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# On the ultimate uncertainty of the top quark pole mass

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ABSTRACT

We combine the known asymptotic behaviour of the QCD perturbation series expansion, which relates the pole mass of a heavy quark to the $\overline{\text{MS}}$ mass, with the exact series coefficients up to the four-loop order to determine the ultimate uncertainty of the top-quark pole mass due to the renormalon divergence. We perform extensive tests of our procedure by varying the number of colours and flavours, as well as the scale of the strong coupling and the $\overline{\text{MS}}$ mass. Including an estimate of the internal bottom and charm quark mass effect, we conclude that this uncertainty is around 110 MeV. We further estimate the additional contribution to the mass relation from the five-loop correction and beyond to be around 300 MeV.



## 1. Introduction

The top quark mass is a fundamental parameter of the Standard Model (SM). Due to its large size, it has non-negligible impact in the precision tests of the SM. After the discovery of the Higgs boson and the measurement of its mass, the values of the $W$ and top mass are strongly correlated, such that a precise determination of both parameters would lead to a SM test of unprecedented precision [1]. Indeed, there is presently some tension between the value of the top mass $177 \pm 2.1$ GeV fitted from electroweak data and from its direct measurement [1], for which the combination of the Tevatron and LHC data yields the 1.6 $\sigma$ lower value of $173.34 \pm 0.27 \pm 0.71$ GeV [2]. The value of the top mass is also crucial to the issue of stability of the SM vacuum (see [3] for a recent analysis). The Higgs quartic coupling decreases at high scales, eventually becoming negative. This evolution is very sensitive to the top mass value. For example, a top mass near 171 GeV would imply that the quartic coupling may vanish at the Planck scale, rather than turn negative.

The standard direct determination of the top mass at hadron colliders, being based upon observables that are related to the mass of the system comprising the top decay products, are quoted as measurements of the pole mass. On the other hand, it seems more natural to use the $\overline{\text{MS}}$ mass in both precision electroweak observables and in vacuum stability studies. In [4] the relation between the $\overline{\text{MS}}$ and pole mass for a heavy quark (the "mass conversion formula" from now on) has been computed to the fourth order in the strong coupling $\alpha_s$. Assuming the value of 163.643 GeV for the top-quark $\overline{\text{MS}}$ mass $\overline{m}_t = m_t(\overline{m_t})$, and assuming $\alpha_s^{(6)}(m_t) = 0.1088$, we have [4]

$$m_P = 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \text{GeV} \tag{1.1}$$

for the series expansion of the mass conversion formula. The last term from the fourth order correction is less than one half of the third order one.

It is also known that the mass conversion formula is affected by infrared (IR) renormalons [5–7]. This means that there are factorially growing terms of infrared origin in the perturbative expansion, such that the expansion starts to diverge at some order. If the series is treated as an asymptotic expansion, the ambiguity in its resummation is of order of a typical hadronic scale. Because of this, it is often stated that the ultimate accuracy of top pole mass cannot be below a few hundred MeV. One of the goals of this work is to make this estimate more precise.

It is remarkable that the perturbative relation between the pole and $\overline{\text{MS}}$ mass of a heavy quark appears to be dominated by the leading infrared renormalon already in low orders [8,9]. This observation was used in previous work [10–12], and more recently

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in [14,15] to estimate the unknown normalization of the leading IR renormalon, and mostly applied in the context of bottom physics. In the context of top physics, the importance of this issue was raised recently in [16]. The purpose of this work is to combine the newly available four-loop coefficient [4] in the mass conversion formula with the known structure of the first infrared renormalon singularity [7] to determine the normalization constant and discuss its impact on top physics. We also perform an analysis of the dependence on the number of colours and flavours, which is by itself of interest, and stability tests with respect to variations of the scale of the strong coupling and $\overline{\text{MS}}$ mass. This leads to an expression for the mass conversion factor including an estimate of the contributions beyond four loops, and an estimate of the irreducible error.

## 2. Reminder

The renormalon divergence is a manifestation of the fact that the mass conversion formula, while infrared finite is sensitive to small loop momentum. In the case of the pole mass this sensitivity is particularly strong, namely linear, resulting in rapid divergence of the perturbative expansion, and an infrared sensitivity of order $\Lambda_{\text{QCD}}$ [5,6]. The ambiguity in defining the pole mass is therefore of similar size. This is not surprising as the pole mass of a quark is not an observable due to confinement and the difference with the physical heavy meson masses is also of order $\Lambda_{\text{QCD}}$. Unlike other heavy quarks, the top quark decays on hadronic time scales, and thus the propagator pole position acquires an imaginary part. The renormalon divergence is not altered [17] by the fact that the top quark is unstable with a width larger than $\Lambda_{\text{QCD}}$ and hence does not form bound states. The finite width simplifies the perturbative treatment of top quarks, since it provides a natural IR cut-off, and there exists no quantity for which the pole mass would ever be relevant. But the infrared sensitivity of the QCD corrections to the mass conversion factor, which causes the divergence, remains unaffected by the width.

Slightly more technically, the divergence arises from logarithmic enhancements of the loop integrand. Heuristically, this can be understood by noticing that the running coupling evaluated at the scale $l$ of the loop momentum has the expansion

$$\alpha_s(l) = \frac{1}{b_0 \ln l^2/\Lambda_{\text{QCD}}^2} = \frac{\alpha_s(m)}{1-\alpha_s(m) b_0 \ln m^2/l^2} = \sum_1^\infty \alpha_s^n(m)\, b_0^n \ln^n \frac{m^2}{l^2}\,. \tag{2.1}$$

The IR contribution to the last loop integration in the $(n+1)$-loop order then takes the form

$$\delta m^{(n+1)} \propto \alpha_s^{n+1}(m) \int^m \mathrm{d}l\, b_0^n \ln^n \frac{m^2}{l^2} = m\,(2b_0)^n\, \alpha_s^{n+1}(m)\, n!\,. \tag{2.2}$$

With this behaviour the series of mass corrections reaches a minimal term of order

$$\begin{aligned} m\,(2b_0)^n \alpha_s^{n+1} n! &\approx m\, \alpha_s\, n^{-n}\, (\sqrt{2\pi} n^{n+1/2} e^{-n}) \\ &\approx m \sqrt{\frac{\pi \alpha_s}{b_0}}\, \exp\left(-\frac{1}{2b_0\alpha_s}\right) \\ &\approx \sqrt{\frac{\pi\alpha_s}{b_0}}\, \Lambda_{\text{QCD}}, \end{aligned} \tag{2.3}$$

when $n \approx 1/(2b_0\alpha_s)$ and then diverges. Asymptotic expansions can sometimes be summed using the Borel transform. Given a power series

$$f(\alpha_s) = \sum_{n=1}^\infty c_n \alpha_s^n\,, \tag{2.4}$$

the corresponding Borel transform is defined by

$$B[f](t) = \sum_{n=0}^\infty c_{n+1} \frac{t^n}{n!}\,. \tag{2.5}$$

The Borel integral

$$\int_0^\infty dt\, e^{-t/\alpha_s}\, B[f](t) \tag{2.6}$$

has the same series expansion as $f(\alpha_s)$ and provides the exact result under suitable conditions. However, for the case of (2.2), where $c_{n+1} = (2b_0)^n n!$, the Borel integral

$$\int_0^\infty dt\, e^{-t/\alpha_s}\, \frac{1}{1-2b_0 t} \tag{2.7}$$

cannot be performed because of the pole at $t = 1/(2b_0)$. We can introduce some prescription for handling the pole in the integral, as, for example, the principal value prescription. Whether or not this reconstructs the exact result, an ambiguity remains, quantified by the imaginary part of the integral when going above or below the singular point. A commonly used procedure is to define this ambiguity to be equal to the imaginary part of the integral divided by Pi (see, e.g., [18], section 5.2). For (2.7), this yields

$$\Lambda_{\text{QCD}}/(2b_0)\,. \tag{2.8}$$

In the range of $\alpha_s$ values considered in this paper, the ambiguity is close to the size of the smallest term in (2.3).[1]

It can be shown [7] that while the precise asymptotic behaviour of the mass conversion formula differs from the simple ansatz employed in this section for illustration, as discussed below, the ambiguity is exactly proportional to $\Lambda_{\text{QCD}}$, which evaluates to about 250 MeV in the $\overline{\text{MS}}$ scheme. In the remainder of this work, we aim to quantify the proportionality factor.

## 3. The leading pole mass renormalon

We write the perturbative expansion of the mass conversion formula as

$$m_P = m(\mu_m)\left(1 + \sum_{n=1}^\infty c_n(\mu, \mu_m, m(\mu_m))\, \alpha_s^n(\mu)\right). \tag{3.1}$$

Here $\alpha_s(\mu)$ is the $\overline{\text{MS}}$ coupling in the $n_l$ light flavours theory, and $m(\mu_m)$ stands for the $\overline{\text{MS}}$ mass evaluated at the scale $\mu_m$. (In the following we will consider different scale choices for the heavy quark mass and the strong coupling. We also use $\overline{m}$ to denote the $\overline{\text{MS}}$ mass evaluated self-consistently at a scale equal to the mass itself, i.e.

$$\overline{m} = m(\overline{m}). \tag{3.2}$$

[1] Note, however, the different parametric dependence on $\alpha_s$ of (2.3) and (2.8). The correct dependence is that of (2.8), for the following reason: The typical width of the region where the minimal term is attained grows parametrically as $\sqrt{1/(2b_0\alpha_s)}$. The accuracy of an asymptotic series is better estimated by the minimal term times the factor accounting for the number of terms in this region, which makes (2.3) parametrically consistent with (2.8). Numerically, this factor turns out to be of order one for the applications considered in this paper, as will be confirmed in section 4 below. In case of doubt, the estimate from the ambiguity of the Borel integral should be the preferred choice.

**Table 1**
The values of $N$ obtained from the coefficients of the perturbative expansion up to the fourth order for several values of $n_l$. Three values of the renormalization scale are considered.

| $n_l$ | $c_1/c_1^{(as)}$ | $c_2/c_2^{(as)}$ | $c_3/c_3^{(as)}$ | $c_4/c_4^{(as)}$ | $\Delta_{34}$ |
|---|---|---|---|---|---|
| $\mu/\overline{m}=1$ | | | | | |
| −1000000 | 0.6953 | 0.9624 | 0.9349 | 0.9714 | 0.038 |
| −10 | 0.4744 | 0.7152 | 0.6898 | 0.7005 ± 0.0002 | 0.015 ± 0.000 |
| 0 | 0.4377 | 0.6357 | 0.6130 | 0.5977 ± 0.0006 | 0.025 ± 0.001 |
| 3 | 0.3954 | 0.6150 | 0.5723 | 0.5370 ± 0.0011 | 0.064 ± 0.002 |
| 4 | 0.3633 | 0.6120 | 0.5522 | 0.5056 ± 0.0015 | 0.088 ± 0.003 |
| 5 | 0.3143 | 0.6119 | 0.5244 | 0.4616 ± 0.0020 | 0.127 ± 0.004 |
| 6 | 0.2436 | 0.6089 | 0.4818 | 0.3942 ± 0.0028 | 0.200 ± 0.007 |
| 7 | 0.1474 | 0.5378 | 0.4084 | 0.2786 ± 0.0042 | 0.378 ± 0.015 |
| 8 | 0.0098 | 0.0379 | 0.2719 | 0.0564 ± 0.0068 | 1.312 ± 0.068 |
| 10 | 0.2684 | −0.0916 | −0.1108 | −1.7228 ± 0.0271 | 1.758 ± 0.004 |
| $\mu/\overline{m}=0.5$ | | | | | |
| −1000000 | 1.3907 | 1.3554 | 0.6952 | 1.0773 | 0.431 |
| −10 | 0.9487 | 0.9410 | 0.6701 | 0.7110 ± 0.0003 | 0.059 ± 0.000 |
| 0 | 0.8753 | 0.7907 | 0.6149 | 0.5807 ± 0.0012 | 0.057 ± 0.002 |
| 3 | 0.7908 | 0.7343 | 0.5659 | 0.5030 ± 0.0023 | 0.118 ± 0.005 |
| 4 | 0.7266 | 0.7159 | 0.5370 | 0.4631 ± 0.0030 | 0.148 ± 0.006 |
| 5 | 0.6286 | 0.6975 | 0.4943 | 0.4078 ± 0.0040 | 0.192 ± 0.010 |
| 6 | 0.4872 | 0.6704 | 0.4267 | 0.3243 ± 0.0056 | 0.273 ± 0.017 |
| 7 | 0.2948 | 0.5640 | 0.3117 | 0.1845 ± 0.0084 | 0.513 ± 0.043 |
| 8 | 0.0196 | 0.0370 | 0.1123 | −0.0768 ± 0.0135 | 10.676 ± 5.676 |
| 10 | 0.5367 | −0.0621 | −0.2877 | −2.1014 ± 0.0541 | 1.518 ± 0.011 |
| $\mu/\overline{m}=2$ | | | | | |
| −1000000 | 0.3477 | 0.6235 | 0.8631 | 0.9409 | 0.086 |
| −10 | 0.2372 | 0.4800 | 0.5883 | 0.6576 ± 0.0001 | 0.111 ± 0.000 |
| 0 | 0.2188 | 0.4380 | 0.5217 | 0.5698 ± 0.0003 | 0.088 ± 0.001 |
| 3 | 0.1977 | 0.4314 | 0.4947 | 0.5247 ± 0.0006 | 0.059 ± 0.001 |
| 4 | 0.1817 | 0.4330 | 0.4831 | 0.5026 ± 0.0007 | 0.040 ± 0.001 |
| 5 | 0.1572 | 0.4376 | 0.4681 | 0.4724 ± 0.0010 | 0.009 ± 0.002 |
| 6 | 0.1218 | 0.4413 | 0.4452 | 0.4262 ± 0.0014 | 0.044 ± 0.003 |
| 7 | 0.0737 | 0.3968 | 0.4038 | 0.3460 ± 0.0021 | 0.154 ± 0.006 |
| 8 | 0.0049 | 0.0286 | 0.3177 | 0.1877 ± 0.0034 | 0.515 ± 0.017 |
| 10 | 0.1342 | −0.0761 | −0.0083 | −1.1238 ± 0.0135 | 1.971 ± 0.000 |

The leading IR renormalon divergence implies the following large-$n$ behaviour of the perturbative coefficients [7] (and [18],[2] eq. (5.90))

$$c_n(\mu, \mu_m, m(\mu_m)) \underset{n\to\infty}{\longrightarrow} N c_n^{(as)}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)} \tilde{c}_n^{(as)}, \tag{3.3}$$

where

$$\tilde{c}_{n+1}^{(as)} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \times \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right). \tag{3.4}$$

It is remarkable that $b = b_1/(2b_0^2)$ and the $s_i$ coefficients of the sub-leading $\mathcal{O}(1/n^i)$ behaviour can all be given in terms of the coefficients of the beta-function [7]. The relevant expressions are collected in appendix Appendix A. We also note that the scale $\mu_m$ at which $m$ is evaluated does not appear explicitly on the right-hand side of (3.3) and hence is irrelevant in (3.1) as far as the large-$n$ behaviour is concerned. The dependence on the scale $\mu$ of the strong coupling is compensated by the factor $\mu$ in front of $\tilde{c}_{n+1}^{(as)}$ in (3.3). With these definitions the normalization $N$ is independent of $\mu$ and $\mu_m$. It cannot however be computed rigorously with present perturbative techniques in general, but in the limit of large negative or positive $n_l$ it assumes the value [5]

$$\lim_{|n_l|\to\infty} N = \frac{C_F}{\pi} \times e^{\frac{5}{6}}, \tag{3.5}$$

which equals 0.97656 for $n_c = 3$ ($C_F = 4/3$).

In the following we compare the exactly known low-order coefficients of the perturbative expansion in the mass conversion relation with their expected asymptotic behaviour. By definition (see (3.3)) the normalization $N$ is given by

$$N = \lim_{n\to\infty} \frac{c_n(\mu, \mu_m, m(\mu_m))}{c_n^{(as)}(\mu, m(\mu_m))}. \tag{3.6}$$

We now determine $N$ by evaluating the above expression for $n = 1, 2, 3, 4$, for which $c_n(\mu, \mu_m, m(\mu_m))$ is known. To this end the result of [4] for the four-loop coefficient has been expressed in terms of the strong coupling constant with $n_l$ flavours rather than $n_l + 1$, since the asymptotic expression refers to the $n_l$ massless flavour theory. We also use results from [19] for the $n_l$, $n_c$, $\mu$ and $\mu_m$ dependence of the four-loop coefficient. In addition to the ratio $c_n/c_n^{(as)}$ for $n$ from 1 to 4 we consider the relative difference between the $N$ estimates performed using the third and the fourth order coefficients, defined as

$$\Delta_{34} = 2\,\frac{|c_3/c_3^{(as)} - c_4/c_4^{(as)}|}{|c_3/c_3^{(as)} + c_4/c_4^{(as)}|}. \tag{3.7}$$

The value of $\Delta_{34}$ can be considered to be an estimate of how close is the third order coefficient to the asymptotic value. It is likely to be an overestimate of the deviation of the fourth order coefficient from the asymptotic formula and should not be taken as an error on the normalization $N$.

[2] The perturbative coefficients $r_n$ in this reference are related to those employed here by $r_n = c_{n+1}$. With this notation the number of loops contributing to $c_n$ is $n$.

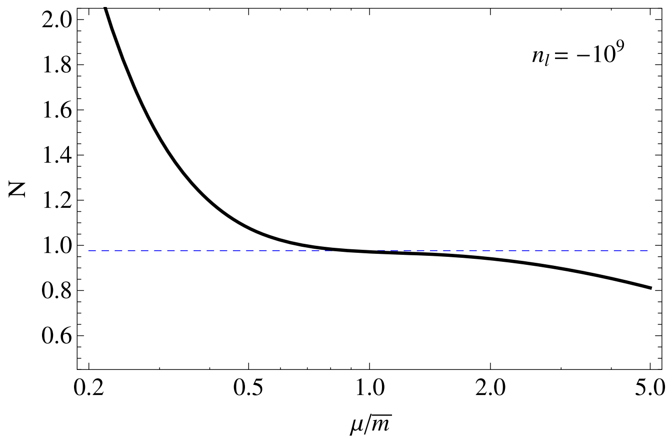


**Fig. 1.** The normalization $N$ as a function of $\mu/\overline{m}$ varied by a factor five around the central scale. The dashed line shows the exact value $4e^{5/6}/(3\pi) = 0.97656\ldots$.

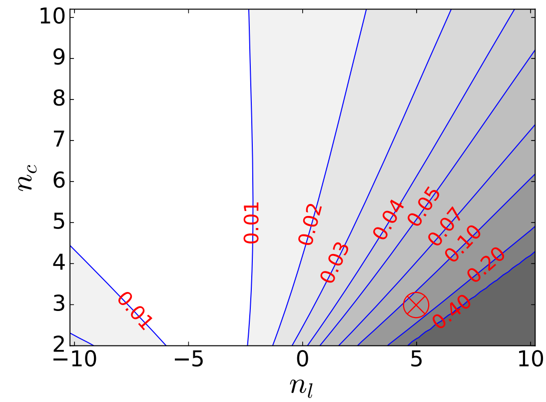


**Fig. 2.** $\Delta_{34}$ as a function of $n_c$ and $n_l$, for $\mu = \mu_m = \overline{m}$. The cross corresponds to the case relevant for top, i.e. $n_c = 3$ and $n_l = 5$.

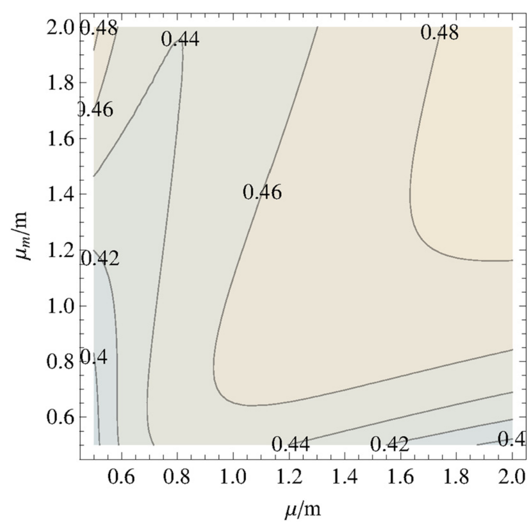


**Fig. 3.** The normalization $N$ as a function of $\mu/m(\mu_m)$ and $\mu_m/m(\mu_m)$.

We report our results in Table 1 for $\mu_m = \overline{m}$ and the three values $\mu = \overline{m}$, $\mu = \overline{m}/2$ and $\mu = 2\overline{m}$ of the coupling renormalization scale. The number of colours has been fixed to $n_c = 3$ in this table, and the number of light flavours was varied from a very large negative value (equivalent to the large-$n_l$ limit) up to $n_l = 10$. In columns 2 to 5 we show the ratios $c_n/c_n^{(\mathrm{as})}$, that correspond to an estimate of $N$ according to (3.6) for finite $n$. In the last column we give $\Delta_{34}$. The $\pm$ numbers account for the change in $N$ due to the numerical uncertainty in the calculation of the exact four-loop conversion coefficient, which is about 0.1% on the $n_l$ independent term for $\mu = \mu_m = m(\mu_m)$.

We first discuss the result for $\mu = \overline{m}$. For $n_l$ very large and negative the value of $N$ is close to the one predicted by (3.5). The value of $\Delta_{34}$ corresponds to a 4% deviation of the third order coefficient from the asymptotic result, which is indeed the case, and the fourth-order value is already much closer.[3] As $n_l$ increases, the value of $N$ decreases, reaching 0.506(2) and 0.462(2) for $n_l = 4$ and 5, respectively, with a 9 and 13% variation when going from the third to the fourth order coefficient. As $n_l$ increases, $\Delta_{34}$ also increases, so that for $n_l$ above 7 the $N$ values obtained from the third and fourth order coefficients differ by factors of order 1. This behaviour is not unexpected: by increasing the number of light flavours the first coefficient of the $\beta$ function, $b_0$, decreases (it vanishes for $n_l = 33/2$), hence the renormalon dominance is delayed to higher orders. We shall comment further on the $n_l$ dependence below.

When considering different choices of the renormalization scale, we see that the $\mu = \overline{m}/2$ case leads to larger variations than $\mu = 2\overline{m}$. The large $n_l$ limit yields a value that is about 10% higher than the exact result but the associated value $\Delta_{34} \approx 40\%$ is also large, indicating that the series is not as close to the asymptotic regime as for $\mu = \overline{m}$. For the interesting cases $n_l = 4$ and $n_l = 5$, $\Delta_{34}$ is also more than a factor of two larger than for $\mu = \overline{m}$. Again, this behaviour is not unexpected. The coefficients $c_n$ depend only on logarithms of $\mu/m$ up to the $(n-1)$th power. Eq. (3.3) shows that these logarithms must asymptotically exponentiate to $\mu/m$, which clearly happens less efficiently at finite order when $\ln(\mu/m)$ is larger. Hence we expect the best approximation to the asymptotic behaviour to occur when $\mu \approx \overline{m}$. Fig. 1 shows that this is indeed the case for large $-n_l$. It further shows a plateau around $\mu \approx \overline{m}$ and a more rapid departure from the exact result for $\mu$ smaller then $\overline{m}$ than for larger $\mu$, as also seen in Table 1.

We also determine the normalization $N$ for different values of $n_c$ and show the result for $\Delta_{34}$ in Fig. 2. We generically find $\Delta_{34} < 0.1$ except in regions where $b_0$ is small, where we do not expect our method to work. Fig. 2 therefore demonstrates that the exact four-loop coefficient indeed matches the asymptotic formula (3.1) in the expected range of $n_c$ and $n_l$ values, comprising those of physical interest.

For the following a reliable determination of $N$ and an estimate of its error is particularly important for $n_c = 3$, $n_l = 5$, corresponding to the case of the top quark. We determine the error by varying the two renormalization scales independently, that is we vary $\mu/m(\mu_m)$ and $\mu_m/m(\mu_m)$ independently between 0.5 and 2, compute $N$ from $c_4/c_4^{(\mathrm{as})}$ as above, and determine the error on $N$ from the maximal variation. The dependence of $N$ on the two scale ratios is shown in Fig. 3. With this definition our error estimate on $N$ neither depends on the value of the heavy quark mass nor the one of the strong coupling. We find

$$N = 0.4616^{+0.027}_{-0.070}\,(\mu \text{ and } \mu_m) \pm 0.002\,(c_4)\,. \tag{3.8}$$

[3] We may note that the contribution from sub-leading renormalon poles to $c_n$ is of order $1/2^n$ relative to the leading one, but there is a further suppression for the case at hand due to a small numerical coefficient, at least in the large-$n_l$ limit, see [18].

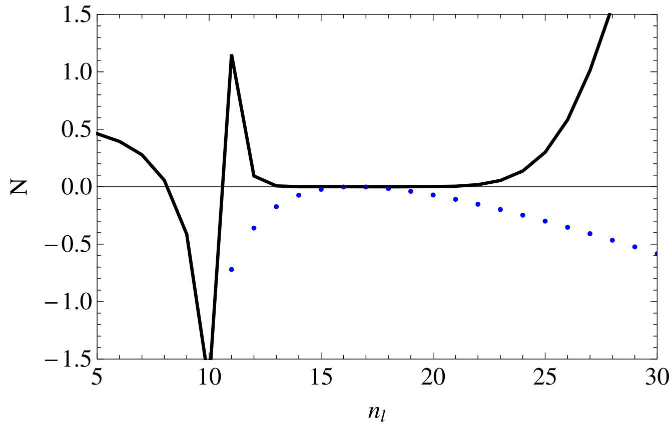


**Fig. 4.** The normalization $N$ (for $n_c=3$, $\mu=\mu_m=\overline{m}$) as a function of $n_l$. The dots show $1/b$.

As a further check we note that when the subleading term $s_2$ ($s_1$ and $s_2$) is removed in (3.4), the central value changes very little to 0.4573 (0.4584).[4]

A similar method to determine the normalization of the leading pole mass renormalon, albeit without variations of $\mu_m$ and $n_c$, has already been used in [14]. More precisely, instead of the four-loop pole mass considered here the three-loop static potential was employed to arrive at the best estimate, based on the fact that the pole mass and static potential leading renormalon normalizations are rigorously related by a factor of $-1/2$. Their values are indeed in good agreement with ours, though deteriorating with increasing $n_l$. The approach to the exact value for large negative $n_l$ was also observed in [14].

The authors of [14] also determined the normalization $N$ as a function of $n_l$ and noted that it tends to zero in the range $n_l = 12\ldots 23$ close to the conformal window. We confirm this behaviour in our analysis, see Fig. 4. To understand why the normalization of the leading renormalon is forced to be small in this $n_l$ region, we look at the explicit expression of for $c_n^{(\rm as)}$ from (3.3) for $n=4$,

$$c_4^{(\rm as)} = (2b_0)^3\,(1+b)(2+b)(3+b) \times\left(1+\frac{s_1}{3+b}+\frac{s_2}{(3+b)(2+b)}+\cdots\right). \tag{3.9}$$

The region $n_l = 12\ldots 23$ is approximately centred around the value of $n_l$, where $b_0$ vanishes, hence $b=b_1/(2b_0^2)$ becomes large. As soon as $b \gg n_0$, where $n_0$ is the order from which $N$ is determined (here $n_0=4$), the individual terms in the above expression behave as

$$c_4^{(\rm as)} = (2b_0)^{n_0}\left(\frac{b_1}{2b_0^2}\right)^{n_0}\left(1+\frac{s_1}{b}+\frac{s_2}{b^2}+\cdots\right) \sim \frac{1}{(2b_0)^{n_0}}\left(1+\frac{\#}{b_0^2}+\frac{\#}{b_0^4}+\cdots\right), \tag{3.10}$$

from which we conclude a) that $c_4^{(\rm as)} \sim 1/(2b_0)^{n_0}$ becomes very large, hence $N$ *must* become small to fit the given value of the exact four-loop coefficient $c_4$, and b) the series of sub-leading asymptotic terms $s_1$, $s_2$, etc. breaks down, hence the extracted value of $N$ is completely unreliable. The smallness of $N$ is therefore a technical artifact of the method, which ceases to be valid when $b$ becomes large compared to $n_0$, and the question whether $N$ is small in the conformal window cannot be answered. In fact, while small $b_0$ makes renormalon behaviour less relevant to low orders due to the diminished $(2b_0)^n$ factor, there seems to be no reason why the normalization $N$ should vanish when the theory becomes conformal non-perturbatively.

[4] Using the five-loop beta-function coefficient from [13], which appeared after this analysis was finished, allows us to compute the next sub-asymptotic term $s_3$ in (3.4) (see appendix). We find that $N$ changes by a negligible amount to 0.4606.

## 4. The $m_P$–$\overline{m}$ conversion factor to all orders and the ultimate top pole mass uncertainty

In the following we use two methods to estimate the remainder of the mass conversion relation beyond the exactly known four-loop accuracy and to estimate the intrinsic ambiguity of summing the assumed asymptotic expansion. The first relies on truncation of the expansion and an estimate of the minimal term. The second on Borel summation. We restrict ourselves to the case of the top quark mass ($n_c=3$, $n_l=5$) and choose $\mu=\mu_m=\overline{m}$.

We begin by writing

$$m_P(n) = \overline{m}\left(1+\sum_{k=1}^{n} c_k\alpha_s^k\right), \tag{4.1}$$

where the coefficients are the exact ones up to the fourth order in $\alpha_s$, and determined from the asymptotic formula (3.4) (with normalization fitted to the fourth order term) for the terms of order 5 and higher. We would like to define the best value of $m_P$ as the value at which its increment with $n$ is minimal. More precisely, we define

$$\Delta(n+1/2) = m_P(n+1) - m_P(n), \tag{4.2}$$

which is a decreasing function of $n$ up to a certain value $n_0$ beyond which it begins to increase due to the renormalon divergence of the series expansion. By interpolating $\Delta$ with a quadratic form in the three points $n_0-1/2$, $n_0+1/2$, $n_0+3/2$, we find its minimum at (generally non-integer)

$$n_{\rm min} = n_0+1/2 - \frac{\Delta(n_0+3/2)-\Delta(n_0-1/2)}{2(\Delta(n_0+3/2)+\Delta(n_0-1/2)-2\Delta(n_0+1/2))}. \tag{4.3}$$

By interpolating linearly the value of $m_P(n_{\rm min})$ between $n_0$ and $n_0+1$ we get

$$m_P^{\rm c} = \frac{m_P(n_0)(\Delta(n_0+3/2)-\Delta(n_0+1/2))}{\Delta(n_0+3/2)+\Delta(n_0-1/2)-2\Delta(n_0+1/2)} + \frac{m_P(n_0+1)(\Delta(n_0-1/2)-\Delta(n_0+1/2))}{\Delta(n_0+3/2)+\Delta(n_0-1/2)-2\Delta(n_0+1/2)} \tag{4.4}$$

as the best value of the pole mass. We note that with this prescription, if $\Delta(n_0-1/2)=\Delta(n_0+3/2)$, then $m_P^{\rm c}$ corresponds to $(m_P(n_0)+m_P(n_0+1))/2$, as one would intuitively expect, while for $\Delta(n_0-1/2) \gg \Delta(n_0+3/2)$ ($\Delta(n_0-1/2) \ll \Delta(n_0+3/2)$), we obtain $m_P(n_0+1)$ ($m_P(n_0)$).

We now estimate the correction to the top pole mass due to terms of order higher than four by

$$\delta^{(5+)}m_P = N\mu\,\overline{\sum_{k=5}}\,\tilde{c}_k^{(\rm as)}\alpha_s^k(\mu), \tag{4.5}$$

where $\tilde{c}_j^{(\rm as)}$ is defined in (3.4), and the barred sum represents the procedure we have just outlined for the evaluation of the (divergent) sum. We report in Table 2 the values of $\tilde{c}_j^{(\rm as)}$ beyond the fourth order term. Eq. (4.5) can be easily computed for any value of $\alpha_s$ and $\mu$ and is well approximated by the second-order Taylor series around the reference value:

**Table 2**
The coefficients $\tilde{c}_j^{(\mathrm{as})}$ above the fourth order. Their value multiplied by the corresponding power of $\alpha_s = 0.108531$ is also reported.

| $j$ | $\tilde{c}_j^{(\mathrm{as})}$ | $\tilde{c}_j^{(\mathrm{as})}\alpha_s^j$ |
|---|---|---|
| 5 | $0.985499 \times 10^2$ | 0.001484 |
| 6 | $0.641788 \times 10^3$ | 0.001049 |
| 7 | $0.495994 \times 10^4$ | 0.000880 |
| 8 | $0.443735 \times 10^5$ | 0.000854 |
| 9 | $0.451072 \times 10^6$ | 0.000942 |
| 10 | $0.513535 \times 10^7$ | 0.001164 |
| 11 | $0.647283 \times 10^8$ | 0.001593 |
| 12 | $0.894824 \times 10^9$ | 0.002390 |
| 13 | $0.134620 \times 10^{11}$ | 0.003902 |
| 14 | $0.218949 \times 10^{12}$ | 0.006888 |
| 15 | $0.382818 \times 10^{13}$ | 0.013070 |

$$\delta^{(5+)}m_P = N\mu \times 10^{-3}\left(3.604 + 14.69\left(\frac{\alpha_s(\mu)}{0.1085} - 1\right) + 9.54\left(\frac{\alpha_s(\mu)}{0.1085} - 1\right)^2\right). \tag{4.6}$$

For typical values of $N \approx 0.5$ and $\mu \approx 160$ GeV the formula is accurate at the sub-MeV level for a $\pm 5\%$ variation of the strong coupling constant.

We now adopt the PDG value $\alpha_s(M_Z) = 0.1181 \pm 0.0013$, and take $\mu = \overline{m} = 163.508$ GeV for definiteness. With this input we find $\alpha_s(\mu) = 0.108531$ for the (five flavour) strong coupling constant and 173.34 GeV for the top pole mass using the four-loop conversion formula. From the values reported in the table and the value of $N$ given in (3.8) we obtain for the series remainder

$$\delta^{(5+)}m_P = 0.272^{+0.016}_{-0.041}\,(N) \pm 0.001\,(c_4) \pm 0.011\,(\alpha_s) \pm 0.066\,(\text{ambiguity})\,\text{GeV}\,, \tag{4.7}$$

where we show the error due to the uncertainty in the normalization $N$, the four-loop coefficient $c_4$, and $\alpha_s(M_Z)$. For the irreducible renormalon ambiguity we tentatively estimate the size of the first omitted term by the value of $\Delta(n_0 - 1/2)$. For the top mass conversion factor we find

$$m_P^c/\overline{m} = 1.06177^{+0.00010}_{-0.00025}\,(N) \pm 0.00001\,(c_4) \pm 0.00087\,(\alpha_s) \pm 0.00041\,(\text{ambiguity})\,. \tag{4.8}$$

We also computed the change of the conversion factor under variations of $\mu/\overline{m}$ and $\mu_m/\overline{m}$, simultaneously in the exact four-loop part and the remainder, accounting for the dependence of $N$ on $\mu$ and $\mu_m$ (Fig. 3). This leads to $^{+0.00025}_{-0.00041}$, which we do not include above, since it is strongly correlated with the uncertainty of the same order from $N$ alone.

In the second method we first compute the Borel transform of the asymptotic series coefficients $\tilde{c}^{(\mathrm{as})}$ in (3.3), which gives

$$B[\tilde{c}^{(\mathrm{as})}](t) = \frac{1}{(1-2b_0t)^{1+b}} + \frac{s_1}{b}\,\frac{1}{(1-2b_0t)^b} + \frac{s_2}{b(b-1)}\,\frac{1}{(1-2b_0t)^{-1+b}} + \ldots, \tag{4.9}$$

and then the Borel sum

$$BS[\tilde{c}^{(\mathrm{as})}](\alpha_s) = \int_0^\infty dt\, e^{-t/\alpha_s}\, B[\tilde{c}^{(\mathrm{as})}](t)\,. \tag{4.10}$$

Since the series is not Borel-summable due to the IR renormalon singularity at $t = 1/(2b_0)$, we define the sum as the principal value and estimate the ambiguity as the imaginary part of the integral when the contour is deformed into the upper complex plane, divided by Pi. This procedure is known to usually give a reliable estimate [18], close to the sum to the minimal term and the estimate of the summation ambiguity by the smallest term in the series. The Borel sum can easily be computed analytically, since (with the contour deformed into the upper complex plane)

$$\int_0^\infty dt\, e^{-t/\alpha_s}\,\frac{1}{(1-2b_0t)^\gamma} = \frac{\alpha_s}{(-2b_0\alpha_s)^\gamma}\, e^{-1/(2b_0\alpha_s)}\,\Gamma(1-\gamma, -1/(2b_0\alpha_s))\,, \tag{4.11}$$

where $\Gamma(a, z)$ denotes the incomplete Gamma function. The remainder of the mass conversion formula is obtained by subtracting the first four coefficients, resulting in

$$\delta^{(5+)}m_P = N\mu\left(BS[\tilde{c}^{(\mathrm{as})}](\alpha_s(\mu)) - \sum_{k=1}^{4}\tilde{c}_k^{(\mathrm{as})}\alpha_s(\mu)^k\right). \tag{4.12}$$

With parameter input as above, we find

$$\delta^{(5+)}m_P = 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s) \pm 0.071\,(\text{ambiguity})\,\text{GeV}\,, \tag{4.13}$$

which is close to the result (4.7) from the previous method. For any value of $\alpha_s$ and $\mu$ the result can again be determined accurately in the phenomenologically relevant region according to the fit formula

$$\delta^{(5+)}m_P = N\mu \times 10^{-3}\left(3.315 + 12.71\left(\frac{\alpha_s(\mu)}{0.1085} - 1\right) + 4.55\left(\frac{\alpha_s(\mu)}{0.1085} - 1\right)^2\right). \tag{4.14}$$

For the top mass conversion factor itself, we find

$$m_P^c/\overline{m} = 1.06164^{+0.00009}_{-0.00023}\,(N) \pm 0.00001\,(c_4) \pm 0.00086\,(\alpha_s) \pm 0.00043\,(\text{ambiguity}). \tag{4.15}$$

In this case, the scale variation is $^{+0.00013}_{-0.00028}$.

The ultimate uncertainty on the top quark pole mass, which we identify with the ambiguity of about 70 MeV, is smaller than estimates from the large-$n_l$ limit, because the normalization $N$ is smaller. We also note that dividing the imaginary part of the Borel integral by Pi to obtain the ambiguity is a convention that has proven reliable in contexts where the quantity in question is amenable of a non-perturbative definition [18]. This is not the case for the pole mass, so that we cannot ask how well the divergent series approximates the exact, non-perturbative result. The point is rather that the pole mass can in principle be used as a reasonable perturbative reference parameter, as long as computing additional orders does not require increasingly larger shifts in the reference value. The dividing-by-Pi convention therefore appears reasonable, since, if the imaginary part of the Borel transform was instead used to estimate the ambiguity, it would be almost as large as the known four-loop term, where the series is clearly still in the regime of decreasing terms. We observe that, in any case, even if the ambiguity were taken to be the imaginary part of the Borel integral itself, the resulting estimate of would still be significantly below the uncertainty that can conceivably be achieved at hadron colliders.

## 5. Internal bottom and charm mass effect

The analysis assumed up to now that the five lighter quarks are massless. Since the typical loop momentum at order $\alpha_s^{n+1}$ is of order $m_t e^{-n}$ in the regime where the series is dominated by the leading renormalon divergence, we expect internal quark mass effects from the bottom and charm quark to become more important in higher orders. Furthermore, the minimal term is attained when the typical loop momentum is of order $\Lambda_{\rm QCD}$, hence the ambiguity should be determined by $\Lambda$-parameter $\Lambda_{\rm QCD}^{(3)}$ in the three-flavour scheme, excluding the bottom and charm quark. In this section we estimate the effect of the finite bottom and charm quark mass on the top mass conversion factor and the ultimate uncertainty.

The decoupling of internal quark loops from quarks with masses $m_q \gg \Lambda_{\rm QCD}$ in the renormalon asymptotic behaviour was studied analytically and numerically in the large-$n_l$ limit [9]. The analysis showed that the asymptotic behaviour of the series in a theory with $n_l$ quarks of which $n_m$ are massive, approaches the series of the theory with $n_l - n_m$ massless quarks when both are expressed in terms of the $\overline{\rm MS}$ coupling $\alpha_s^{(n_l-n_m)}(m_t)$ in the $n_l - n_m$ flavour scheme.[5] Based on this observation it has been argued [14] that the bottom mass conversion factor should be expressed in terms of $\alpha_s^{(3)}(m_b)$ rather than the four-flavour coupling $\alpha_s^{(4)}(m_b)$. For the two- and three-loop coefficients, for which the mass dependence is known [20,21], it was shown that this substitution indeed renders the charm mass effect almost negligible.

This procedure does not work for top, however, since the masses of the bottom and charm quark are too small in relation to $m_t$ to express the *entire* series in terms of the four- or three-flavour coupling. Instead, we switch from the five- to the four-flavour scheme at the order, where the typical internal loop momentum is of order $m_b$, which is $\mathcal{O}(\alpha_s^5)$, and from the four- to the three-flavour scheme at $\mathcal{O}(\alpha_s^6)$. Since the mass effect is not known for $c_4$ at the four-loop order, and since $c_n$ beyond the four-loop order can only be estimated assuming dominance of the first renormalon (as done above), this implies the following procedure: (a) at two- and three-loops we include the known mass dependence, but $c_4$ is approximated by the massless value. For given top $\overline{\rm MS}$ mass, this increases the top pole mass by 11 (2-loop) + 16 (3-loop) MeV, adopting $\overline{m}_b = 4.2$ GeV and $\overline{m}_c = 1.3$ GeV. Since the $c_n$ increase as $n_l$ decreases, the mass effect is also expected to be positive in higher orders. Hence approximating $c_4$ by its massless value *underestimates* the mass effect. (b) At five-loop, we use $c_5^{(\rm as)}[\alpha_s^{(4)}(m_t)]^5$ with $c_5^{(\rm as)}$ determined as described in sect. 3, but with the normalization $N_m = 0.5056$ and beta-function coefficients for the four-flavour theory, $n_l = 4$. (c) Beyond five loops, the remainder and the ambiguity is calculated according to (4.12) (with obvious modification, since we sum the terms from six rather than five loops), but with the three-flavour scheme coupling $\alpha_s^{(3)}(m_t)$ and normalization $N_m = 0.5370$. Since the bottom and charm quarks are not yet completely decoupled at the five- to seven-loop order, and since an extra quark flavour decreases the $c_n$, we expect that (b) and (c) *overestimate* the mass effect, since the approximation assumes that bottom and charm are already decoupled completely. The sum of (b) and (c) adds another 53 MeV to the top pole mass, such that the total mass effect is estimated to be 80 MeV. Since the bottom is neither heavy enough to be decoupled in low orders, nor light enough to be ignored, where in both cases a massless approximation can be justified, there is an inherent uncertainty in the above estimate. However, as argued above, the errors in the approximations are expected to go in opposite directions, hence we consider $(80 \pm 30)$ MeV a conservative estimate of the internal bottom and charm quark mass effect on the top pole mass. The 30 MeV error estimate arises from an estimate of the neglected mass effect on $c_4$ by extrapolation from the known lower orders. We have also checked that the approximation described here works well in models for the series inspired by the large-$n_l$ limit.

Including the internal mass effect into the massless results (4.13) and (4.15), we obtain for the series remainder from the five-loop order

$$\delta^{(5+)} m_P = 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \pm 0.009\,(\alpha_s) \pm 0.108\,(\text{ambiguity})\,\text{GeV}\,, \tag{5.1}$$

where we now dropped the negligible uncertainty from the massless four-loop coefficient $c_4$. Apart from the shift of the value of $\delta^{(5+)} m_P$ the ambiguity has increased to 108 MeV, which is mainly due to the fact that $\Lambda_{\rm QCD}^{(3)}$ is larger than $\Lambda_{\rm QCD}^{(5)}$. Note that the ambiguity is independent of the precise value of the bottom and charm mass, as long as $m_b, m_c \gg \Lambda_{\rm QCD}$. This also implies that it is the same for any heavy quark, including the bottom quark, since it depends only on the infrared properties of the theory, which is QCD with three approximately massless flavours.

For the top mass conversion factor itself, we find

$$m_P^c/\overline{m} = 1.06213^{+0.00007}_{-0.00038}\,(N) \pm 0.00018\,(m_{b,c}) \pm 0.00086\,(\alpha_s) \pm 0.00066\,(\text{ambiguity}). \tag{5.2}$$

The scale variation remains as for (4.15). We adopt (5.1) and (5.2) as our final results. Given the $\overline{\rm MS}$ mass, the top quark pole mass is determined by this relation with an accuracy of 1.1 per mil, half of which is due to the irreducible uncertainty of the relation itself.

## 6. Conclusions

We employed the four-loop coefficient in the pole-$\overline{\rm MS}$ quark mass relation, which has recently become available [4], and knowledge of the leading asymptotic behaviour of the series expansion of the mass conversion factor [7] to estimate the remainder of the series from terms above the four-loop order and the intrinsic ambiguity due to the asymptotic nature of the series. For the case of the top quark we find about 300 MeV for the former, including an estimate of the effect of the internal bottom and charm quark mass, and 110 MeV for the ambiguity, which also represents the ultimate precision that can be obtained for the pole mass. The ambiguity of 110 MeV is far below the accuracy that can conceivably be achieved at the Large Hadron Collider, but larger than the one foreseen in theoretical and experimental studies [22,23] of a scan of the top pair production threshold at a high-energy $e^+e^-$ collider. In this case the pole mass ceases to be a useful concept and other mass definitions must be employed.


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[5] Note that (4.14) in [9] does not apply term by term, but only as a transformation of the entire series. Term by term the approximation holds, if the right-hand side of (4.14) is multiplied by the factor

$$\exp\left(\frac{1}{12\pi\beta_0^{(3)}}\ln\frac{m_b^2}{m_c^2}\right) \Big/ \left(1 - \frac{\alpha_s^{(3)}}{6\pi}\ln\frac{m_b^2}{m_c^2}\right)^{n+1},$$

which follows from (4.16) in [9]. Here we put $\beta_0^{(3)}$ into the exponent rather than $\beta_0^{(4)}$ as in (4.16), since in the presence of a massive quark, the leading singularity is slightly shifted to $u = 1/2 \times \beta_0^{(4)}/\beta_0^{(3)}$ when $u$ is defined as $-\beta_0^{(4)} t$.

## Appendix A. Summary of formulae

In this Appendix, in order to make contact with the notation of [7,18], we define the QCD beta-function as

$$\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s(\mu)}{\partial \mu^2} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \ldots . \tag{A.1}$$

With this convention $\beta_0 = -(11n_c/3 - 2n_l/3)/(4\pi)$, while in the main text we used $b_i = -\beta_i > 0$ (for small $n_l$). We adopt the $\overline{\text{MS}}$ scheme with $n_l$ massless quark flavours. (The heavy quark whose mass is considered here is decoupled.) The constants that appear in (3.4) are given by [7,18] $b = -\beta_1/(2\beta_0^2)$ and

$$s_1 = \left(-\frac{1}{2\beta_0}\right)\left(-\frac{\beta_1^2}{2\beta_0^3} + \frac{\beta_2}{2\beta_0^2}\right), \tag{A.2}$$

$$s_2 = \left(-\frac{1}{2\beta_0}\right)^2 \left(\frac{\beta_1^4}{8\beta_0^6} + \frac{\beta_1^3}{4\beta_0^4} - \frac{\beta_1^2\beta_2}{4\beta_0^5} - \frac{\beta_1\beta_2}{2\beta_0^3} + \frac{\beta_2^2}{8\beta_0^4} + \frac{\beta_3}{4\beta_0^2}\right), \tag{A.3}$$

$$\begin{aligned} s_3 = \left(-\frac{1}{2\beta_0}\right)^3 \Bigg( & -\frac{\beta_1^6}{48\beta_0^9} - \frac{\beta_1^5}{8\beta_0^7} - \frac{\beta_1^4}{6\beta_0^5} \\ & + \frac{\beta_1^4\beta_2}{16\beta_0^8} + \frac{3\beta_1^3\beta_2}{8\beta_0^6} + \frac{\beta_1^2\beta_2}{2\beta_0^4} - \frac{\beta_1^2\beta_2^2}{16\beta_0^7} \\ & - \frac{\beta_1^2\beta_3}{8\beta_0^5} - \frac{\beta_1\beta_2^2}{4\beta_0^5} - \frac{\beta_1\beta_3}{3\beta_0^3} + \frac{\beta_2^3}{48\beta_0^6} \\ & - \frac{\beta_2^2}{6\beta_0^3} + \frac{\beta_2\beta_3}{8\beta_0^4} + \frac{\beta_4}{6\beta_0^2}\Bigg). \end{aligned} \tag{A.4}$$

Note that we have corrected some misprints in the expression for $b$ and $s_2$ given in [18] (eqs. (5.91) and (5.92)) as already noted in [10]. The result for $s_3$ was not given explicitly in [18].

## References


[1] Particle Data Group Collaboration, K.A. Olive, et al., Review of Particle Physics, Chin. Phys. C 38 (2014) 090001.
[2] ATLAS, CDF, CMS, D0 collaboration, First combination of Tevatron and LHC measurements of the top-quark mass, arXiv:1403.4427.
[3] D. Buttazzo, G. Degrassi, P.P. Giardino, G.F. Giudice, F. Sala, A. Salvio, A. Strumia, Investigating the near-criticality of the Higgs boson, J. High Energy Phys. 1312 (2013) 089, arXiv:1307.3536.
[4] P. Marquard, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, Quark mass relations to four-loop order in perturbative QCD, Phys. Rev. Lett. 114 (2015) 142002, arXiv:1502.01030.
[5] M. Beneke, V.M. Braun, Heavy quark effective theory beyond perturbation theory: renormalons, the pole mass and the residual mass term, Nucl. Phys. B 426 (1994) 301–343, arXiv:hep-ph/9402364.
[6] I.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev, A.I. Vainshtein, The Pole mass of the heavy quark. Perturbation theory and beyond, Phys. Rev. D 50 (1994) 2234–2246, arXiv:hep-ph/9402360.
[7] M. Beneke, More on ambiguities in the pole mass, Phys. Lett. B 344 (1995) 341–347, arXiv:hep-ph/9408380.
[8] M. Beneke, V.M. Braun, Naive nonabelianization and resummation of fermion bubble chains, Phys. Lett. B 348 (1995) 513–520, arXiv:hep-ph/9411229.
[9] P. Ball, M. Beneke, V.M. Braun, Resummation of $(\beta_0\alpha_s)^n$ corrections in QCD: techniques and applications to the tau hadronic width and the heavy quark pole mass, Nucl. Phys. B 452 (1995) 563–625, arXiv:hep-ph/9502300.
[10] A. Pineda, Determination of the bottom quark mass from the Upsilon(1S) system, J. High Energy Phys. 06 (2001) 022, arXiv:hep-ph/0105008.
[11] T. Lee, Heavy quark mass determination from the quarkonium ground state energy: a pole mass approach, J. High Energy Phys. 10 (2003) 044, arXiv:hep-ph/0304185.
[12] A.H. Hoang, A. Jain, I. Scimemi, I.W. Stewart, Infrared renormalization group flow for heavy quark masses, Phys. Rev. Lett. 101 (2008) 151602, arXiv:0803.4214.
[13] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, Five-loop running of the QCD coupling constant, Phys. Rev. Lett. 118 (2017) 082002, arXiv:1606.08659.
[14] C. Ayala, G. Cvetič, A. Pineda, The bottom quark mass from the $\Upsilon(1S)$ system at NNNLO, J. High Energy Phys. 09 (2014) 045, arXiv:1407.2128.
[15] T. Lee, Flavor dependence of normalization constant for an infrared renormalon, Phys. Lett. B 742 (2015) 327–329, arXiv:1502.02698.
[16] P. Nason, Theory Summary, in: 8th International Workshop on Top Quark Physics (TOP2015), 14–18 September 2015 Ischia, Italy, PoS TOP (2015) 056, arXiv:1602.00443.
[17] M.C. Smith, S.S. Willenbrock, Top quark pole mass, Phys. Rev. Lett. 79 (1997) 3825–3828, arXiv:hep-ph/9612329.
[18] M. Beneke, Renormalons, Phys. Rep. 317 (1999) 1–142, arXiv:hep-ph/9807443.
[19] P. Marquard, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, D. Wellmann, $\overline{MS}$-on-shell quark mass relation up to four loops in QCD and a general SU(N) gauge group, Phys. Rev. D 94 (2016) 074025, arXiv:1606.06754.
[20] N. Gray, D.J. Broadhurst, W. Grafe, K. Schilcher, Three loop relation of quark (modified) Ms and pole masses, Z. Phys. C 48 (1990) 673–680.
[21] S. Bekavac, A. Grozin, D. Seidel, M. Steinhauser, Light quark mass effects in the on-shell renormalization constants, J. High Energy Phys. 0710 (2007) 006, arXiv:0708.1729.
[22] M. Beneke, Y. Kiyo, P. Marquard, A. Penin, J. Piclum, M. Steinhauser, Next-to-next-to-next-to-leading order QCD prediction for the top antitop $S$-wave pair production cross section near threshold in $e^+e^-$ annihilation, Phys. Rev. Lett. 115 (2015) 192001, arXiv:1506.06864.
[23] F. Simon, A first look at the impact of NNNLO theory uncertainties on top mass measurements at the ILC, in: International Workshop on Future Linear Colliders (LCWS15), Whistler, B.C., Canada, November 2–6, 2015, 2015, arXiv:1603.04764.