{i\sigma\langle v,k\rangle}\,d\mathcal{H}^{d-1}(k)\Big)\,d\tau$$

by the method of stationary phase. As we will see at the end of this section, such estimates yield pointwise bounds for $K_2^\varepsilon(x,y)$ when $\sigma = |x-y|$ and $v = \frac{x-y}{|x-y|}$. The main technical difficulties come from the fact that our estimates have to be uniform with respect to ε and that the presence of the singular prefactor requires to estimate both $a(\lambda)$ and $a(\lambda+t)-a(\lambda)$ where

$$a(\tau) = \chi(\tau-\lambda)\int_{F_\lambda} h(k)e^{i\sigma\langle v,k\rangle}\,d\mathcal{H}^{d-1}(k) \tag{33}$$

for $\tau \in (\lambda-\rho,\lambda+\rho)$. This fact will be proved first.

Proposition 7. *Let $a:\mathbb{R}\to\mathbb{R}$ be measurable such that $|a(\lambda+t)-a(\lambda)| \le \omega(|t|)$ where $t \mapsto \omega(t)/t$ is integrable over $\mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}$. Then the following inequalities hold for $\varepsilon > 0$:*

(i) $$\Big|\int_{\mathbb{R}} \frac{a(\tau)}{\tau-\lambda\mp i\varepsilon}\,d\tau - p.v.\int_{\mathbb{R}}\frac{a(\tau)}{\tau-\lambda}\,d\tau \mp i\pi a(\lambda)\Big| \le \int_0^\infty \frac{\varepsilon}{\sqrt{t^2+\varepsilon^2}}\frac{\omega(t)}{t}\,dt,$$

(ii) $$\Big|\int_{\mathbb{R}} \frac{a(\tau)}{\tau-\lambda\mp i\varepsilon}\,d\tau\Big| \le \pi\cdot\Big(\int_0^\infty \frac{\omega(t)}{t}\,dt + |a(\lambda)|\Big).$$

Proof. Without loss of generality we assume $\lambda = 0$. Then we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}} \frac{a(\tau)}{\tau \mp i\varepsilon}\, d\tau - p.v. \int_{\mathbb{R}} \frac{a(\tau)}{\tau}\, d\tau \mp i\pi a(0) \Big| \\
&= \Big| \lim_{r \to 0^+} \int_{|\tau|>r} \Big(\frac{a(\tau)}{\tau \mp i\varepsilon} - \frac{a(\tau)}{\tau} \mp \frac{i\varepsilon(\tau \pm i\varepsilon) a(0)}{\tau(\tau^2+\varepsilon^2)} \Big) d\tau \Big| \\
&\le \liminf_{r \to 0^+} \int_{|\tau|>r} \Big| \frac{a(\tau)}{\tau \mp i\varepsilon} - \frac{a(\tau)}{\tau} \mp \frac{i\varepsilon(\tau \pm i\varepsilon) a(0)}{\tau(\tau^2+\varepsilon^2)} \Big| d\tau \\
&= \liminf_{r \to 0^+} \int_{|\tau|>r} \Big| \pm \frac{i\varepsilon(\tau \pm i\varepsilon) a(\tau)}{\tau(\tau^2+\varepsilon^2)} \mp \frac{i\varepsilon(\tau \pm i\varepsilon) a(0)}{\tau(\tau^2+\varepsilon^2)} \Big| d\tau \\
&= \liminf_{r \to 0^+} \int_{|\tau|>r} \frac{\varepsilon |a(\tau) - a(0)|}{|\tau| \sqrt{\tau^2+\varepsilon^2}}\, d\tau \\
&\le \int_0^\infty \frac{\varepsilon}{\sqrt{t^2+\varepsilon^2}} \frac{\omega(t)}{t}\, dt.
\end{aligned}$$

This proves (i) and (ii) follows from

$$\begin{aligned}
\Big| \int_{\mathbb{R}} \frac{a(\tau)}{\tau \mp i\varepsilon}\, d\tau \Big| &\le \Big| p.v. \int_{\mathbb{R}} \frac{a(\tau)}{\tau}\, d\tau - i\pi a(0) \Big| + \int_0^\infty \frac{\varepsilon}{\sqrt{t^2+\varepsilon^2}} \frac{\omega(t)}{t}\, dt \\
&\le \int_{\mathbb{R}} \frac{|a(\tau) - a(0)|}{|\tau|}\, d\tau + \pi |a(0)| + \int_0^\infty \frac{\omega(t)}{t}\, dt \\
&\le \pi \Big(\int_0^\infty \frac{\omega(t)}{t}\, dt + |a(0)| \Big).
\end{aligned}$$

□

Variants of the above result are usually attributed to Plemelj and Sokhotski. According to this proposition we investigate functions a of the type (33) for suitable integrands h. In order to derive estimates for such a we will perform a change of coordinates in order to reduce the estimates over the Fermji surfaces F_τ to estimates over $\mathbb{R}^{d-1}$. The estimates over those pieces of the F_τ where the phase function $k \mapsto \langle v, k \rangle$ is nonstationary will be estimated later with the aid of the following result. In the following $I \subset \mathbb{R}$ will be a bounded interval with $0 \in I$ and we will write $\Phi_t(x)$ instead of $\Phi(t,x)$ or $\Phi(t)(x)$ for $t \in I, x \in \mathbb{R}^{d-1}$.

Proposition 8. *Let $K \subset \mathbb{R}^{d-1}$ be a compact set and let $\Phi \in C^1(I; W^{N,\infty}(\mathbb{R}^{d-1}))$ satisfy $|\nabla \Phi_t| \ge c > 0$ on K. Then for all $\alpha \in (0,1)$ there is a $C > 0$ such that for all $\sigma \ge 1$ and $f \in C^1(I; W^{N,1}(\mathbb{R}^{d-1}))$ with* $\operatorname{supp}(f_t) \subset K$ *we have*

$$\Big| \int_{\mathbb{R}^{d-1}} f_t(x) e^{i\sigma \Phi_t(x)}\, dx \Big| \le C |\sigma|^{-N} \|f_t\|_{W^{N,1}(\mathbb{R}^{d-1})},$$

$$\Big| \int_{\mathbb{R}^{d-1}} f_t(x) e^{i\sigma \Phi_t(x)}\, dx - \int_{\mathbb{R}^{d-1}} f_0(x) e^{i\sigma \Phi_0(x)}\, dx \Big| \le C |t|^\alpha |\sigma|^{\alpha - N} \|f\|_{C^1(I; W^{N,1}(\mathbb{R}^{d-1}))}.$$

Proof. Without loss of generality we assume $\nabla \Phi_t(x) \cdot \xi \ge c > 0$ on K for some unit vector $\xi \in \mathbb{S}^{d-1}$ and all $t \in I$, otherwise consider a partition of unity of a suitable covering of $K \times I$

where the corresponding inequalities hold for unit vectors $\xi^1,\ldots,\xi^M$ for some $M\in\mathbb{N}$. We define the differential operators D_t and the formal adjoints D_t^* via

$$(D_t\psi)(x) := \frac{1}{i\sigma}\frac{\langle\nabla\psi(x),\xi\rangle}{\langle\nabla\Phi_t(x),\xi\rangle}, \qquad (D_t^*\psi)(x) = \frac{i}{\sigma}\xi\cdot\nabla\Big(\frac{\psi(\cdot)}{\langle\nabla\Phi_t(\cdot),\xi\rangle}\Big)(x).$$

This definition is motivated by $D_t(e^{i\sigma\Phi_t}) = e^{i\sigma\Phi_t}$. By induction one can show that

$$((D_t^*)^N\psi)(x) = \Big(\frac{i}{\sigma}\Big)^N \frac{P_N(\psi(x),\ldots,\nabla^N\psi(x),\nabla\Phi_t(x),\ldots,\nabla^N\Phi_t(x))}{\langle\nabla\Phi_t(x),\xi\rangle^{N+1}}$$

and P_N is a polynomial of degree $N+1$ that is 1-homogeneous with respect to the ψ-components ($(D_t^*)^N$ is linear) and N-homogeneous with respect to the Φ_t-components. Therefore, integrating by parts N times gives

$$\begin{aligned}\Big|\int_{\mathbb{R}^{d-1}} f_t(x)e^{i\sigma\Phi_t(x)}\,dx\Big| &= \Big|\int_{\mathbb{R}^{d-1}} f_t(x)D_t^N(e^{i\sigma\Phi_t(x)})\,dx\Big| \\ &= \Big|\int_{\mathbb{R}^{d-1}}((D_t^*)^N f_t)(x)e^{i\sigma\Phi_t(x)}\,dx\Big| \\ &\le C|\sigma|^{-N}\int_{\mathbb{R}^{d-1}}\big(|\nabla\Phi_t|+\ldots+|\nabla^N\Phi_t|\big)^N\big(|f_t|+\ldots+|\nabla^N f_t|\big)\,dx \\ &\le C|\sigma|^{-N}\|f_t\|_{W^{N,1}(\mathbb{R}^{d-1})}\end{aligned}$$

and the first inequality is proved. The proof of the second inequality is similar. Proceeding as above we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{d-1}} f_t(x)e^{i\sigma\Phi_t(x)}\,dx - \int_{\mathbb{R}^{d-1}} f_t(x)e^{i\sigma\Phi_0(x)}\,dx\Big| \\ &= \Big|\int_{\mathbb{R}^{d-1}} e^{i\sigma\Phi_0(x)}\big(((D_t^*)^N f_t)(x) - ((D_0^*)^N f_0)(x)\big)\,dx\Big| \\ &+ \Big|\int_{\mathbb{R}^{d-1}}\big(e^{i\sigma(\Phi_0(x)-\Phi_t(x))}-1\big)((D_t^*)^N f_t)(x)e^{i\sigma\Phi_t(x)}\,dx\Big|.\end{aligned}$$

The first integral is estimated as follows:

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} e^{i\sigma\Phi_0(x)}\big(((D_t^*)^N f_t)(x) - ((D_0^*)^N f_0)(x)\big)\,dx\Big| \\ &\le |\sigma|^{-N}\int_{\mathbb{R}^d}\Big|\frac{P_N(f_t(x),\ldots,\nabla^N f_t(x),\nabla\Phi_t(x),\ldots,\nabla^N\Phi_t(x))}{\langle\nabla\Phi_t(x),\xi\rangle^{N+1}} \\ &\qquad - \frac{P_N(f_0(x),\ldots,\nabla^N f_0(x),\nabla\Phi_0(x),\ldots,\nabla^N\Phi_0(x))}{\langle\nabla\Phi_0(x),\xi\rangle^{N+1}}\Big|\,dx \\ &= |\sigma|^{-N}\int_{\mathbb{R}^d}\Big|\int_0^t\frac{d}{ds}\Big(\frac{P_N(f_s(x),\ldots,\nabla^N f_s(x),\nabla\Phi_s(x),\ldots,\nabla^N\Phi_s(x))}{\langle\nabla\Phi_s(x),\xi\rangle^{N+1}}\Big)\,ds\Big|\,dx \\ &\le C|t||\sigma|^{-N}\|f\|_{C^1(I;W^{N,1}(\mathbb{R}^{d-1}))}.\end{aligned}$$

The estimate for the second integral follows from the estimates used in the proof of the first inequality and from the global α-Hölder-continuity of sine and cosine. □

While the above proposition will be used for the estimates of integrals over those regions where the phase is nonstationary, the following propositions deal with the remaining parts. To this end we use the Fourier transformation

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} f(x) e^{-i\langle x,\xi\rangle}\,dx,$$

which is, as usual, defined for all Schwartz functions in $\mathcal{S}(\mathbb{R}^{d-1})$ and, by duality, for all tempered distributions in $\mathcal{S}'(\mathbb{R}^{d-1})$. The dual pairing will be denoted by the symbol $\langle\cdot,\cdot\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}$. First we calculate the Fourier transform of the tempered distribution given by the function $x \mapsto e^{i\sigma\langle x,Ax\rangle}$. Since we did not find a reference for these computations, we present the proof of this well-known result.

Proposition 9. *Let $\sigma > 0$ and $A \in \mathbb{R}^{(d-1)\times(d-1)}$ symmetric and invertible. Then we have*

$$\mathcal{F}(e^{i\sigma\langle x,Ax\rangle})(\xi) = (2\sigma)^{\frac{1-d}{2}} |\det(A)|^{-\frac{1}{2}} e^{i\frac{\pi}{4}\operatorname{sgn}(A)} e^{-i\frac{\langle \xi, A^{-1}\xi\rangle}{4\sigma}}$$

where $\operatorname{sgn}(A)$ denotes the signature of A, i.e. the number of its positive eigenvalues minus the number of its negative eigenvalues.

Proof. Let (K_R) be a sequence of compact sets with $K_R \nearrow \mathbb{R}^{d-1}$ as $R \to \infty$. Then we have for all $h \in \mathcal{S}(\mathbb{R}^{d-1})$ by Fubini's Theorem

$$\begin{aligned}
\langle \mathcal{F}(e^{i\sigma\langle x,Ax\rangle}), h\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})} &= \langle e^{i\sigma\langle x,Ax\rangle}, \mathcal{F}^{-1}h\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})} \\
&= \int_{\mathbb{R}^{d-1}} e^{i\sigma\langle x,Ax\rangle} (\mathcal{F}^{-1}h)(x)\,dx \\
&= \lim_{R\to\infty} \int_{K_R} e^{i\sigma\langle x,Ax\rangle} (\mathcal{F}^{-1}h)(x)\,dx \\
&= (2\pi)^{\frac{1-d}{2}} \lim_{R\to\infty} \int_{\mathbb{R}^{d-1}} h(\xi) \Big(\int_{K_R} e^{i(\sigma\langle x,Ax\rangle + \langle x,\xi\rangle)}\,dx \Big)\,d\xi.
\end{aligned}$$

We will show that the integral over K_R converges as $R \to \infty$. To this end we write $A = Q^T D Q$ for an orthogonal matrix Q and a diagonal matrix D. On the diagonal of D we have the eigenvalues $\mu_1, \ldots, \mu_m, -\mu_{m+1}, \ldots, -\mu_{d-1}$ of A where $m \in \{1, \ldots, d\}$ and all μ_j are positive. Then we have

$$\operatorname{sgn}(A) = m - (d-1-m) = 2m + 1 - d$$

and the matrix $S := Q^T \operatorname{diag}(|\mu_1|^{-1/2}, \ldots, |\mu_{d-1}|^{-1/2})\sigma^{-1/2}$ satisfies

$$\det(S) = \sigma^{\frac{1-d}{2}} |\det(A)|^{-\frac{1}{2}}, \qquad \sigma\langle Sx, ASx\rangle = \sum_{j=1}^{d-1} \frac{\mu_j}{|\mu_j|} x_j^2 = \sum_{j=1}^{m} x_j^2 - \sum_{j=m+1}^{d-1} x_j^2 =: |x'|^2 - |x''|^2.$$

From this we obtain

$$\begin{aligned}
&\int_{K_R} e^{i(\sigma\langle x,Ax\rangle + \langle x,\xi\rangle)}\,dx \\
&= \int_{S^{-1}K_R} |\det(S)| e^{i(\sigma\langle Sx,ASx\rangle + \langle Sx,\xi\rangle)}\,dx \\
&= \sigma^{\frac{1-d}{2}} |\det(A)|^{-\frac{1}{2}} \int_{S^{-1}K_R} e^{i(|x'|^2 - |x''|^2 + \langle x, S^T\xi\rangle)}\,dx
\end{aligned}$$

$$= \sigma^{\frac{1-d}{2}}|\det(A)|^{-\frac12}\int_{S^{-1}K_R} e^{i(|x'+\frac12(S^T\xi)'|^2-|x''-\frac12(S^T\xi)''|^2-\frac{|(S^T\xi)'|^2-|(S^T\xi)''|^2}{4})}\,dx$$
$$= \sigma^{\frac{1-d}{2}}|\det(A)|^{-1/2}e^{-i\frac{|(S^T\xi)'|^2-|(S^T\xi)''|^2}{4}}\int_{K_R'} e^{i(|y'|^2-|y''|^2)}\,dy$$
$$= \sigma^{\frac{1-d}{2}}|\det(A)|^{-\frac12}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}}\int_{K_R'} e^{i(|y'|^2-|y''|^2)}\,dy,$$

where the compact set K_R' is defined by $K_R' := S^{-1}K_R - \frac12((S^T\xi)',-(S^T\xi)'')^T$. From

$$\int_{M_1}^{M_2} e^{\pm iz^2}\,dz \to \sqrt{\pi}e^{\pm i\frac{\pi}{4}} \quad \text{as } M_1\to-\infty, M_2\to\infty$$

we deduce

$$\lim_{R\to\infty}\int_{K_R} e^{i(\sigma\langle x,Ax\rangle-\langle x,\xi\rangle)}\,dx = \sigma^{\frac{1-d}{2}}|\det(A)|^{-\frac12}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}}\cdot\left(\sqrt{\pi}e^{i\frac{\pi}{4}}\right)^m\left(\sqrt{\pi}e^{-i\frac{\pi}{4}}\right)^{d-m-1}$$
$$= \sigma^{\frac{1-d}{2}}|\det(A)|^{-\frac12}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}}\cdot\pi^{\frac{d-1}{2}}e^{i\frac{\pi}{4}(2m+1-d)}$$
$$= \left(\frac{\pi}{\sigma}\right)^{\frac{d-1}{2}}|\det(A)|^{-\frac12}e^{i\frac{\pi}{4}\operatorname{sgn}(A)}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}}.$$

Hence,

$$\langle\mathcal{F}(e^{i\sigma\langle x,Ax\rangle}),h\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})} = (2\pi)^{\frac{1-d}{2}}\int_{\mathbb{R}^{d-1}} h(\xi)\Big(\left(\frac{\pi}{\sigma}\right)^{\frac{d-1}{2}}|\det(A)|^{-\frac12}e^{i\frac{\pi}{4}\operatorname{sgn}(A)}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}}\Big)\,d\xi$$
$$= \langle(2\sigma)^{\frac{1-d}{2}}|\det(A)|^{-\frac12}e^{i\frac{\pi}{4}\operatorname{sgn}(A)}e^{-i\frac{\langle\xi,A^{-1}\xi\rangle}{4\sigma}},h\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})},$$

which is all we had to show. □

Two further technical estimates are needed.

Proposition 10. *Let $A\in C^1(I;\mathbb{R}^{(d-1)\times(d-1)})$ be a family of symmetric matrices such that $\|A_t^{-1}\|+\|A_t\|+\|\frac{d}{dt}A_t\|$ is bounded on I. Then, for all $s>\frac{d-1}{2},\alpha\in(0,1)$ there is a positive number C such that for all $f\in C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))$ and $\sigma\ge1$ we have*

$$\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_t(\xi)\,d\xi\Big| \le C\sigma^{-\alpha}\|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})},$$
$$\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_t(\xi)\,d\xi - \int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_0^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_0(\xi)\,d\xi\Big| \le C|t|\sigma^{-\alpha}\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))}.$$

Proof. From $|e^{it}-1|\le C|t|^\alpha$ we get for all $\sigma\ge1$

$$\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_t(\xi)\,d\xi\Big|$$
$$\le \int_{\mathbb{R}^{d-1}}\big|e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1\big||\hat{f}_t(\xi)|\,d\xi$$
$$\le \int_{\mathbb{R}^{d-1}} C\sigma^{-\alpha}|\xi|^{2\alpha}|\hat{f}_t(\xi)|\,d\xi$$
$$\le C\sigma^{-\alpha}\Big(\int_{\mathbb{R}^{d-1}}(1+|\xi|^2)^{-s}\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^{d-1}}(1+|\xi|^2)^{s+2\alpha}|\hat{f}_t(\xi)|^2\,d\xi\Big)^{1/2}$$

$$\le C\sigma^{-\alpha}\|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})}.$$

At this point, the assumption $s > \frac{d-1}{2}$ was used. The difference of the integrals is estimated as follows:

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_t(\xi)\,d\xi-\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_0^{-1}\xi\rangle}{4\sigma}}-1)\hat{f}_0(\xi)\,d\xi\Big|\\
&=\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,(A_t^{-1}-A_0^{-1})\xi\rangle}{4\sigma}}-1)e^{-i\frac{\langle\xi,A_0^{-1}\xi\rangle}{4\sigma}}\hat{f}_t(\xi)\,d\xi\Big|\\
&\quad+\Big|\int_{\mathbb{R}^{d-1}}(e^{-i\frac{\langle\xi,A_0^{-1}\xi\rangle}{4\sigma}}-1)(\hat{f}_t(\xi)-\hat{f}_0(\xi))\,d\xi\Big|\\
&\le\int_{\mathbb{R}^{d-1}}\big|e^{-i\frac{\langle\xi,(A_t^{-1}-A_0^{-1})\xi\rangle}{4\sigma}}-1\big||\hat{f}_t(\xi)|\,d\xi\\
&\quad+\int_{\mathbb{R}^{d-1}}\big|e^{-i\frac{\langle\xi,A_0^{-1}\xi\rangle}{4\sigma}}-1\big||\hat{f}_t(\xi)-\hat{f}_0(\xi)|\,d\xi.
\end{aligned}$$

Using

$$\|A_t^{-1}-A_0^{-1}\|\le C|t|,\quad |t|\|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})}+\|f_t-f_0\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})}\le|t|\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))}$$

we get the second estimate. □

In the next step we use the above propositions in order to prove estimates for

$$\Delta_t(f):=\int_{\mathbb{R}^{d-1}}f_t(x)e^{i\sigma\langle x,A_tx\rangle}\,dx-f_t(0)\Big(\frac{\pi}{\sigma}\Big)^{\frac{d-1}{2}}|\det(A_t)|^{-\frac12}e^{i\frac{\pi}{4}\operatorname{sgn}(A_t)}. \tag{34}$$

Proposition 11. *Let $A\in C^1(I;\mathbb{R}^{(d-1)\times(d-1)})$ be a family of symmetric matrices such that $\|A_t^{-1}\|+\|A_t\|+\|\frac{d}{dt}A_t\|$ is bounded on I. Then, for all $s>\frac{d-1}{2},\alpha\in(0,1)$ there is a positive number C such that for all $f\in C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))$ and $\sigma\ge1$ we have*

$$|\Delta_t(f)|\le C\sigma^{\frac{1-d}{2}-\alpha}\|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})},$$
$$|\Delta_t(f)-\Delta_0(f)|\le C|t|\sigma^{\frac{1-d}{2}-\alpha}\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))}.$$

Proof. We set $m(t):=(2\sigma)^{\frac{1-d}{2}}|\det(A_t)|^{-\frac12}e^{i\frac{\pi}{4}\operatorname{sgn}(A_t)}$. Notice that $\operatorname{sgn}(A_t)$ is constant since no eigenvalue of A_t approaches zero as t varies over I because $\|A_t^{-1}\|+\|A_t\|$ is bounded. Moreover, the assumptions on the family (A_t) imply

$$|m(t)|\le C\sigma^{\frac{1-d}{2}},\qquad |m(t)-m(0)|\le C|t|\sigma^{\frac{1-d}{2}}. \tag{35}$$

From Proposition 9 we get

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}}f_t(x)e^{i\sigma\langle x,A_tx\rangle}\,dx&=\langle e^{i\sigma\langle x,A_tx\rangle},f_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\\
&=\langle\mathcal{F}(e^{i\sigma\langle x,A_tx\rangle}),\hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\\
&=\langle m(t)e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}},\hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\\
&=m(t)\Big(\langle1,\hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}+\langle e^{-i\frac{\langle\xi,A_t^{-1}\xi\rangle}{4\sigma}}-1,\hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\Big)
\end{aligned}$$

$$= m(t)(2\pi)^{\frac{d-1}{2}} f_t(0) + m(t)\langle e^{-i\frac{\langle \xi, A_t^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}.$$

Therefore, (34) implies

$$\Delta_t(f) = m(t)\langle e^{-i\frac{\langle \xi, A_t^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}$$

so that the estimate for $|\Delta_t(f)|$ follows from Proposition 10 and (35). Similarly, we get again from Proposition 10

$$\begin{aligned}
|\Delta_t(f) - \Delta_0(f)| &= \Big| m(t)\langle e^{-i\frac{\langle \xi, A_t^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})} - m(0)\langle e^{-i\frac{\langle \xi, A_0^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_0\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\Big| \\
&\le |m(t) - m(0)| \big|\langle e^{-i\frac{\langle \xi, A_t^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\big| \\
&\quad + |m(0)| \big|\langle e^{-i\frac{\langle \xi, A_t^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_t\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})} - \langle e^{-i\frac{\langle \xi, A_0^{-1}\xi\rangle}{4\sigma}} - 1, \hat{f}_0\rangle_{\mathcal{S}'(\mathbb{R}^{d-1})}\big| \\
&\le C|t|\sigma^{\frac{1-d}{2}} \cdot C\sigma^{-\alpha}\|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})} + C\sigma^{\frac{1-d}{2}} \cdot C|t|\sigma^{-\alpha}\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))} \\
&\le C|t|\sigma^{\frac{1-d}{2}-\alpha}\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))}.
\end{aligned}$$

□

Proof of Proposition 3: We write $\lambda_s(k) = \Lambda(k + 2\pi s), \psi_s(x,k) = \Psi(x, k + 2\pi s)$ for all $s \in \mathbb{Z}^d$ with $\chi(\lambda_s(k) - \lambda) \neq 0$, see assumption (A2) and (18). Then the coarea formula yields for all $x, y \in \mathbb{R}^d$

$$\begin{aligned}
K_2^\varepsilon(x,y) &= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\lambda_s(k) - \lambda)}{\lambda_s(k) - \lambda - i\varepsilon} \psi_s(x,k)\overline{\psi_s(y,k)}\, dk \\
&= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\Lambda(k + 2\pi s) - \lambda)}{\Lambda(k + 2\pi s) - \lambda - i\varepsilon} \Psi(x, k + 2\pi s)\overline{\Psi(y, k + 2\pi s)}\, dk \\
&= \int_{\mathbb{R}^d} \frac{\chi(\Lambda(k) - \lambda)}{|B|(\Lambda(k) - \lambda - i\varepsilon)} \Psi(x,k)\overline{\Psi(y,k)}\, dk \\
&= \int_{\mathbb{R}} \frac{\chi(\tau - \lambda)}{\tau - \lambda - i\varepsilon} \Big(\int_{F_\tau} \frac{\Psi(x,k)\overline{\Psi(y,k)}}{|B||\nabla\Lambda(k)|}\, d\mathcal{H}^{d-1}(k) \Big)\, d\tau \\
&= \int_{\mathbb{R}} \frac{\chi(\tau - \lambda)}{\tau - \lambda - i\varepsilon} \Big(\int_{F_\tau} h_{x,y}(k) e^{i\sigma_{x,y}\langle v_{x,y}, k\rangle}\, d\mathcal{H}^{d-1}(k) \Big)\, d\tau \\
&= \int_{\mathbb{R}} \frac{a_{x,y}(\tau)}{\tau - \lambda - i\varepsilon}\, d\tau
\end{aligned}$$

where we used the shorthand notations $\sigma_{x,y} := |x - y|$, $v_{x,y} := \frac{x-y}{|x-y|}$ as well as

$$a_{x,y}(\tau) := \chi(\tau - \lambda) \int_{F_\tau} h_{x,y}(k) e^{i\sigma_{x,y}\langle v_{x,y}, k\rangle}\, d\mathcal{H}^{d-1}(k), \tag{36}$$

$$h_{x,y}(k) := \frac{\Psi(x,k)\overline{\Psi(y,k)} e^{-i\langle x-y, k\rangle}}{|B||\nabla\Lambda(k)|}. \tag{37}$$

Notice that this formula is of the form (33). In view of this formula and Proposition 7 (i) we define

$$K_2^{\pm}(x,y) := p.v. \int_{\mathbb{R}} \frac{a_{x,y}(\tau)}{\tau-\lambda}\,d\tau \pm i\pi a_{x,y}(\lambda).$$

From these formulas and Proposition 7 (i) and (ii) we deduce that it remains to prove an estimate of the form

$$|a_{x,y}(\lambda)| \le C_1(1+|x-y|)^{\frac{1-d}{2}}, \qquad |a_{x,y}(\lambda+t)-a_{x,y}(\lambda)| \le C_2(1+|x-y|)^{\frac{1-d}{2}} t^{\alpha} \tag{38}$$

for some $\alpha, C_1, C_2 > 0$ and $|t| \le \rho$. Notice that $a_{x,y}(\tau) = 0$ whenever $|\tau-\lambda| > \rho$ by the definition of χ. Indeed, once this estimate is shown, we get

$$\begin{aligned}
|K_2^{\pm}(x,y)| &\le \pi \cdot \Big(\int_0^{\infty} \frac{|a_{x,y}(\lambda+t)-a_{x,y}(\lambda)|}{t}\,dt + |a_{x,y}(\lambda)| \Big) \\
&\le \pi \cdot \Big(C_2(1+|x-y|)^{\frac{1-d}{2}} \int_0^{\tau} t^{\alpha-1}\,dt + C_1(1+|x-y|)^{\frac{1-d}{2}} \Big) \\
&\le \pi(C_2\alpha^{-1}(|\lambda|+\rho)^{\alpha} + C_1)(1+|x-y|)^{\frac{1-d}{2}}, \\
|K_2^{\varepsilon}(x,y) - K_2^{\pm}(x,y)| &\le \int_0^{\infty} \frac{\varepsilon}{\sqrt{\varepsilon^2+t^2}} \frac{|a_{x,y}(\lambda+t)-a_{x,y}(\lambda)|}{t}\,dt \\
&\le C_2(1+|x-y|)^{\frac{1-d}{2}} \int_0^{\infty} \frac{\varepsilon}{\sqrt{\varepsilon^2+t^2}} t^{\alpha-1}\,dt \\
&= o(1)(1+|x-y|)^{\frac{1-d}{2}}.
\end{aligned}$$

The estimates (38) will be achieved via the method of stationary phase.

We only prove the (much more difficult) estimates for $\sigma_{x,y} \ge 1$. For notational convenience we drop the subscripts, i.e. $\sigma = \sigma_{x,y}, v = v_{x,y}, h = h_{x,y}$. Using (A2)(a) and (A2)(c) we find that for $t \in I := (-\rho,\rho)$ and $\rho > 0$ small enough the Fermi surfaces $F_{\lambda+t} = \{k \in U : \Lambda(k) = \lambda+t\}$ admit local graphical representations. After a permutation of the coordinates these representations may be assumed to be of the form $k = (z, \phi_t(z))$ for z in some open bounded set $V \subset \mathbb{R}^{d-1}$ where $(t,z) \mapsto \phi_t(z)$ is of class C^{N+1} because $\Lambda \in C^{N+1}(U)$. This is a consequence of the Implicit Function Theorem. So it suffices to prove the correspondig estimates for the integrals

$$I_{t,v} := \int_{\mathbb{R}^{d-1}} f_t(z) e^{i\sigma\Phi_{t,v}(z)}\,dz \qquad \text{where}$$
$$f_t(z) := \eta(z) h(z,\phi_t(z)) \sqrt{1+|\nabla\phi_t(z)|^2},$$

for a cutoff function $\eta \in C_0^{\infty}(\mathbb{R}^{d-1})$. By assumption (A2)(b) the Gaussian curvature of S_t is uniformly positive, which implies the lower bound $\det(D^2\phi_t(z)) \ge c > 0$ whenever $z \in \operatorname{supp}(\eta), t \in I$. Recall that the Gassian curvature at such a point of $F_{\lambda+t}$ is given by the formula

$$\mathcal{K}_t(z,\phi_t(z)) = \frac{\det(D^2\phi_t(z))}{(1+|\nabla\phi_t(z)|^2)^{\frac{d+1}{2}}}. \tag{39}$$

So the Implicit Function Theorem implies that, possibly after redefining η, we can find open neighbourhoods V, V_*, V_{**} of $\mathrm{supp}(\eta)$ such that $V \subset\subset V_* \subset\subset V_{**}$ and $\nabla\phi_0$ is a diffeomorphism on each of these sets as well as

$$\det(D^2\phi_t(z)) \geq c > 0 \quad \text{for all } z \in V_* \text{ and } t \in I. \tag{40}$$

We now estimate the integrals $I_{t,v}$ for unit vectors v belonging to two different regimes.

1st case: $-v'/v_d \notin \nabla\phi_0(V)$ *(this includes the case* $v_d = 0$*).* This assumption implies $|\nabla\Phi_{t,v}(z)| \geq c > 0$ on $\mathrm{supp}(\eta) \supset \mathrm{supp}(f_t)$ for all sufficiently small $|t|$ and all such v. So Proposition 8 yields for any given $\alpha \in (0,1)$ and all $x, y \in \mathbb{R}^d$ the estimates

$$|I_{t,v}| \leq C\sigma^{-N}\|f_t\|_{W^{N,1}(\mathbb{R}^{d-1})} \leq C\sigma^{-N}\|h\|_{C^N(U)} \leq C\sigma^{-N},$$

as well as

$$|I_{t,v} - I_{0,v}| \leq C|t|^\alpha\sigma^{\alpha-N}\|f\|_{C^1(I;W^{N,1}(\mathbb{R}^{d-1}))} \leq C|t|^\alpha\sigma^{\alpha-N}\|h\|_{C^{N+1}(U)} \leq C|t|^\alpha\sigma^{\alpha-N},$$

see the definition of h from (36) and assumption (A2)(a). Recall that $(x,y) \mapsto h_{x,y}(\cdot) = h(\cdot)$ is $\mathbb{Z}^d \times \mathbb{Z}^d$ periodic by the quasiperiodic extensions of the eigenfunctions ψ_s to $\mathbb{R}^d \times B$ mentioned in the introduction. The constant C in the above estimates absorbs C^{N+1}-norms of the ϕ_t and thus depends on the C^{N+1}-norm of Λ as well as estimates from below for $|\nabla\Lambda|$.

2nd case: $-v'/v_d \in \nabla\phi_0(V)$. For sufficiently small $|t|$ we have in this case $-v'/v_d \in \nabla\phi_t(V_*)$ because V_* is slightly larger than V. Since $\nabla\phi_t : V_* \to \nabla\phi_t(V_*)$ is a diffeomorphism, we may set $z_{t,v} := (\nabla\phi_t|_{V_*})^{-1}(-v'/v_d) \in V_*$ so that the following holds:

$$\nabla\Phi_{t,v}(z_{t,v}) = 0 \qquad \text{whenever } -v'/v_d \in \nabla\phi_0(V),\ |t| \text{ small.}$$

Having thus determined the unique point of stationary phase in V_* we now make a local coordinate transformation around that point which makes the phase function $\Phi_{t,v}$ look like a 2-homogeneous (in particular quadratic) polynomial in order to apply Proposition 9. The map $(t,v) \mapsto z_{t,v}$ is $N \geq 2$ times continuously differentiable. Hence, Morse's lemma provides a positive δ and local C^{N-2}-diffeomorphisms $\psi_{t,v} : B_\delta(0) \to \psi_{t,v}(B_\delta(0))$ as well as matrices $A_{t,v} \in \mathbb{R}^{(d-1)\times(d-1)}$ such that $\psi_{t,v}(0) = 0$, $z_{t,v} + \psi_{t,v}(B_\delta(0)) \subset V_{**}$ and

$$\Phi_{t,v}(z_{t,v} + \psi_{t,v}(y)) = \langle y, A_{t,v}y\rangle \qquad \text{whenever } y \in B_\delta(0), -v'/v_d \in \nabla\phi_0(V),\ |t| \text{ small.}$$

Notice that δ may be chosen independently of t, v because of (40). This identity gives

$$A_{t,v} = \frac{1}{2}\psi'_{t,v}(0)^T D^2\Phi_{t,v}(z_{t,v})\psi'_{t,v}(0) \tag{41}$$

so that (40) implies $\det(A_{t,v}) \geq c > 0$ for $|t|$ small and all v as above. In order to exploit the locally quadratic form of the phase function $\Phi_{t,v}$ around $z_{t,v}$ we choose a cut-off function $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{d-1})$ such that

$$\tilde{\chi} \equiv 1 \text{ near } 0, \qquad \mathrm{supp}(\tilde{\chi}) \subset \psi_{t,v}(B_\delta(0)) \quad \text{for all small } |t| \text{ and } -v'/v_d \in \nabla\phi_0(V)$$

and observe $I_{t,v} = I^1_{t,v} + I^2_{t,v}$ where

$$I^1_{t,v} = \int_{\mathbb{R}^{d-1}} f_t(z)(1 - \tilde{\chi}(z - z_{t,v}))e^{i\sigma\Phi_{t,v}(z)}\,dz,$$

$$I^2_{t,v} = \int_{\mathbb{R}^{d-1}} f_t(z)\tilde{\chi}(z - z_{t,v})e^{i\sigma\Phi_{t,v}(z)}\, dz.$$

We first deal with $I^1_{t,v}$. As in the first case the phase function $\Phi_{t,v}$ is uniformly nonstationary on the support of $z \mapsto f_t(z)(1 - \tilde{\chi}(z - z_{t,v}))$ since the only stationary point in the support of f_t is $z_{t,v}$ and $\tilde{\chi} \equiv 1$ near zero. So the same estimates as above yield

$$|I^1_{t,v}| \le C\sigma^{-N}, \qquad |I^1_{t,v} - I^1_{0,v}| \le C|t|^{\alpha}\sigma^{\alpha-N}. \tag{42}$$

The estimates for $I^2_{t,v}$ are based on Proposition 9 and Proposition 11. After a change of variables we obtain

$$I^2_{t,v} = \int_{\mathbb{R}^d} g_{t,v}(y)e^{i\sigma\langle y, A_{t,v}y\rangle}\, dy$$
$$\text{where}\quad g_{t,v}(y) := f_t(z_{t,v} + \psi_{t,v}(y))\tilde{\chi}(\psi_{t,v}(y))|\det(\psi'_{t,v}(y))|.$$

In view of (34) we first calculate $g_{t,v}(0)|\det(A_{t,v})|^{-\frac{1}{2}}$. Exploiting

$$1 + |\nabla\phi_t(z_{v,t})|^2 = 1 + \frac{|v'|^2}{|v_d|^2} = |v_d|^{-2}$$

we get

$$\begin{aligned}
g_{t,v}(0)|\det(A_{t,v})|^{-\frac{1}{2}} &= f_t(z_{t,v})\tilde{\chi}(0)|\det(\psi'_{t,v}(0))|\Big|\det\big(\frac{1}{2}\psi'_{t,v}(0)^T D^2\Phi_{t,v}(z_{t,v})\psi'_{t,v}(0)\big)\Big|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} f_t(z_{t,v})|\det(\psi'_{t,v}(0))|\Big|\det\big(\psi'_{t,v}(0)^T D^2\Phi_{t,v}(z_{t,v})\psi'_{t,v}(0)\big)\Big|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} f_t(z_{t,v})|\det(D^2\Phi_{t,v}(z_{t,v}))|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} \eta(z_{t,v})h(z_{t,v},\phi_t(z_{t,v}))|v_d|^{-1}|\det(v_d D^2\phi_t(z_{t,v}))|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} \eta(z_{t,v})h(z_{t,v},\phi_t(z_{t,v}))\Big||v_d|^{d+1}\det(D^2\phi_t(z_{t,v}))\Big|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} \eta(z_{t,v})h(z_{t,v},\phi_t(z_{t,v}))\Big|(1+|\nabla\phi_t(z_{t,v})|^2)^{-\frac{d+1}{2}}\det(D^2\phi_t(z_{t,v}))\Big|^{-\frac{1}{2}}\\
&= 2^{\frac{d-1}{2}} \eta(z_{t,v})h(z_{t,v},\phi_t(z_{t,v}))\mathcal{K}_t(z_{t,v},\phi_t(z_{t,v}))^{-1/2},
\end{aligned}$$

see (39). Moreover, (41) implies

$$\operatorname{sgn}(A_{t,v}) = \operatorname{sgn}(D^2\Phi_{t,v}(z_{t,v})) = \operatorname{sgn}(v_d D^2\phi_t(z_{t,v})) = \operatorname{sign}(v_d)\operatorname{sgn}(D^2\phi_t(z_{t,v}))$$

and the latter factor is constant with respect to t, v, see (40). So the definition of $\Delta_{t,v}$ implies

$$I^2_{t,v} = \Delta_{t,v}(g) + \mu_v\Big(\frac{2\pi}{\sigma}\Big)^{\frac{d-1}{2}}\eta(z_{t,v})h(z_{t,v},\phi_t(z_{t,v}))\mathcal{K}_t(z_{t,v},\phi_t(z_{t,v}))^{-1/2} \tag{43}$$

for $\mu_v = e^{i\frac{\pi}{4}\operatorname{sign}(v_d)\operatorname{sgn}(D^2\phi_t(z_{t,v}))}$ Proposition 11 yields for $\sigma \ge 1$ and $s \in \mathbb{R}, \alpha \in (0,1)$ such that $\frac{d-1}{2} < s < s + 2\alpha < N$ and $\beta := \min\{N - \frac{d-1}{2}, 1\}$

$$\begin{aligned}
|I^2_{t,v}| &\le |I^2_{t,v} - \Delta_{t,v}(g_{t,v})| + |\Delta_{t,v}(g_{t,v})|\\
&\le C\sigma^{\frac{1-d}{2}}\|h\|_{C(U)} + C\sigma^{\frac{1-d}{2}-\alpha}\|g_{t,v}\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})}\\
&\le C\sigma^{\frac{1-d}{2}}\big(\|h\|_{C(U)} + \|f_t\|_{H^{s+2\alpha}(\mathbb{R}^{d-1})}\big)
\end{aligned}$$

$$\leq C\sigma^{\frac{1-d}{2}}\|h\|_{C^N(U)}, \tag{44}$$

$$|I^2_{t,v} - I^2_{0,v}| \leq C|t|\sigma^{\frac{1-d}{2}}\|h\|_{C^1(U)} + C|t|\sigma^{\frac{1-d}{2}-\alpha}\|f\|_{C^1(I;H^{s+2\alpha}(\mathbb{R}^{d-1}))}$$

$$\leq C|t|\sigma^{\frac{1-d}{2}}\|h\|_{C^N(U)}. \tag{45}$$

Combining the estimates for $I^1_{t,v}, I^2_{t,v}$ from (42),(44),(45) we get the result for small $|t|$ and thus for all t. □

5. Proof of Proposition 5

Proposition 12. *Let R_j be defined as in* (23) *and set*

$$g_j(\xi) := \sum_{m\in R_j} e^{im\xi} \qquad \text{for } \xi \in \mathbb{R}^d,\ j \in \mathbb{N}.$$

Then for all compact subsets $K \subset \mathbb{R}^{d-1}$ and $\delta > 0$ there is a $C > 0$ such that for all $\xi' \in \mathbb{R}^{d-1}, s, t \in \mathbb{R}$ and $j \in \mathbb{N}$ the following estimates hold:

$$\int_K |g_j(\xi', s)|\, d\xi' \leq C2^{j(1+\delta)}, \qquad \int_K |g_j(\xi', s) - g_j(\xi', t)|\, d\xi' \leq C2^{j(1+\delta)}|s-t|^{\delta/2}.$$

Proof. By definition of R_j the function g_j can be written as

$$g_j(\xi) = \prod_{p=1}^d D_j(\xi_p) - \prod_{p=1}^d D_{j-1}(\xi_p), \qquad \text{where } D_j(z) := \sum_{m=-2^j}^{2^j} e^{imz} = \frac{\sin((2^j+\frac{1}{2})z)}{\sin(\frac{z}{2})}.$$

The Dirichlet kernels satisfy the estimates $|D_j(z)| \leq C2^j$ as well as

$$|D_j(s) - D_j(t)| \leq \sum_{p=-2^j}^{2^j} |e^{ipt}(e^{ip(s-t)} - 1)| \leq \sum_{p=-2^j}^{2^j} 2|p(s-t)|^{\delta/2} \leq C2^{j(1+\delta/2)}|s-t|^{\delta/2}.$$

These estimates for D_j imply ($\xi' = (\xi_1, \ldots, \xi_{d-1}) \in K$)

$$|g_j(\xi', s)| \leq \sum_{\iota\in\{j-1,j\}} \Big(\prod_{p=1}^{d-1} |D_\iota(\xi_p)|\Big)|D_\iota(s)| \leq C2^j \sum_{\iota\in\{j-1,j\}} \prod_{p=1}^{d-1} |D_\iota(\xi_p)|$$

as well as

$$\begin{aligned}
|g_j(\xi', s) - g_j(\xi', t)| &= \Big|\Big(\prod_{p=1}^{d-1} D_j(\xi_p)\Big)D_j(s) - \Big(\prod_{p=1}^{d-1} D_{j-1}(\xi_p)\Big)D_{j-1}(s) \\
&\quad - \Big(\prod_{p=1}^{d-1} D_j(\xi_p)\Big)D_j(t) + \Big(\prod_{p=1}^{d-1} D_{j-1}(\xi_p)\Big)D_{j-1}(t)\Big| \\
&\leq \sum_{\iota\in\{j-1,j\}} \Big(\prod_{p=1}^{d-1} |D_\iota(\xi_p)|\Big)|D_\iota(s) - D_\iota(t)| \\
&\leq C2^{j(1+\delta/2)}|s-t|^{\delta/2} \sum_{\iota\in\{j-1,j\}} \Big(\prod_{p=1}^{d-1} |D_\iota(\xi_p)|\Big).
\end{aligned}$$

Integrating the first estimate with respect to ξ' over $K \subset [-M,M]^{d-1}$ gives

$$\int_K |g_j(\xi',s)|\,d\xi' \le C2^j \sum_{\iota\in\{j-1,j\}} \Big(\int_{-M}^{M} |D_\iota|\Big)^{d-1} \le C2^j \sum_{\iota\in\{j-1,j\}} \iota^{d-1} \le C2^{j(1+\delta)}$$

where the final C depends on M and thus on the compact set K, but not on j. The estimate for the integral of the Dirichlet kernel over $[-M,M]$ can be found in [15] (Lemma 7). Performing the corresponding estimates for the other term we get the asserted estimates. □

Proof of Proposition 5: We have to show that for all $\delta > 0$ there is a $C_\delta > 0$ such that for all $\varepsilon \in \mathbb{R}\setminus\{0\}$ the following inequality holds:

$$\sup_{x,y\in\Omega, l\in B} \big|U(K_2^{\varepsilon,j}(\cdot,y))(x,l)\big| \le C_\delta 2^{j(1+\delta)} \qquad \text{for all } j\in\mathbb{N}_0 \text{ and}$$

$$\sup_{x,y\in\Omega, l\in B} \big|U(K_2^{\varepsilon,j}(\cdot,y) - K_2^{\pm,j}(\cdot,y))(x,l)\big| = o(1)2^{j(1+\delta)} \qquad \text{for all } j\in\mathbb{N}_0 \text{ as } \varepsilon\to 0^\pm.$$

We only prove the first inequality in detail. Indeed, the formulas for $K_2^\varepsilon, K_2^{\varepsilon,j}$ from (20),(23) yield for all $x,y\in\Omega$ and $l\in B$

$$\begin{aligned}
U(K_2^{\varepsilon,j}(\cdot,y))(x,l) &= \sum_{m\in\mathbb{Z}^d} e^{iml} K_2^\varepsilon(x-m,y) 1_{R_j}([x-m]-[y]) \\
&= \sum_{m\in\mathbb{Z}^d} e^{iml} 1_{R_j}(-m) \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon} \psi_s(x-m,k)\overline{\psi_s(y,k)}\,dk \\
&= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon} \psi_s(x,k)\overline{\psi_s(y,k)} \Big(\sum_{m\in R_j} e^{im(l-k)}\Big)\,dk \\
&= \fint_B \sum_{s\in\mathbb{Z}^d} \frac{\chi(\lambda_s(k)-\lambda)}{\lambda_s(k)-\lambda-i\varepsilon} \psi_s(x,k)\overline{\psi_s(y,k)} g_j(l-k)\,dk.
\end{aligned}$$

In order to simplify this expression further we use assumption (A2). Writing $\lambda_s(k) = \Lambda(k+2\pi s), \psi_s(x,k) = \Psi(x,k+2\pi s)$ for all $s\in\mathbb{Z}^d$ with $\chi(\lambda_s(k)-\lambda)\ne 0$, see (A2) and (18), and using the $2\pi\mathbb{Z}^d$-periodicity of g_j we arrive at

$$\begin{aligned}
U(K_2^{\varepsilon,j}(\cdot,y))(x,l) &= \sum_{s\in\mathbb{Z}^d} \int_{B+2\pi s} \frac{\chi(\Lambda(k)-\lambda)}{|B|(\Lambda(k)-\lambda-i\varepsilon)} \Psi(x,k)\overline{\Psi(y,k)} g_j(l-k)\,dk \\
&= \int_{\mathbb{R}^d} \frac{\chi(\Lambda(k)-\lambda)}{|B|(\Lambda(k)-\lambda-i\varepsilon)} \Psi(x,k)\overline{\Psi(y,k)} g_j(l-k)\,dk \\
&= \int_{\mathbb{R}} \frac{\chi(\tau-\lambda)}{\tau-\lambda-i\varepsilon} \Big(\int_{F_\tau} \frac{\Psi(x,k)\overline{\Psi(y,k)} g_j(l-k)}{|B||\nabla\Lambda(k)|}\,d\mathcal{H}^{d-1}(k)\Big)\,d\tau \\
&= \int_{\mathbb{R}} \frac{\chi(\tau-\lambda)}{\tau-\lambda-i\varepsilon} \Big(\int_{F_\tau} h_{x,y}(k)\,d\mathcal{H}^{d-1}(k)\Big)\,d\tau
\end{aligned}$$

where

$$h_{x,y}(k) := \frac{\Psi(x,k)\overline{\Psi(y,k)} g_j(l-k)}{|B||\nabla\Lambda(k)|}.$$

In the third equality above we used the coarea formula. Assumption (A2)(a) and (c) imply

$$|h_{x,y}(k)| \le C|g_j(l-k)|, \qquad |h_{x,y}(k) - h_{x,y}(\tilde{k})| \le C|g_j(l-k) - g_j(l-\tilde{k})|$$

for some $C > 0$ and all $x, y \in \mathbb{R}^d, l \in B$ and $j \in \mathbb{N}$. In view of Proposition 7 (ii) we may bound the expression $U(K_2^{\varepsilon,j}(\cdot,y))(x,l)$ by estimating the difference

$$\int_{F_{\lambda+t}} h_{x,y}(k)\, d\mathcal{H}^{d-1}(k) - \int_{F_\lambda} h_{x,y}(k)\, d\mathcal{H}^{d-1}(k).$$

As in the proof of Proposition 3 we may content ourselves with proving the estimates on pieces of the Fermi surfaces that are paramtrized over the first $d-1$ Euclidean coordinates according to $k = (z, \phi_t(z))$ for $z \in V \subset \mathbb{R}^{d-1}$ where $(t,z) \mapsto \phi_t(z)$ is of class C^{N+1}. In particular, we have $\|\phi_t - \phi_0\|_{C^1(V)} \le C|t|$ for all $t \in I$. So we get

$$\begin{aligned}
&\int_V \Big| h_{x,y}(z,\phi_t(z))(1+|\nabla\phi_t(z)|^2)^{1/2} - h_{x,y}(z,\phi_0(z))(1+|\nabla\phi_0(z)|^2)^{1/2}\Big|\, dz \\
&\le \int_V \Big(|h_{x,y}(z,\phi_t(z))| \big|(1+|\nabla\phi_t(z)|^2)^{1/2} - (1+|\nabla\phi_0(z)|^2)^{1/2}\big| \\
&\qquad + \big|h_{x,y}(z,\phi_t(z)) - h_{x,y}(z,\phi_0(z))\big| (1+|\nabla\phi_0(z)|^2)^{1/2}\Big)\, dz \\
&\le C\int_V |h_{x,y}(z,\phi_t(z))||\nabla\phi_t(z) - \nabla\phi_0(z)|\, dz + C\int_V |h_{x,y}(z,\phi_t(z)) - h_{x,y}(z,\phi_0(z))|\, dz \\
&\le C|t| \int_V |g_j(l'-z, l_d - \phi_t(z)))|\, dz + C\int_V |g_j(l'-z, l_d-\phi_t(z)) - g_j(l'-z, l_d - \phi_0(z))|\, dz \\
&\le C(2^{j(1+\delta)}|t| + 2^{j(1+\delta)}|t|^{\delta/2}) \\
&\le C 2^{j(1+\delta)}|t|^{\delta/2}.
\end{aligned}$$

In the second last inequality we used Proposition 12. Using similar estimates we get from the same proposition

$$\int_V |h_{x,y}(z,\phi_t(z))|(1+|\nabla\phi_t(z)|^2)^{1/2}\, dz \le C\int_V |g_j(l'-z, l_d-\phi_t(z)))|\, dz \le C2^{j(1+\delta)}.$$

Combining Proposition 7 (ii) and the above estimates we arrive at

$$\begin{aligned}
|U(K_2^{\varepsilon,j}(\cdot,y))(x,l)| &\le C\Big|\int_{F_\lambda} h_{x,y}(k)\, d\mathcal{H}^{d-1}(k)\Big| \\
&\quad + C\int_0^\rho \frac{1}{t}\Big|\int_{F_{\lambda+t}} h_{x,y}(k)\, d\mathcal{H}^{d-1}(k) - \int_{F_\lambda} h_{x,y}(k)\, d\mathcal{H}^{d-1}(k)\Big|\, dt \\
&\le C2^{j(1+\delta)}.
\end{aligned}$$

This implies the first of the asserted estimates. The second estimate is proved in the same manner where the estimate of Proposition 7 (ii) is replaced by the one from (i). □

6. Appendix - On the equiboundedness of eigenfunctions

Here, we prove that assumption (A3) holds for separable potentials $V \in L^\infty(\mathbb{R}^d)$ given by $V(x) = V_1(x_1) + \ldots + V_d(x_d)$. The main ingredient is the corresponding result for one-dimensional eigenvalue problems due to Il'in and Joo [14], see also Theorem 2.1 in [17] for a short proof of this result.

Proposition 13 (Teorema 1, [14])**.** *Let $q \in L^1(a,b)$. Then there is a $C > 0$ such that all solutions $u \in W^{2,1}(a,b)$ of*

$$-u'' + qu = \lambda u \quad \text{in } (a,b)$$

with $\lambda \geq 0$ satisfy $\|u\|_\infty \leq C\|u\|_2$.

This result does not admit straightforward extensions to eigenfunctions on general bounded domains. As pointed out in Example 2.7 in [17], the radially symmetric eigenfunctions of the Laplacian on a three-dimensional ball associated with homogeneous Dirichlet boundary conditions do not satisfy such a uniform bound. In the special case of a separable potential, however, it is possible to use the one-dimensional result to conclude the uniform boundedness of the Floquet-Bloch eigenfunctions.

Lemma 3. *Let $V \in L^\infty_{loc}(\mathbb{R}^d)$ be a separable $\mathbb{Z}^d$-perodiodic potential and $(\psi_s(\cdot,k))$ an orthonormal basis associated with the eigenvalue problem (4) in $L^2(\Omega;\mathbb{C})$. Then there is a positive $C > 0$ such that*

$$\|\psi_s(\cdot,k)\|_{L^\infty(\Omega;\mathbb{C})} \leq C \qquad \text{for all } s \in \mathbb{Z}^d, k \in B.$$

Proof. Since V is separable, we can write $V(x) = V_1(x_1) + \ldots + V_d(x_d)$ and each eigenpair $(\lambda_s(k), \psi_s(\cdot,k))$ of (4) is given by $\lambda_s(k) = \mu_1 + \ldots + \mu_d$ and $\psi_s(x,k) = \phi_1(x_1) \cdot \ldots \cdot \phi_d(x_d)$ where (μ_i, ϕ_i) satisfy

$$-\phi_i'' + V_i\phi_i = \mu_i\phi_i \quad \text{in } (0,1), \quad \phi_i(1) = e^{ik_i}\phi_i(0).$$

From Proposition 13 we get $\|\phi_i\|_\infty \leq C_i\|\phi\|_2$ for $i = 1,\ldots,d$ where each of the C_i depends only on V_i. Notice that the restriction $\lambda \geq 0$ from Proposition 13 is in fact not needed since only finitely many eigenfunctions associated with negative eigenvalues exist. So in total we get

$$\|\psi_s(\cdot,k)\|_{L^\infty(\Omega;\mathbb{C})} = \|\phi_1\|_\infty \cdot \ldots \cdot \|\phi_d\|_\infty \leq C\|\phi_1\|_2 \cdot \ldots \cdot \|\phi_d\|_2 = C\|\psi_s(\cdot,k)\|_{L^2(\Omega;\mathbb{C})},$$

which is all we had to show. □

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R. Mandel
Karlsruhe Institute of Technology
Institute for Analysis
Englerstrasse 2
D-76131 Karlsruhe, Germany
E-mail address:
Rainer.Mandel@kit.edu