Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

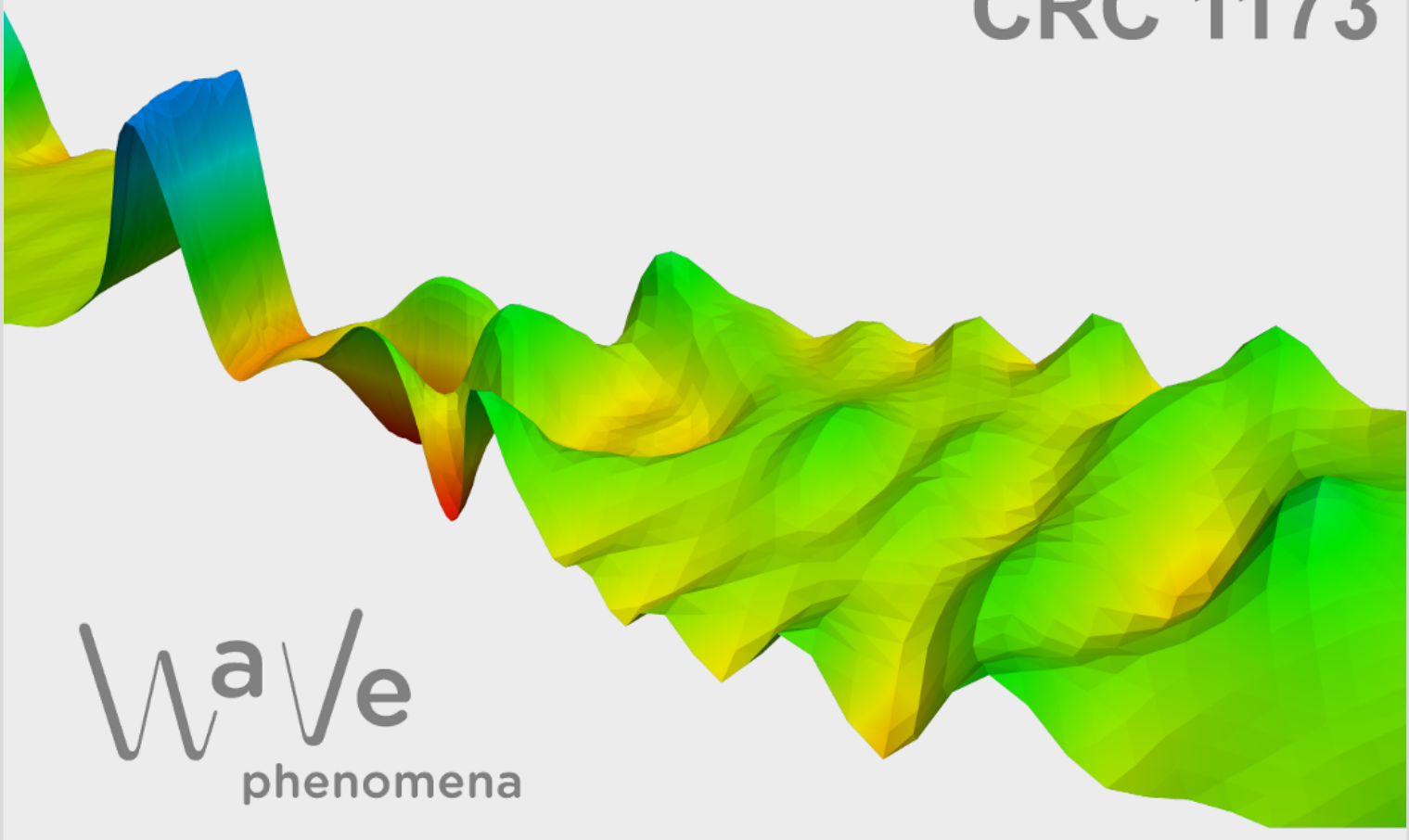
Biharmonic wave maps into spheres

Sebastian Herr, Tobias Lamm, Roland Schnaubelt

CRC Preprint 2018/45, December 2018

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Funded by

ISSN 2365-662X

BIHARMONIC WAVE MAPS INTO SPHERES

SEBASTIAN HERR, TOBIAS LAMM, AND ROLAND SCHNAUBELT

Abstract. A global weak solution of the biharmonic wave map equation in the energy space for spherical targets is constructed. The equation is reformulated as a conservation law and solved by a suitable Ginzburg-Landau type approximation.

1. Introduction

We study biharmonic wave maps $u : I \times \mathbb{R}^n \to S^l$, where S^l is the l-dimensional unit sphere in $\mathbb{R}^{l+1}$, and $I \subset \mathbb{R}$ is an open interval. These maps are critical points of the action functional

$$\Phi(u) := \frac{1}{2} \int_{I \times \mathbb{R}^n} (|\partial_t u|^2 - |\Delta u|^2)\, d(t,x)$$

acting on functions with values in S^l. Here $\Delta u = (\Delta u^i)_i$ is the extrinsic Laplacian; i.e., the Laplacian w.r.t. x when considering u as a map into $\mathbb{R}^{l+1}$. In our main Theorem 1.1 we construct a global weak solution for all data in the energy space.

We introduce two equivalent versions of the biharmonic wave map system for regular solutions. Sufficiently smooth critical points $u : I \times \mathbb{R}^n \to S^l$ of Φ satisfy

$$(\partial_t^2 + \Delta^2) u \perp T_u S^l \tag{1.1}$$

which can be viewed as the geometric version of biharmonic wave map equation To show this claim, for $\varphi \in C_c^\infty(I \times \mathbb{R}^n, \mathbb{R}^{l+1})$ we consider the variation $u_\tau = \pi(u + \tau\varphi)$, where $\pi : \mathbb{R}^{l+1} \setminus \{0\} \to S^l$ denotes the retraction $\pi(y) = \frac{y}{|y|}$, and $\tau > 0$ is small enough. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \Phi(u_\tau)\Big|_{\tau=0} &= \int_{I \times \mathbb{R}^n} \langle \partial_t u, \partial_t D\pi(u)\varphi \rangle - \langle \Delta u, \Delta D\pi(u)\varphi \rangle d(t,x) \\ &= - \int_{I \times \mathbb{R}^n} \langle \partial_t^2 u + \Delta^2 u, D\pi(u)\varphi \rangle d(t,x) \end{aligned}$$

Choose a smooth orthonormal frame $\{v_1(t,x), \dots, v_l(t,x)\}$ for $T_{u(t,x)} S^l$, a scalar function $\eta \in C_c^\infty(I \times \mathbb{R}^n, \mathbb{R})$ and define $\varphi_j = v_j \eta$, for $j = 1, \dots, l$. Since $D\pi(u)\varphi_j = \varphi_j$, for a critical point u of Φ we obtain

$$0 = \int_{I \times \mathbb{R}^n} \langle \partial_t^2 u + \Delta^2 u, v_j \rangle \eta d(t,x).$$

We conclude that $\langle \partial_t^2 u + \Delta^2 u, v_j \rangle = 0$ for any $j = 1, \dots, l$, which shows (1.1).

Date: December 10, 2018.

2000 *Mathematics Subject Classification.* 35L75, 58J45.

Key words and phrases. Global weak solutions, geometric plate equation, conservation law.

TL and RS gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

For smooth $u : I \times \mathbb{R}^n \to S^l$, equation (1.1) is equivalent to the PDE-version

$$(\partial_t^2 + \Delta^2)u = \big[\, |\Delta u|^2 - |\partial_t u|^2 - \Delta|\nabla u|^2 - 2\,\mathrm{div}\langle \Delta u, \nabla u\rangle\big]\, u \tag{1.2}$$

of the biharmonic wave map system. Here and below, for any $A \in \mathbb{R}^{(l+1)\times(l+1)}$ the expression $\mathrm{div}\langle \Delta u, A\nabla u\rangle$ is shorthand for $\sum_{i=1}^n \partial_i \langle \Delta u, A\partial_i u\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^{l+1}$. We also write $\langle \nabla\Delta u, \nabla u\rangle = \sum_{i=1}^d \langle \partial_i\Delta u, \partial_i u\rangle$ etc.. Moreover, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{l+1}$ and in $\mathbb{R}^{(l+1)(l+1)}$.

We show the above mentioned equivalence. Equation (1.1) means that there is a function $\lambda_u : I \times \mathbb{R}^n \to \mathbb{R}$ such that $(\partial_t^2 + \Delta^2)u = \lambda_u u$. A solution to (1.2) of course satisfies this identity with $\lambda_u = [\ldots]$. To see the converse, we multiply $(\partial_t^2 + \Delta^2)u = \lambda_u u$ by u and use the product rule. It follows that

$$\lambda_u = \langle \partial_t^2 u, u\rangle + \langle \Delta^2 u, u\rangle = \partial_t\langle \partial_t u, u\rangle - \langle \partial_t u, \partial_t u\rangle + \mathrm{div}\langle \nabla\Delta u, u\rangle - \langle \nabla\Delta u, \nabla u\rangle.$$

By $|u|^2 = 1$, we have $2\langle \partial_k u, u\rangle = \partial_k |u|^2 = 0$ for $k \in \{t, 1, \ldots, n\}$. We then compute

$$\begin{aligned}\lambda_u &= -|\partial_t u|^2 + \Delta\langle \Delta u, u\rangle - 2\,\mathrm{div}\langle \Delta u, \nabla u\rangle + \langle \Delta u, \Delta u\rangle\\ &= -|\partial_t u|^2 + \Delta\,\mathrm{div}\langle \nabla u, u\rangle - \Delta\langle \nabla u, \nabla u\rangle - 2\,\mathrm{div}\langle \Delta u, \nabla u\rangle + |\Delta u|^2\\ &= |\Delta u|^2 - |\partial_t u|^2 - \Delta|\nabla u|^2 - 2\,\mathrm{div}\langle \Delta u, \nabla u\rangle,\end{aligned}$$

as asserted.

The energy corresponding to Φ is given by

$$E[u](t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^n} (|\partial_t u|^2 + |\Delta u|^2)\, dx.$$

We thus introduce the space for (global) weak solutions of our problems as

$$Z = \{u \in L^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{l+1}) \,|\, u(t,x) \in S^l \text{ a.e., } \partial_t u, \Delta u \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n, \mathbb{R}^{l+1}))\}.$$

As above, one observes that

$$\langle \Delta u, u\rangle = \mathrm{div}\langle \nabla u, u\rangle - \langle \nabla u, \nabla u\rangle = -|\nabla u|^2,$$

so that each $u \in Z$ satisfies

$$|\nabla u|^2 \le |\Delta u| \qquad \text{and} \qquad |\nabla u| \in L^4_{\mathrm{loc}}(\mathbb{R}; L^4(\mathbb{R}^n)). \tag{1.3}$$

A *weak solution* of (1.1) is defined as a map $u \in Z$ fulfilling

$$0 = \int_{\mathbb{R}\times\mathbb{R}^n} \big(-\langle \partial_t u, \partial_t v\rangle + \langle \Delta u, \Delta v\rangle\big)\, dt\, dx \tag{1.4}$$

for all functions v belonging to

$$\begin{aligned}V := \{v \in L^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{l+1}) \,|\, |\partial_t v|, |\nabla v|^2, |\Delta v| \in L^2(\mathbb{R}\times\mathbb{R}^n);\ &\mathrm{supp}_t\, v \text{ is compact,}\\ &\text{and } v \text{ satisfies} \quad v(t,x) \in T_{u(t,x)}S^l \ \text{ for a.e. } (t,x) \in \mathbb{R}\times\mathbb{R}^n\},\end{aligned}$$

where supp_t denotes the support of $v : \mathbb{R} \to L^\infty(\mathbb{R}^n, \mathbb{R}^{l+1})$, $t \mapsto v(t,\cdot)$. Moreover, $u \in Z$ is a *weak solution* of (1.2) if

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}^n} (-\langle \partial_t u, \partial_t \phi\rangle + \langle \Delta u, \Delta\phi\rangle)\, dt\, dx\\ &= \int_{\mathbb{R}\times\mathbb{R}^n} \big((|\Delta u|^2 - |\partial_t u|^2)\langle u, \phi\rangle - |\nabla u|^2\, \Delta\langle u, \phi\rangle + 2\langle \Delta u, \nabla u\rangle\, \nabla\langle u, \phi\rangle\big) dt\, dx\end{aligned} \tag{1.5}$$

for all $\phi \in W$, where we put

$$W := \{v \in L^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{l+1}) \,|\, |\partial_t v|, |\nabla v|^2, |\Delta v| \in L^2(\mathbb{R}\times\mathbb{R}^n);\ \mathrm{supp}_t\, v \text{ is compact}\}.$$

Note that the terms on the right hand side in this definition are integrable by (1.3). In Lemma 2.1 we prove the equivalence of the weak solvability of (1.1) and of (1.2).

The fourth order system (1.2) is analogous to the (second order) wave maps system, see e.g. [7]. In this situation global weak solutions in the energy space have been constructed by Shatah [6] for spherical targets and by Freire [2] for target manifolds being homogeneous spaces. Our main result is a variant of the result of Shatah for biharmonic wave maps.

Theorem 1.1. *Let $(u_0, u_1) \in L^\infty(\mathbb{R}^n, \mathbb{R}^{l+1}) \times L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ satisfy $\Delta u_0 \in L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ as well as $u_0(x) \in S^l$ and $u_1(x) \in T_{u_0(x)}S^l$ for a.e. $x \in \mathbb{R}^n$. Then there is a global weak solution $u \in Z$ of (1.2) with $u(0) = u_0$ and $\partial_t u(0) = u_1$. Moreover, the maps $\Delta u, \partial_t u : \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ are continuous and bounded, we have $u(t,\cdot) \in S^l$ and $\partial_t u(t,\cdot) \in T_{u(t,\cdot)}S^l$ a.e. for each $t \in \mathbb{R}$, and the difference $u - u_0$ belongs to $C(\mathbb{R}; H^\theta(\mathbb{R}^n, \mathbb{R}^{l+1}))$ for all $\theta \in [0, 2)$. Finally, for all $t \in \mathbb{R}$ the solution satisfies the energy identity*

$$E[u](t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^n} (|\partial_t u|^2 + |\Delta u|^2)\, dx = E[u](0) = \frac{1}{2}\int_{\mathbb{R}^n} (|u_1|^2 + |\Delta u_0|^2)\, dx.$$

As in the case of the above mentioned results for wave maps, our construction uses a suitable Ginzburg-Landau type approximation of (1.2). We refer the reader to [3] for a survey on scalar fourth order wave-type equations.

We note that there is a second functional which also deserves to be called the action functional corresponding to biharmonic wave maps, namely

$$\Psi(u) := \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^n} (|\partial_t u|^2 - |(\Delta u)^T|^2)\, dt\, dx = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^n} (|\partial_t u|^2 - |\Delta u|^2 + |\nabla u|^4)\, dt\, dx,$$

where $(\Delta u)^T = \Delta u + u|\nabla u|^2$ is the tangential component of the Laplacian. In this case critical points satisfy the PDE

$$\partial_t^2 u + \Delta^2 u + 2\,\mathrm{div}(|\nabla u|^2 \nabla u) \perp T_u S^l$$

or equivalently

$$\begin{aligned}\partial_t^2 u + \Delta^2 u + 2\,\mathrm{div}(|\nabla u|^2 \nabla u) = u\big(&|\Delta u|^2 - |\partial_t u|^2 - \Delta|\nabla u|^2 \\ &- 2\,\mathrm{div}\langle \Delta u, \nabla u\rangle - 2|\nabla u|^4\big).\end{aligned} \tag{1.6}$$

Due to the additional nonlinear term, our proof of Theorem 1.1 does not extend to this equation.

2. The conservation law

As a first result we show that the systems (1.1) and (1.2) are also equivalent in the weak sense and that they can be can be written in divergence form (2.1). The latter fact will be crucial for our global existence result.

Lemma 2.1. *For $u \in Z$ the following assertions are equivalent.*

1. *The map u is a weak solution of (1.1).*
2. *The map u is a weak solution of (1.2).*
3. *For all skew-symmetric matrices $\Lambda \in \mathbb{R}^{(l+1)\times(l+1)}$ the map u is a weak solution of the system*

$$0 = \partial_t\langle \partial_t u, \Lambda u\rangle + \Delta\langle \Delta u, \Lambda u\rangle - 2\,\mathrm{div}\langle \Delta u, \Lambda \nabla u\rangle \tag{2.1}$$

on $\mathbb{R}\times\mathbb{R}^n$ *with test functions in* W_s*, where*

$$W_s := \{v \in L^\infty(\mathbb{R}\times\mathbb{R}^n) \,|\, |\partial_t v|, |\nabla v|^2, |\Delta v| \in L^2(\mathbb{R}\times\mathbb{R}^n);\ \operatorname{supp}_t v \text{ is compact}\}.$$

Proof. (1)⇒(3). Let u be a weak solution of (1.1). Take $\varphi \in V_s$ and $\Lambda \in \mathbb{R}^{(l+1)\times(l+1)}$ with $\Lambda^T = -\Lambda$. The function $v = \varphi\Lambda u$ belongs to V by (1.3) and takes values in $T_u S^l$ since Λ is skew-symmetric. We thus obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}\times\mathbb{R}^n} \left(-\langle \partial_t u, \partial_t(\varphi\Lambda u)\rangle + \langle \Delta u, \Delta(\varphi(\Lambda u)\rangle\right) dt\,dx \\ &= \int_{\mathbb{R}\times\mathbb{R}^n} \left(-\partial_t\varphi\langle \partial_t u, \Lambda u\rangle + \Delta\varphi\langle \Delta u, \Lambda u\rangle + 2\nabla\varphi\langle \Delta u, \Lambda\nabla u\rangle\right) dt\,dx, \end{aligned}$$

using that $\langle \partial_t u, \Lambda\partial_t u\rangle = 0 = \langle \Delta u, \Lambda\Delta u\rangle$. Hence, u is a weak solution of (2.1).

(3)⇒(2). Let u be a weak solution of (2.1). We employ for $1 \le i < j \le l+1$ and $\omega \in S^l$ the tangent vectorfields

$$\Lambda_{ij}\omega = (e_i \otimes e_j - e_j \otimes e_i)\omega = \omega^i e_j - \omega^j e_i \in T_\omega S^l.$$

These vectorfields span $T_\omega S^l$ since each $\xi \in T_\omega S^l$ has the representation

$$\xi = (\omega\otimes\xi - \xi\otimes\omega)\omega = \sum_{1\le i<j\le l+1} (\omega^i\xi^j - \omega^j\xi^i)\Lambda_{ij}\omega.$$

For a given function $\phi \in W$ we deduce

$$\begin{aligned} \phi &= \langle \phi, u\rangle u + \sum_{1\le i<j\le l+1} \varphi_{ij}\Lambda_{ij}u, \qquad \text{where} \\ \varphi_{ij} &:= u^i(\phi^j - \langle \phi, u\rangle u^j) - u^j(\phi^i - \langle \phi, u\rangle u^i). \end{aligned} \tag{2.2}$$

Note that all maps $\varphi_{ij}\Lambda_{ij}u$ and $\langle \phi, u\rangle u$ belong to W, and φ_{ij} to W_s. Assertion (3) then yields

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}^n} \Big(- \langle \partial_t u, \partial_t(\varphi_{ij}\Lambda_{ij}u)\rangle + \langle \Delta u, \Delta(\varphi_{ij}\Lambda_{ij}u\rangle\Big)\, dt\,dx \\ &= \int_{\mathbb{R}\times\mathbb{R}^n} \Big(- \partial_t\varphi_{ij}\langle \partial_t u, \Lambda_{ij}u\rangle + \Delta\varphi_{ij}\langle \Delta u, \Lambda_{ij}u\rangle + 2\nabla\varphi_{ij}\langle \Delta u, \Lambda_{ij}\nabla u\rangle\Big)\, dt\,dx \\ &= 0, \end{aligned}$$

where two terms vanish because of the skew-symmetry of Λ_{ij}. For the normal component, we compute

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}^n} \left(-\langle \partial_t u, \partial_t(\langle \phi, u\rangle u)\rangle + \langle \Delta u, \Delta(\langle \phi, u\rangle u)\rangle\right) dt\,dx \\ &= \int_{\mathbb{R}\times\mathbb{R}^n} \big(-|\partial_t u|^2\langle u, \phi\rangle + |\Delta u|^2\langle u, \phi\rangle + 2\langle \Delta u, \nabla u\rangle\,\nabla\langle \phi, u\rangle + \langle \Delta u, u\rangle\,\Delta\langle \phi, u\rangle\big)\, dt\,dx \\ &= \int_{\mathbb{R}\times\mathbb{R}^n} \big((|\Delta u|^2 - |\partial_t u|^2)\langle u, \phi\rangle - |\nabla u|^2\,\Delta\langle u, \phi\rangle + 2\langle \Delta u, \nabla u\rangle\,\nabla\langle u, \phi\rangle\big) dt\,dx \end{aligned}$$

since $\langle \partial_t u, u\rangle = 0$ and $\langle \Delta u, u\rangle = -|\nabla u|^2$. Summing up, the decomposition (2.2) implies that u solves (1.2) weakly.

(2)⇒(1). Let u be a weak solution of (1.2). For test functions $v \in W$ taking values in $T_{u(t,x)}S^l$ equation (1.4) follows from (1.5) since then $\langle u, v\rangle = 0$. □

Note that the conservation law can also be obtained via Noether's theorem. For any map u the action functional Φ is invariant under rotations $R(\tau)u = \exp(\tau\Lambda)u$. This fact implies

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi(R(\tau)u,\Omega)\Big|_{\tau=0} &= \int_\Omega \left(\langle \partial_t u, \partial_t(\Lambda u)\rangle - \langle \Delta u, \Delta(\Lambda u)\rangle\right) dt\,dx \\ &= \int_\Omega \left(\partial_t\langle \partial_t u, \Lambda u\rangle + \Delta\langle \Delta u, \Lambda u\rangle - 2\,\mathrm{div}\langle \Delta u, \Lambda\nabla u\rangle\right) dt\,dx \\ &\quad - \int_\Omega \langle (\partial_t^2 + \Delta^2)u, \Lambda u\rangle\, dt\,dx. \end{aligned}$$

for each subset $\Omega \subset \mathbb{R}\times\mathbb{R}^n$. The second integral vanishes since $(\partial_t^2+\Delta^2)u \perp T_u S^l$, and hence we have derived again the conservation law (2.1).

Remark 2.2. *Similar to Lemma 2.1 one can prove that a (smooth) map u is a solution of (1.6) iff for all $\Lambda \in \mathbb{R}^{(l+1)\times(l+1)}$ with $\Lambda^T = -\Lambda$ we have*

$$0 = \partial_t\langle \partial_t u, \Lambda u\rangle + \Delta\langle \Delta u, \Lambda u\rangle - 2\,\mathrm{div}\langle \Delta u, \Lambda\nabla u\rangle + 2\,\mathrm{div}\langle |\nabla u|^2\nabla u, \Lambda u\rangle. \tag{2.3}$$

3. Existence of a global weak solution

In this section we construct a global weak solution of (1.2) using a penalization method as in [2]. To this end, we fix an increasing function $\chi \in C^\infty([0,\infty))$ with $\chi(s) = 1$ for all $s \geq \frac{1}{2}$ and $\chi(s) = s$ for all $s \leq \frac{1}{4}$. We then define the smooth map $F : \mathbb{R}^{l+1} \to \mathbb{R}$ by

$$F(x) = \chi \circ (|x|^2 - 1)^2.$$

Observe that F is bounded, its derivatives are compactly supported, $F^{-1}(0) = S^l$, and $\nabla F(x) = 0$ if $|x| = 1$.

For $\varepsilon > 0$ and initial functions $(u_0, u_1) \in L^\infty(\mathbb{R}^n, \mathbb{R}^{l+1}) \times L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ with $\Delta u_0 \in L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ and $u_0(x) \in S^l$ for almost every $x \in \mathbb{R}^n$, we look at the auxiliary system

$$\begin{aligned} &\partial_t^2 u_\varepsilon + \Delta^2 u_\varepsilon + \tfrac{1}{\varepsilon}\nabla F(u_\varepsilon) = 0 \\ &u_\varepsilon(0,\cdot) = u_0, \quad \partial_t u_\varepsilon(0,\cdot) = u_1, \end{aligned} \tag{3.1}$$

without requiring that $u(t,x) \in S^l$ if $t \neq 0$ a.e.. We point out that the initial value u_0 here (and below) is not square-integrable, which causes technical difficulties. In contrast to the wave map case in [2], solutions of (3.1) do not possess finite speed of propagation so that standard cut-off arguments cannot be used. Instead we look for (distributional) solutions of the form $u_\varepsilon = u_0 + v_\varepsilon$ for a function v_ε solving the shifted system

$$\begin{aligned} \partial_t^2 v_\varepsilon + \Delta^2 v_\varepsilon + \tfrac{1}{\varepsilon}\nabla F(u_0 + v_\varepsilon) + \Delta^2 u_0 = 0 \\ v_\varepsilon(0,\cdot) = 0, \quad \partial_t v_\varepsilon(0,\cdot) = u_1, \end{aligned} \tag{3.2}$$

weakly, with test functions in W. For brevity, we sometimes write H^k instad of $H^k(\mathbb{R}^n, \mathbb{R}^{l+1})$ for $k \in \mathbb{Z}$, and analogously for other function spaces.

We use the following fact. Let $Z \hookrightarrow Y$ be reflexive Banach spaces, Z be dense in Y, and $f : \mathbb{R} \to Y$ be a weakly continuous function which is essentially bounded with values in Z. Then f is bounded and weakly continuous as a map into Z.

Lemma 3.1. *Let $\varepsilon > 0$ and (u_0, u_1) belong to $L^\infty(\mathbb{R}^n, \mathbb{R}^{l+1}) \times L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ with $\Delta u_0 \in L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ and $u_0(x) \in S^l$ for almost every $x \in \mathbb{R}^n$. Then there is a distributional solution $u_\varepsilon = u_0 + v_\varepsilon$ of (3.1) such that $\Delta v_\varepsilon, \partial_t v_\varepsilon \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n, \mathbb{R}^{l+1}))$, the functions $v_\varepsilon : \mathbb{R} \to H^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ and $\partial_t v_\varepsilon : \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$ are weakly continuous, and v_ε solves (3.2) weakly with test functions in W. For all $t \in \mathbb{R}$, we have the energy inequality*

$$\begin{aligned} E_\varepsilon[u_\varepsilon](t) &:= \int_{\{t\}\times\mathbb{R}^n} \left(\frac{1}{2} |\partial_t u_\varepsilon|^2 + \frac{1}{2} |\Delta u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) dx \\ &\le E_\varepsilon[u_\varepsilon](0) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|u_1|^2 + |\Delta u_0|^2 \right) dx =: E_0. \end{aligned} \tag{3.3}$$

Proof. 1) To construct the function $v = u_\varepsilon - u_0$, we first study a regularized problem (and we drop the subscript ε). Let u_0 and u_1 be the given data. By means of standard mollifiers, we obtain functions $u_{1,i}$ in H^2 converging to u_1 in L^2 as $i \to \infty$, as well as $u_{0,j} \in L^\infty$ such that $\Delta u_{0,j}$ and $\Delta^2 u_{0,j}$ belong to L^2, the maps $u_{0,j}$ tend to u_0 pointwise a.e. and with a uniform bound, and $(\Delta u_{0,j})$ converges to Δu_0 in L^2 as $j \to \infty$. Finally, let α_k be the characteristic function of the ball $B(0,k)$ in $\mathbb{R}^n$. We now introduce the modified equation

$$\begin{aligned} \partial_t^2 v + \Delta^2 v + \tfrac{1}{\varepsilon} \alpha_k \nabla F(u_{0,j} + v) + \Delta^2 u_{0,j} = 0, \\ v(0,\cdot) = 0, \quad \partial_t v(0,\cdot) = u_{1,i}. \end{aligned} \tag{3.4}$$

Define $X = H^2(\mathbb{R}^n, \mathbb{R}^{l+1}) \times L^2(\mathbb{R}^n, \mathbb{R}^{l+1})$. We have $(v(t), \partial_t v(t)) \in X$ and we look at the operator matrix

$$A = \begin{pmatrix} 0 & -I \\ \Delta^2 & 0 \end{pmatrix}, \qquad D(A) = H^4(\mathbb{R}^n, \mathbb{R}^{l+1}) \times H^2(\mathbb{R}^n, \mathbb{R}^{l+1}) \subset X.$$

Using the group version of the Lumer-Phillips theorem, see Corollary II.3.6 of [1], one checks that $-A$ generates a strongly continuous (unbounded) group. Moreover, the map

$$G : X \to X,\ G(\varphi, \psi) = \begin{pmatrix} 0 \\ \frac{1}{\varepsilon} \alpha_k \nabla F(u_{0,j} + \varphi) + \Delta^2 u_{0,j} \end{pmatrix}$$

is globally Lipschitz and C^1. (For the differentiability one can employ the Sobolev embedding $H^2 \hookrightarrow L^p$ for some $p > 2$.). Slight variants of Theorems 6.1.2 and 6.1.5 in [5] hence provide a unique global solution $v = v_{i,j,k} \in C(\mathbb{R}, H^4) \cap C^1(\mathbb{R}, H^2) \cap C^2(\mathbb{R}, L^2)$ of the system (3.4) in this case. We can now differentiate the energy

$$\tilde{E}_{j,k}[v](t) := \int_{\{t\}\times\mathbb{R}^n} \left(\frac{1}{2} |\partial_t v|^2 + \frac{1}{2} |\Delta(u_{0,j} + v)|^2 + \frac{1}{\varepsilon} \alpha_k F(u_{0,j} + v) \right) dx$$

with respect to $t \in \mathbb{R}$. Integration by parts yields

$$\begin{aligned} \partial_t \tilde{E}_{j,k}[v](t) &= \int_{\{t\}\times\mathbb{R}^n} \big(\langle \partial_t v, \partial_{tt} v \rangle + \langle \Delta^2 (u_{0,j} + v), \partial_t v \rangle + \langle \tfrac{1}{\varepsilon} \alpha_k \nabla F(u_{0,j} + v), \partial_t v \rangle \big) dx \\ &= 0, \\ \tilde{E}_{j,k}[v](t) &= \int_{\mathbb{R}^n} \left(\frac{1}{2} |u_{1,i}|^2 + \frac{1}{2} |\Delta u_{0,j}|^2 + \frac{1}{\varepsilon} \alpha_k F(u_{0,j}) \right) dx \end{aligned}$$

for all $t \in \mathbb{R}$. In the next steps, we perform the limits $i, j, k \to \infty$ one after the other. We will not relabel subsequences.

2) As in Theorem 6.1.2 of [5], the solution $(v, \partial_t v)$ of (3.4) depends continuously in X on the initial data. The sequence $(v_{i,j,k})_i$ thus tends in $C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$ to a function $v_{j,k}$ for all $j, k \in \mathbb{N}$, and a subsequence also converges pointwise a.e. in (t, x). Note that, here and throughout the paper, the space $C(\mathbb{R}, X)$ (for a normed space X) consists of all continuous functions $v : \mathbb{R} \to X$, and convergence therein refers to locally uniform convergence (similarly for $C^k(\mathbb{R}, X)$). Consequently, the map $v_{j,k}$ satisfies the initial conditions $v_{j,k}(0, \cdot) = 0$ and $\partial_t v_{j,k}(0, \cdot) = u_1$, and it solves the PDE in (3.4) weakly with test functions in W.

For a fixed $t \in \mathbb{R}$, a further subsequence $(v_{i,j,k}(t, \cdot))_i$ tends to $v_{j,k}(t, \cdot)$ a.e.. Hence, the above energy equality leads to the identity

$$\tilde{E}_{j,k}[v_{j,k}](t) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |u_1|^2 + \frac{1}{2} |\Delta u_{0,j}|^2 + \frac{1}{\varepsilon} \alpha_k F(u_{0,j}) \right) dx =: \tilde{E}_0^{j,k}$$

for all $j, k \in \mathbb{N}$ and $t \in \mathbb{R}$.

3) Now, we pass to the limit $j \to \infty$ for each fixed $k \in \mathbb{N}$. Because of the cut-off α_k and $F(u_0) = 0$, the energies $\tilde{E}_0^{j,k}$ tend to

$$E_0 := \frac{1}{2} \int_{\mathbb{R}^n} \left(|u_1|^2 + |\Delta u_0|^2 \right) dx,$$

so that $\tilde{E}_{j,k}[v_{j,k}](t)$ is dominated by a number c_k for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$. This estimate leads to the convergence

$$\Delta v_{j,k} \rightharpoonup \varphi_k \quad \text{weak}^* \text{ in } L^\infty(\mathbb{R}; L^2) \quad \text{and} \quad \partial_t v_{j,k} \rightharpoonup \psi_k \quad \text{weak}^* \text{ in } L^\infty(\mathbb{R}; L^2)$$

as $j \to \infty$. The functions φ_k and ψ_k inherit the energy bound by E_0. We further obtain the estimate

$$\|v_{j,k}(t)\|_{L^2} = \left\| \int_0^t \partial_s v_{j,k}(s)\, ds \right\|_{L^2} \leq 2 c_k^{1/2} |t| \leq 2 m c_k^{1/2}$$

for all $t \in [-m, m]$ and $j, k \in \mathbb{N}$. The sequence $(v_{j,k})_j$ is thus bounded in the spaces $L^\infty(J; H^2)$ and $W^{1,\infty}(J; L^2)$ for each $k \in \mathbb{N}$ and each bounded interval $J \subseteq \mathbb{R}$. Proposition 1.1.4 in [4] implies the interpolative embedding

$$L^\infty(J; H^2) \cap W^{1,\infty}(J; L^2) \hookrightarrow C^{1-\beta}(J; H^{2\beta}) \tag{3.5}$$

for $\beta \in (0, 1)$. So, by the Arzelà-Ascoli theorem, $(v_{j,k})_j$ tends to a function v_k strongly in $C(J; H^\theta_{\mathrm{loc}})$ for each $\theta \in [0, 2)$ and hence in $C(\mathbb{R}; H^\theta_{\mathrm{loc}})$ and pointwise a.e., for a diagonal sequence. A standard test function argument then yields that $\varphi_k = \Delta v_k$ and $\psi_k = \partial_t v_k$. In particular, v_k belongs to $L^\infty(J; H^2) \cap W^{1,\infty}(J; L^2) \cap C(\mathbb{R}; H^\theta)$, $v_k(0) = 0$, and Δv_k is weakly continuous with values in L^2. Moreover, it satisfies the energy inequality

$$\tilde{E}_k[v_k](t) := \int_{\{t\} \times \mathbb{R}^n} \left(\frac{1}{2} |\partial_t v_k|^2 + \frac{1}{2} |\Delta(u_0 + v_k)|^2 + \frac{1}{\varepsilon} \alpha_k F(u_0 + v_k) \right) dx \leq E_0$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$.

Since the nonlinear term has compact support in space, we next deduce that v_k satisfies the PDE in (3.4) for u_0 instead of $u_{0,j}$ weakly with test functions in W. This equation further shows that the weak derivative $\partial_t^2 v_k$ actually belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}; H^{-2})$ so that $\partial_t v_k$ is continuous from $\mathbb{R}$ to H^{-2} and, as seen above, essentially bounded in L^2. As a result, the map $t \mapsto \partial_t v_k(t)$ is bounded and weakly continuous in L^2. Since $\partial_t v_{j,k}$ converges weak* in $L^\infty(\mathbb{R}; L^2)$ and vanishes at $t = 0$, we conclude that $\partial_t v_k(0) = 0$.

4) In a final step, we let $k \to \infty$. We can proceed as in Step 3) to construct a limit function v with the desired properties. There is only one difference in the derivation of the PDE for v. To apply the dominated convergence theorem, observe that $|\nabla F(u_0+v_k)|$ is bounded by $c\,|v_k|$ and that a converging sequence in L^2 has a subsequence with a majorant in L^2. Finally, the function $u_\varepsilon = u_0 + v$ satisfies the assertions. □

Based on the energy estimate (3.3), we can now pass to the limit $\varepsilon \to 0$ in (3.1). The special form of the penalization term implies that the resulting weak limit u takes values in S^l. As in [2], we employ the equation (2.1) in divergence form to show that u indeed solves of (1.2) weakly. To identify its initial values, we have to assume that u_1 maps into the tangent space of S^l.

Proof of Theorem 1.1. 1) We use the functions $u_j = u_0 + v_j$ from Lemma 3.1, where $v_j = v_{\varepsilon_j}$ for some $\varepsilon_j \to 0^+$. Let $\Lambda \in \mathbb{R}^{(l+1)\times(l+1)}$ be skew-symmetric and $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. We take $\varphi\Lambda u_j$ as a test function for (3.1). (It does not belong to V, in general, but the regularity provided by Lemma 3.1 suffices here.) Since $\nabla F(u_j)$ is a scalar multiple of u_j, we can argue as in the first part of the proof of Lemma 2.1 and conclude that u_j fulfills the equation

$$0 = \partial_t\langle\partial_t u_j, \Lambda u_j\rangle + \Delta\langle\Delta u_j, \Lambda u_j\rangle - 2\,\mathrm{div}\langle\Delta u_j, \Lambda\nabla u_j\rangle. \tag{3.6}$$

in the distributional sense.

2) Starting from the energy estimate (3.3), we can next pass to the limit $\varepsilon_j \to 0$ as in Step 3) of the proof of Lemma 3.1 (again without relabelling subsequences). The functions v_j then converge strongly in $C(\mathbb{R}; H^\theta_{\mathrm{loc}})$ for $\theta < 2$ and pointwise a.e. to a map $v \in C(\mathbb{R}; H^\theta)$. Moreover, $\partial_t v_j$ and Δv_j tend to $\partial_t v$ and Δv weak* in $L^\infty(\mathbb{R}, L^2)$. Combining these facts, we infer that $\Delta v : \mathbb{R} \to L^2$ is bounded and weakly continuous. The limit $u := u_0 + v$ thus satisfies $u(0,\cdot) = u_0$ and $E[u](t) \le E_0$ for all $t \in \mathbb{R}$. Thanks to (3.6) and the convergence of v_j, the function u solves (2.1) distributionally.

The energy bound (3.3) further says that $\|F(u_j(t,\cdot))\|_1 \le \varepsilon_j E_0$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$. For each bounded interval $J \subseteq \mathbb{R}$, Fatou's Lemma now implies that

$$\int_{J\times\mathbb{R}^n} F(u)\,dx\,dt \le \liminf_{j\to\infty} \int_{J\times\mathbb{R}^n} F(u_j)\,dx\,dt = 0.$$

Hence, $F(u) = 0$ and therefore $u(t,x) \in S^l$ for a.e. $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. The continuity of $u - u_0 : \mathbb{R} \to L^2$ then implies that $u(t,x)$ belongs S^l for each $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^n$. Since $u \in Z$, the map ∇u is contained $L^\infty(\mathbb{R}, L^4)$ by (1.3). We can now deduce that u weakly solves (2.1) with test functions in W_s, and so u is a weak solution of (1.2) by Lemma 3.1. Moreover, the equation $|u|^2 = 1$ yields $\langle\partial_t u, u\rangle = 0$ so that $\partial_t u(t,x)$ is contained in the tangent space $T_{u(t,x)}S^l$ for a.e. (t,x).

3) We still have to show the continuity of $\partial_t u, \Delta u : \mathbb{R} \to L^2$ and that $\partial_t u(0,\cdot) = u_1$. So far we know that the first map is essentially bounded. Let $\Lambda \in \mathbb{R}^{(l+1)\times(l+1)}$ again be skew-symmetric. The equation (2.1) and the above stated regularity properties of u imply that $\partial_t\langle\partial_t u, \Lambda u\rangle$ is bounded in $H^{-2} + W^{-1,4/3}$. Hence, the function $\psi : t \mapsto \langle\partial_t u, \Lambda u\rangle$ is continuous in this space. Consequently, ψ is bounded and weakly continuous in L^2. Step 2) implies that for a.e. $t \in \mathbb{R}$ the vector $\partial_t u(t,x)$ belongs to $T_{u(t,x)}S^l$ for a.e. $x \in \mathbb{R}^n$. In view of (2.2), by modifying $\partial_t u(t,\cdot)$ for t in set of measure 0 we obtain a representative $\partial_t u$ which is bounded and weakly continuous as a map from $\mathbb{R}$ to L^2.

Next, we multiply the equations (3.6) for u_j and (2.1) for u by a function $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. We integrate by parts in $t\in[0,\infty)$ with values in $H^{-2}+W^{-1,4/3}$ and subtract the two resulting equations, which yields

$$\begin{aligned}\int_{\{0\}\times\mathbb{R}^n}\langle u_1 - e\partial_t u, \Lambda u_0\rangle\varphi\,dx &= \int_0^\infty\int_{\mathbb{R}^n}\left(-\langle\partial_t u_j,\Lambda u_j\rangle + \langle\partial_t u,\Lambda u\rangle\right)\partial_t\varphi\,dx\,dt\\ &\quad+\int_0^\infty\int_{\mathbb{R}^n}\left(\langle\Delta u_j,\Lambda u_j\rangle - \langle\Delta u,\Lambda u\rangle\right)\Delta\varphi\,dx\,dt\\ &\quad+2\int_0^\infty\int_{\mathbb{R}^n}\left(\langle\Delta u_j,\Lambda\nabla u_j\rangle - \langle\Delta u,\Lambda\nabla u\rangle\rangle\right)\nabla\varphi\,dx\,dt.\end{aligned}$$

By Step 2), the right hand side converges to zero as $j\to\infty$ so that

$$\langle\partial_t u(0) - u_1, \Lambda u_0\rangle = 0.$$

As both u_1 and $\partial_t u(0)$ belong to $T_{u_0}S^l$ a.e., we conclude that $\partial_t u(0) = u_1$.

Finally, in the energy bound obtained in Step 2) we can replace the initial time 0 by any $t\in\mathbb{R}$ since $u(t,\cdot)$ and $\partial_t u(t,\cdot)$ fulfill the conditions for the initial data and hence

$$E[u](0)\le E[u](t)\le E[u](0)$$

which implies the energy identity. Further, in conjunction with the weak continuity, this identity implies the strong continuity of $\partial_t u, \Delta u : \mathbb{R}\to L^2$. □

References

1. Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. MR 1721989
2. A. Freire, *Global weak solutions of the wave map system to compact homogeneous spaces*, Manuscripta Math. **91** (1996), no. 4, 525–533. MR 1421290
3. E. Hebey and B. Pausader, *An introduction to fourth order nonlinear wave equations*, http://www.math.univ-paris13.fr/ pausader/HebPausSurvey.pdf.
4. Alessandra Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 16, Birkhäuser Verlag, Basel, 1995. MR 1329547
5. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR 710486
6. Jalal Shatah, *Weak solutions and development of singularities of the* SU(2) σ*-model*, Comm. Pure Appl. Math. **41** (1988), no. 4, 459–469. MR 933231
7. Jalal Shatah and Michael Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics, vol. 2, New York University Courant Institute of Mathematical Sciences, New York, 1998. MR 1674843 (2000i:35135)

(S. Herr) Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, 33501 Bielefeld, Germany

E-mail address: herr@math.uni-bielefeld.de

(T. Lamm) Department of Mathematics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany

E-mail address: tobias.lamm@kit.edu

(R. Schnaubelt) Department of Mathematics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany

E-mail address: schnaubelt@kit.edu