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Solutions of ordinary differential equations in closed subsets of a Banach space

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Dedicated to Professor Karol Baron on his 70th Birthday

1. Notations and main result. Let E be a real Banach space. For $x \in E$ and $\emptyset \neq B \subseteq E$ we write

$$\operatorname{dist}(x, B) = \inf\{\|x - y\| \mid y \in B\}.$$

For bounded sets $B \subseteq E$ we define

$$\operatorname{diam} B = \sup\{\|x - y\| \mid x, y \in B\}$$

($\operatorname{diam} \emptyset = 0$) and furthermore

$$\alpha(B) = \inf\{\delta \geq 0 \mid B \subseteq B_1 \cup \ldots \cup B_n,\ \operatorname{diam} B_\nu \leq \delta\ (\nu = 1, \ldots, n),\ n \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$; $\alpha(B)$ is the Kuratowski measure of non-compactness of B (cf. Kuratowski [1]).
Finally we use the notation

$$[x, y]_- = \lim_{h \nearrow 0} \frac{1}{h}\{\|x + hy\| - \|x\|\} \quad (x, y \in E).$$

Theorem 1 *Suppose $T > 0$ and $f = g + k$, where $g, k : [0, T] \times E \to E$ are continuous and bounded functions, g satisfying the one-sided Lipschitz condition*

$$[x - y, g(t, x) - g(t, y)]_- \leq L\|x - y\| \quad (0 \leq t \leq T;\ x, y \in E)$$

and k the α-Lipschitz condition

$$\alpha(k([0, T] \times B)) \leq K\alpha(B) \quad (B \subseteq E,\ B \text{ bounded});$$

here K, L are given non-negative numbers.
Moreover let M be a closed subset of the Banach space E such that

$$(1) \qquad \liminf_{h \searrow 0} \frac{1}{h} \operatorname{dist}(x + hf(t, x), M) = 0 \quad (0 \leq t \leq T,\ x \in M).$$

Then for every $(\tau, a) \in [0, T[\times M$ the initial value problem (i.v.p.)

$$(2) \qquad u(\tau) = a, \qquad u'(t) = f(t, u(t)) \quad (\tau \leq t \leq T)$$

has a solution $u : [\tau, T] \to M$.

Proof. Without loss of generality (w.l.o.g.) we assume $L > 0$. We choose $l \in]0, T-\tau]$ according to

$$\frac{1}{L}(e^{Ll}-1) \leq \frac{1}{4(2K+1)}. \tag{3}$$

The function $f : [0,T] \times E \to E$ being continuous, bounded and having property (1), there exist C^1-functions $u_n : [\tau, \tau+l] \to E$ $(n \in \mathbb{N})$ such that

$$u_n(\tau) = a, \quad \|u_n'(t) - f(t, u_n(t))\| \leq \frac{1}{n} \quad (\tau \leq t \leq \tau + l),$$

$$\operatorname{dist}(u_n(t), M) \leq \frac{1}{n} \quad (\tau \leq t \leq \tau + l); \tag{4}$$

the proof of this will be given in the next section.

Schmidt's proof of her Satz 2.3 in [3] shows that a subsequence of $(u_n)_{n\in\mathbb{N}}$ uniformly converges to a solution $u : [\tau, \tau+l] \to E$ of the i.v.p. given by (2). As a consequence of (4) we also get $u : [\tau, \tau+l] \to M$.

If $\tau + l = T$, then we are done. If $\tau + l < T$, then it is sufficient to repeat the foregoing reasoning finitely many times in a standard way.

Remark. Schmidt's form of inequality (3) is a bit different, but in her Satz 2.3 she rather uses the Hausdorff measure of non-compactness instead of α. The here given inequality (3) is appropriate for applying Schmidt's results to our case.

2. Existence of approximate C^1-solutions. The existence of the functions u_n $(n \in \mathbb{N})$ in the proof of Theorem 1 is a consequence of the following Theorem 2.

Theorem 2 *Let $f : [\tau, T] \times E \to E$ be a bounded continuous function, where E is a Banach space and τ, T are reals, $\tau < T$. Let M be a closed subset of E and suppose*

$$\liminf_{h \searrow 0} \frac{1}{h} \operatorname{dist}(x + hf(t,x), M) = 0 \quad (\tau \leq t \leq T,\ x \in M).$$

Finally let $a \in M$ and $\varepsilon > 0$ be given. Then there exists a C^1-function $u : [\tau, T] \to E$ such that

$$u(\tau) = a,\ \|u'(t) - f(t, u(t))\| \leq \varepsilon,\ \operatorname{dist}(u(t), M) \leq \varepsilon\ (\tau \leq t \leq T). \tag{5}$$

Proof. 1. W.l.o.g. we suppose $\varepsilon \leq 1$. According to Martin [2] there is a polygonal line $p : [\tau, T] \to E$ satisfying

$$p(\tau) = a,\ \|p_\pm'(t) - f(t, p(t))\| \leq \frac{\varepsilon}{4},\ \operatorname{dist}(p(t), M) \leq \frac{\varepsilon}{4}\ (\tau \leq t \leq T), \tag{6}$$

where p'_+, p'_- mean right- and left-hand derivatives (with the natural exceptions of $p'_-(\tau), p'_+(T)$).

2. Let (s,c) be a corner of p, where $\tau < s < T,\ c = p(s)$. Then in a small interval $]s-\beta, s+\beta[$ (where $\beta > 0$) there is no second corner of p. We shall show that for sufficiently small positive $\eta < \frac{\beta}{2}$ there is a C^1-function $u :]s-\beta, s+\beta[\to E$ which coincides with p on $]s-\beta, s-\eta] \cup [s+\eta, s+\beta[$ and fulfils the inequalities

$$(7) \quad \|u'(t) - f(t,u(t))\| \le \varepsilon,\ \operatorname{dist}(u(t), M) \le \varepsilon\ (s-\eta \le t \le s+\eta).$$

When changing p in a neighborhood of every corner (s,c) according to the just given description into a C^1-function u, then we get a C^1-function $u : [\tau, T] \to E$ satisfying (5).

3. Now we like to prove the statements of the preceding paragraph. So let (s,c) be a corner of p, w.l.o.g. we assume $s = 0$. Then in a small interval $[-\beta, \beta]$ the function p has the form

$$p(t) = \begin{cases} c + tb_1 & (0 \le t \le \beta) \\ c + tb_2 & (-\beta \le t \le 0), \end{cases}$$

where $\beta > 0$ and $b_1, b_2 \in E$. With $v = \frac{1}{2}(b_1 + b_2), w = \frac{1}{2}(b_1 - b_2)$ we get

$$\begin{aligned} p(t) &= \begin{cases} c + tv + tw & (0 \le t \le \beta) \\ c + tv - tw & (-\beta \le t \le 0), \end{cases} \\ p'(t) &= \begin{cases} v + w & (0 < t < \beta) \\ v - w & (-\beta < t < 0). \end{cases} \end{aligned}$$

Therefore (6) leads to

$$\begin{aligned} \|v + w - f(t, c + tv + tw)\| &\le \frac{\varepsilon}{4} \quad (0 \le t \le \beta), \\ \|v - w - f(t, c + tv - tw)\| &\le \frac{\varepsilon}{4} \quad (-\beta \le t \le 0). \end{aligned}$$

Using these inequalities for $t = 0$ implies

$$2\|w\| = \|v + w - f(0,c) - (v - w - f(0,c))\| \le \frac{\varepsilon}{2},$$

hence

$$\|w\| \le \frac{\varepsilon}{4}, \quad \|v - f(0,c)\| \le \frac{\varepsilon}{2}.$$

The continuity of f at $(0,c)$ shows the existence of an $\eta \in]0, \min\{1, \frac{\beta}{2}\}[$ such that

$$(t,x) \in \mathbb{R} \times E,\ |t| \le \eta,\ \|x\| \le \eta \Rightarrow \|f(t, c + tv + x) - f(0,c)\| \le \frac{\varepsilon}{4}.$$

Hence we have

(8) $$|t| \le \eta,\ \|x\| \le \eta \Rightarrow \|f(t, c + tv + x) - v\| \le \frac{3\varepsilon}{4}.$$

We define $u :]-\beta, \beta[\to E$ by

$$u(t) = \begin{cases} c + tv + tw & (\eta \le t < \beta) \\ c + \left(\frac{1}{2\eta}t^2 + \frac{\eta}{2}\right) w + tv & (|t| \le \eta) \\ c + tv - tw & (-\beta < t \le -\eta). \end{cases}$$

This u is a C^1-function and it coincides with p on $]-\beta, -\eta] \cup [\eta, \beta[$. We finally have to verify (7) (with $s = 0$). For $|t| \le \eta$ we get

$$u'(t) = \frac{t}{\eta} w + v,$$

hence

(9) $$\|u'(t) - v\| \le \|w\| \le \frac{\varepsilon}{4}.$$

Furthermore

$$\|u(t) - c - tv\| = \left(\frac{t^2}{2\eta} + \frac{\eta}{2}\right) \|w\| \le \eta\|w\| \le \eta\frac{\varepsilon}{4} \le \eta.$$

Using (8), we get

$$\|f(t, u(t)) - v\| \le \frac{3\varepsilon}{4}.$$

From this and (9) we derive

$$\|u'(t) - f(t, u(t))\| \le \varepsilon \quad (\text{for } |t| \le \eta),$$

which is the first inequality in (7). Concerning the second one, we observe for $|t| \le \eta$ that

$$p(t) - u(t) = |t|w - \left(\frac{1}{2\eta}t^2 + \frac{\eta}{2}\right) w,$$

hence

$$\|p(t) - u(t)\| \le 2\eta\|w\| \le 2\eta\frac{\varepsilon}{4},$$

and thus

$$\operatorname{dist}(u(t), M) \le \operatorname{dist}(p(t), M) + \|p(t) - u(t)\| \le \frac{\varepsilon}{4} + 2\eta\frac{\varepsilon}{4} = \frac{\varepsilon}{4}(1 + 2\eta) \le \varepsilon.$$

3. A local result. Let us consider Theorem 1 for $M = E$. Then (1) holds for every function $f : [0, T] \times E \to E$. In this case Theorem 1 gives a solution $u : [\tau, T] \to E$ of the i.v.p. (2), which means that we find back Satz 2.3 of Schmidt [3].

As a consequence of Satz 2.3, Schmidt proves a local version of it. Now we are doing the same with our Theorem 1.

Theorem 3 *Suppose $T > 0$ and $M \subseteq D \subseteq E$, where M is a closed and D an open subset of the Banach space E. Consider $f = g + k$, where $g, k : [0,T] \times D \to E$ are continuous and such that*

$$[x-y, g(t,x) - g(t,y)]_- \leq L\|x-y\| \quad (0 \leq t \leq T;\ x, y \in D),$$
$$\alpha(k([0,T] \times B)) \leq K\alpha(B) \quad (B \subseteq D,\ B \text{ bounded}).$$

Let f satisfy condition (1). Then for every $a \in M$ there exists a $\widetilde{T} \in]0,T]$ such that the i.v.p.

$$(10) \qquad u(0) = a,\ u'(t) = f(t, u(t)) \quad (0 \leq t \leq \widetilde{T})$$

has a solution $u : [0, \widetilde{T}] \to M$.

Proof. We simply follow the proof of Satz 2.4 in [3]: We fix $a \in M$ and we choose $T_0, r > 0$ such that g, k are defined and bounded on

$$[0, T_0] \times \{x \mid x \in E,\ \|x-a\| \leq r\};$$

let μ be a positive bound for their norms. We take

$$\widetilde{T} = \min\{T_0, \frac{r}{4\mu}\}.$$

We define $p : [0, \infty[\to [0,1]$ by

$$p(s) = \begin{cases} 1 & (0 \leq s \leq \frac{r}{2}) \\ 2 - \frac{2}{r}s & (\frac{r}{2} \leq s \leq r) \\ 0 & (s \geq r). \end{cases}$$

Then we define $\tilde{f}, \tilde{g}, \tilde{k} : [0, \widetilde{T}] \times E \to E$ by

$$(11) \qquad \tilde{f}(t,x) = \begin{cases} p(\|x-a\|) f(t,x) & (0 \leq t \leq \widetilde{T},\ \|x-a\| \leq r) \\ 0 & (\|x-a\| \geq r), \end{cases}$$

and $\tilde{g}, \tilde{k}$ in an analogous way. We can apply Theorem 1 with T, f, g, k replaced by $\widetilde{T}, \tilde{f}, \tilde{g}, \tilde{k}$. Especially (1) holds after this replacement also for $\tilde{f}$, because in (11) we have $p(\|x-a\|) \geq 0$.
We thus get a solution $u : [0, \widetilde{T}] \to M$ of the i.v.p.

$$u(0) = a,\ u'(t) = \tilde{f}(t, u(t)) \quad (0 \leq t \leq \widetilde{T}).$$

Because of $\|u(t) - a\| \leq 2\mu\widetilde{T} \leq \frac{r}{2}$, the function u also solves (10).

References

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