Automatic Determining Of A Modulating Function

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***Abstract*— This article shows how modulating functions can be generated automatically and used to identify selected parameters of commensurable fractional systems. Existing methods use universal modulating functions. These have to be adjusted to each single problem. The proposed method uses a model based auxiliary system which is derived from the actual system, but has a fixed structure. The problem of applying a suitable modulating function is transformed into a control problem. Taking an additional precondition into account, each parameter can be separately identified. A study of the practical applicability and a numerical example complete the article.**

I. INTRODUCTION

In recent years, complex physical, chemical or biological systems are described more often using fractional order models. Due to the non-locality and memory of fractional integration and differentiation, models are more precisely even though less parameters are used. In addition, these are physically interpretable [1, 2, 3]. Often, it is not possible to calculate or measure all parameters using physical, chemical or biological relations. Remaining parameters have to be identified using measurements.

A lot of identification methods considering fractional order models are based on the modulating function method (see e.g. [4, 5, 6]). The benefit is that no measured signal has to be differentiated. Instead, the modulating function is differentiated. The modulating function can be chosen freely and, therefore, it is not superposed by noise. The drawback is that common modulating functions have a lot of parameters which have to be adapted to the actual problem (see [7]). Another drawback is that no modulating function with fractional boundary terms is known. Therefore, fractional order models using the definition of Riemann-Liouville instead of Caputo are considered.

In this article, the benefit of the modulating function method will be remained unchanged. The drawbacks are eliminated using a model based auxiliary system which is obtained by applying the modulating function method to the original system under consideration of the boundary terms. This idea is described for the integer case in [8]. It is shown that the control input of the model based auxiliary system is connected to the modulating function. So, if an appropriated control input is determined, the modulating function will be known as well. Depending on the chosen control input, it is also possible that each parameter is separately identified. It will also turn out that in case of noisy measurements an energy optimal control will minimize the identification error.

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The article is structured as follows: In Section II, the basics of fractional calculus and of the modulating function method are provided. In Section III, a precondition for separate identification of each parameter is given, first. Afterwards, the model based auxiliary system which is central to determine automatically a modulating function is defined. At the end of Section III, the control based identification is presented. The article is completed with a view at the practical applicability in Section IV and an example in Section V.

II. PRELIMINARIES

A. Fundamentals of Fractional Calculus

In this section, the fractional operators used in this paper are described. For these definitions, the following notations are used: a, b, t, e, $g \in \mathbb{R}^+$ and $a \leq e < t < g \leq b$ and $\alpha \in \mathbb{R}^+$. $\lfloor \cdot \rfloor$ denotes the floor-function and is used to state a modified ceiling-function $\lceil \cdot \rceil := \lfloor \cdot \rfloor + 1$. Through the paper, the functions $f(t)$ and $h(t)$ are assumed to be Lebesgue integrable on the integration interval defined by $[a, b]$ (see [9]). In addition, the assumption is made that $f(t)$ and $h(t)$ are $\lceil \alpha \rceil$-times absolutely continuous on the derivation interval $[a, b]$, where α is the derivation order (see [9]). Furthermore, $\forall \tilde{t} \leq a : f(\tilde{t}) = 0$ and $\forall \tilde{t} \geq b : h(\tilde{t}) = 0$ holds. The uninitialized as well as initialized fractional operators are described in [10] and are given in the following definition. The initialized fractional operators consist of the uninitialized fractional operator and time-variant initialization functions $\eta(\cdot, t)$ resp. $\psi(\cdot, t)$, which have been proven to be necessary in order to describe fractional differential equations (FDEs) correctly. Differing from the used notation in [10], in this paper, an additional operator index on the top left of the fractional operator is used. This index indicates that the function is integrated or differentiated w.r.t. the named variable. The calculus of two-variable functions can be directly extended from the fractional operators for functions depending on one variable (see [11]).

In the following definition, the uninitialized as well as the initialized fractional integration, Riemann-Liouville (RL) and Caputo (C) fractional derivative are summarized.

Definition 1: Fractional Operators.

- Uninitialized Fractional Integral

$$ {}_{e}^{t}i_{t}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{e}^{t} \frac{f(\nu)}{(t-\nu)^{(1-\alpha)}} \mathrm{d}\nu \qquad (1) $$

- Initialized Fractional Integral

$$ {}_{e}^{t}\mathcal{I}_{t}^{\alpha} f(t) := {}_{e}^{t}i_{t}^{\alpha} f(t) + \eta(f, \alpha, a, e, t) \qquad (2) $$

- Uninitialized Right-Sided Fractional Integral

$$ {}_t^t i_g^\alpha h(t) := \frac{1}{\Gamma(\alpha)} \int_t^g \frac{h(\nu)}{(\nu - t)^{(1-\alpha)}} \mathrm{d}\nu \tag{3} $$

- Initialized Right-Sided Fractional Integral

$$ {}_t^t \mathcal{I}_g^\alpha h(t) := {}_t^t i_g^\alpha h(t) + \eta_{\mathrm{rs}}(h, \alpha, g, b, t) \tag{4} $$

- Uninitialized Riemann-Liouville Fractional Derivative

$$ {}_e^t \mathrm{d}RL_t^\alpha f(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lceil\alpha\rceil} \left[{}_e^t i_t^{\lceil\alpha\rceil-\alpha} f(t) \right] \tag{5} $$

- Uninitialized Caputo Fractional Derivative

$$ {}_e^t \mathrm{d}C_t^\alpha f(t) := {}_e^t i_t^{\lceil\alpha\rceil-\alpha} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lceil\alpha\rceil} f(t) \right] \tag{6} $$

- Initialized Fractional Derivative

$$ {}_e^t \mathrm{D}_t^\alpha f(t) := {}_e^t d_t^\alpha f(t) + \psi(f, \alpha, a, e, t) \tag{7} $$

- Uninitialized Right-Sided Riemann-Liouville Fractional Derivative

$$ {}_t^t \mathrm{d}RL_g^\alpha h(t) := (-1)^{\lceil\alpha\rceil} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lceil\alpha\rceil} \left[{}_t^t i_g^{\lceil\alpha\rceil-\alpha} h(t) \right] \tag{8} $$

- Uninitialized Right-Sided Caputo Fractional Derivative

$$ {}_t^t \mathrm{d}C_g^\alpha h(t) := (-1)^{\lceil\alpha\rceil} {}_t^t i_g^{\lceil\alpha\rceil-\alpha} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lceil\alpha\rceil} h(t) \right] \tag{9} $$

- Initialized Right-Sided Fractional Derivative

$$ {}_t^t \mathrm{D}_g^\alpha h(t) := {}_t^t d_g^\alpha h(t) + \psi_{\mathrm{rs}}(h, \alpha, g, b, t) \tag{10} $$

where $\Gamma(\alpha)$ is the Gamma function.

Remark 1: In (2) and in (7), the application of the initialization function $\eta(\cdot, t)$ resp. $\psi(\cdot, t)$ necessitates to know the function $f(t)$ in interval $t \in [a, e]$ (see [10]). To ensure that a system is really at rest, the initialization function implies that the function has to be known from $t \to -\infty$ according to [10]. In the right-sided cases, the function $h(t)$ has to be known for $t \in [g, b]$, so $t \to \infty$. Therefore, exact initialization is not possible in practical application.

Remark 2: In (7), the specific operator depends on the used uninitialized operator (5) or (6) for ${}_e^t d_t^\alpha f(t)$ and is denoted as ${}_e^t \mathrm{D}RL_t^\alpha$ or ${}_e^t \mathrm{D}C_t^\alpha$. According to the uninitialized operator, the related initialization function has to be used (see [10]). This holds also true for the right-sided definition (10).

The left- and right sided fractional operators are connected by a reflection operator $\mathbb{Q}$.

Definition 2: Reflection Operator.
The reflection operator $\mathbb{Q}$ maps a left-sided function onto a right-sided one (see [12]):

$$ \mathbb{Q} f(t) := f(e + g - t) \tag{11} $$

$$:= \tilde{f}(t). \tag{12} $$

B. Fractional Order Models

In this article, a commensurable fractional order model

$$ {}_e^t \mathrm{D}C_t^{n\alpha} y_o(t) + \sum_{i=0}^{n-1} a_i {}_e^t \mathrm{D}C_t^{i\alpha} y_o(t) = \sum_{j=0}^{m} b_j {}_e^t \mathrm{D}C_t^{j\alpha} u_o(t) \tag{13} $$

where $a_i,\ b_j \in \mathbb{R}$ are unknown parameters, $\alpha \in \mathbb{R}^+$ as well as $n, m \in \mathbb{N},\ n > m$ are assumed to be known. Only stable systems will be considered and therefore, $\alpha \in (0, 2)$ is assumed because of the extended Matignon's theorem (see [13]). Nevertheless, the following derivations are done without this restriction. $u_o(t)$ is a noisy observation of the input signal $u(t)$ and $y_o(t)$ of the output signal.

Applying (11) to (13) and using the relation of left-sided and right-sided definitions given in [12], results in

$$ {}_t^t \mathrm{D}C_g^{n\alpha} \tilde{y}_o(t) + \sum_{i=0}^{n-1} a_i {}_t^t \mathrm{D}C_g^{i\alpha} \tilde{y}_o(t) = \sum_{j=0}^{m} b_j {}_t^t \mathrm{D}C_g^{j\alpha} \tilde{u}_o(t) \tag{14} $$

where

$$ \tilde{u}_o(t) = \tilde{u}(t) + \tilde{u}_e(t) \tag{15} $$

$$ \tilde{y}_o(t) = \tilde{y}(t) + \tilde{y}_e(t) \tag{16} $$

with the measurement disturbance $\tilde{u}_e(t)$ and $\tilde{y}_e(t)$ is assumed. Note, that the reflected signals marked by a tilde have to be used. Another assumption regarding the input signal $\tilde{u}(t)$ is that the input signal $\tilde{u}(t)$ describes no homogeneous FDE of order $m\alpha$.

Assumption 1: Non-Vanishing Input Signal.
If not every coefficient $b_j \in \mathbb{R}$ vanishes,

$$ \sum_{j=0}^{m} b_j {}_t^t \mathrm{D}C_g^{j\alpha} \tilde{u}_m(t) \neq 0 \tag{17} $$

is assumed to be true.

Assumption 2: Coprime Polynomials.
It is assumed that the polynomials

$$ a_n \lambda^{\alpha_n} + a_{n-1} \lambda^{\alpha_{n-1}} + \cdots + a_0 \lambda^{\alpha_0} \tag{18} $$

$$ b_m \lambda^{\beta_m} + b_{m-1} \lambda^{\beta_{m-1}} + \cdots + b_0 \lambda^{\beta_0} \tag{19} $$

are coprime.

C. Fractional Integration by Parts

The integration by parts can be extended to functions with non-integer order (see e.g. [5, 14, 15]). Regardless of the type of the fractional operator, fubini's theorem for fractional integration [12] and common integration by parts has to be applied to the integral of a product [16].

Regarding a derivative of a function $h(t)$ described in right-sided Caputo definition (6), the resulting fractional integration by parts is

$$ \begin{aligned} \int_a^b {}_t^t \mathrm{d}C_b^\alpha h(t) f(t) \mathrm{d}t = & \int_a^b h(t) {}_a^t \mathrm{d}RL_t^\alpha f(t) \mathrm{d}t \\ & + \sum_{k=0}^{\lfloor\alpha\rfloor} (-1)^k \left[\frac{\mathrm{d}^{\lfloor\alpha\rfloor-k} h(t)}{\mathrm{d}t^{\lfloor\alpha\rfloor-k}} \frac{\mathrm{d}^k {}_a^t i_t^{\lceil\alpha\rceil-\alpha} f(t)}{\mathrm{d}t^k} \right]_a^b . \end{aligned} \tag{20} $$

Remark 3: The type of the fractional derivative has switched from right-sided C- to left-sided RL-derivative.

D. Fractional Modulating Function Method

The modulating function method is based on the integration by parts (see [17]). Considering the fractional integration by parts, boundary terms occur. To eliminate these boundary terms, the modulating function has to fulfill two properties (see [17]). The integration by parts as well as the properties are generalized for the fractional case if the system is assumed to be of RL-type (see e.g. [5, 6, 14]). In this article, a C-type system (14) is assumed and, therefore, the fractional modulating function method is adjusted to this type of system.

Assumption 3: Properties of the Modulating Functions. The modulating function $\varphi(t)$ fulfills the following properties:

$$(P1) \;:\; \varphi(t) \in \mathcal{C}^{n\alpha}([e,g]),$$
$$(P2) \;:\; \frac{\mathrm{d}^k\, {}_e i_t^{\lceil i\alpha\rceil - i\alpha}\varphi(e)}{\mathrm{d}t^k} = \frac{\mathrm{d}^k\, {}_e i_t^{\lceil i\alpha\rceil - i\alpha}\varphi(g)}{\mathrm{d}t^k} = 0$$
$$\forall k = 0,1,\ldots,\lfloor i\alpha\rfloor \text{ and } i = 0,1,...,n.$$

Remark 4: $(P2)$ eliminates the boundary terms of (20).

Remark 5: The modulating function for system using Caputo definition needs fractional order boundary terms. In present methods, system (13) is usually described using RL-derivatives. Therefore, integer order boundary terms occur such as an integer order system is considered. Multiple modulating functions are known for the integer case (see e.g. [7]).

In [14], another property is given. So, the initialization function of system (14) does not need to be considered. The property is adapted to the used system (14):

$$(P3) \;:\; {}_e^t\mathrm{d}RL_t^{i\alpha}\varphi(t) = 0 \quad \forall i = 0,1,...,n$$

where $t \in [a,e]$.

Applying the modulating function method to system (13) and considering that Ass. 3 and $(P3)$ are fulfilled, results in

$$\int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t + \sum_{i=0}^{n-1} a_i \int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{i\alpha}\varphi(t)\,\mathrm{d}t = \sum_{j=0}^{m} b_j \int_e^g \tilde{u}(t)\, {}_e^t\mathrm{d}RL_t^{j\alpha}\varphi(t)\,\mathrm{d}t. \tag{21}$$

III. AUTOMATIC DETERMINING OF THE MODULATING FUNCTION

The drawback of the known modulating functions is the parameterization which has to be adapted to every considered system. To overcome this drawback, a control based method for automatic determining of a modulating function is described in this section. In subsection III-A, a precondition is given which allows the separate identification of any parameter of system (14). Using the properties given in subsection II-D and the formulated precondition, a model based auxiliary system is described in subsection III-B. In subsection III-C, it is shown that the energy-optimal control described in [18] minimizes the identification error occurring through additional noise.

A. Parameter Specific Modulating Function

In this subsection, a precondition is given to identify separately every parameter. Therefore, system (21) is rewritten as vector equation

$$\int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t$$
$$= \begin{bmatrix} \int_e^g \tilde{y}(t)\varphi(t)\,\mathrm{d}t & \ldots & \int_e^g \tilde{u}(t)\, {}_e^t\mathrm{d}RL_t^{m\alpha}\varphi(t)\,\mathrm{d}t \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ b_m \end{bmatrix} \tag{22}$$
$$= \underline{m}^{\top}(g)\,\underline{p}. \tag{23}$$

If the modulating function is chosen such that

$$\begin{bmatrix} \int_e^g \tilde{y}(t)\varphi_s(t)\,\mathrm{d}t & \ldots & \int_e^g \tilde{u}(t)\, {}_e^t\mathrm{d}RL_t^{m\alpha}\varphi_s(t)\,\mathrm{d}t \end{bmatrix} = \underline{e}_s^{\top} \tag{24}$$

where $\underline{e}_s \in \mathbb{R}^{\kappa\times 1}$ and $\kappa = n+m+1$ and

$$\{\underline{e}_s\}_k = \begin{cases} 1 & \text{for } k = s \\ 0 & \text{else} \end{cases} \tag{25}$$

is fulfilled, the s-th parameter p_s of $\underline{p}$ is identified.

Definition 3: Parameter Specific Modulating Function. A modulating function $\varphi(t)$ which fulfills $(P1)$, $(P2)$ and (24) is called parameter specific modulating function $\varphi_s(t)$.

B. Model Based Auxiliary System

In this subsection, a model based auxiliary system is defined in order to calculate the parameter specific modulating function defined in previous section. The idea is to replace every expression of the modulating function $\varphi(t)$ in $(P2)$ and (21) as a combination of ${}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)$ and a fractional integration with corresponding order. Then, the model based auxiliary system has to represent the connection of the different derivatives of the modulating function $\varphi(t)$, the applied modulating function method (21) and the boundary terms depending on ${}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)$.

This subsection is structured as follows. First, the boundary terms depending on ${}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)$ are given. Next, the structure of the model based auxiliary system composed of four subsystems is defined. Each subsystem is explained afterwards. Finally, the complete structure of the system matrix and the input vector of the model based auxiliary system are stated.

First, $(P2)$ is rewritten in

$$\frac{\mathrm{d}^k\, {}_e i_t^{\lceil i\alpha\rceil - i\alpha}\varphi(t)}{\mathrm{d}t^k} = {}_e^{t,\lambda\alpha+\lceil(n-\lambda)\alpha\rceil-k} i_t\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t) \tag{26}$$

where $k = 0,1,...,\lfloor(n-\lambda)\alpha\rfloor$ and $\lambda = 1,2,..,n$. A vector $\underline{\gamma}$ is defined to take the rewritten boundary terms which are given in (26) into account for $i = 0,1,...,n-1$.

Definition 4: Order Vector of Boundary Terms.

$$\underline{\gamma} := \begin{bmatrix} \underline{\beta}_1^{\top} & \underline{\beta}_2^{\top} & \ldots & \underline{\beta}_n^{\top} \end{bmatrix} \tag{27}$$

where

$$\underline{\beta}_k^{\mathsf{T}} := \begin{bmatrix} \lceil (n-k)\,\alpha \rceil + k\alpha & \lceil (n-k)\,\alpha \rceil - 1 + k\alpha \\ \dots & 1 + k\alpha \end{bmatrix} \tag{28}$$

and $z = r \sum_{k=1}^{n} \lceil (n-k)\,\alpha \rceil$ where

$$r := \min \left\{ k \in \mathbb{N}^+ \left| \frac{\max\{\underline{\gamma}\}}{k} < 1 \right. \right\} \tag{29}$$

and $\underline{\gamma} \in \mathbb{R}^{1\times z}$.

The next step is the definition of the model based auxiliary system which is based on (22) and (27). Following, the fractional pseudo state vector is defined.

Definition 5: Model Based Auxiliary System.

$$\begin{bmatrix} {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\alpha}{w}}\,\underline{x}^{\varphi}(t) \\ \dot{\underline{x}}^{\times}(t) \\ \dot{\underline{x}}^{\partial_n}(t) \\ {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\gamma}{r}}\,\underline{x}^{\partial_r}(t) \end{bmatrix} + \begin{bmatrix} \underline{\psi}\left(\underline{x}^{\varphi}(t), \frac{\alpha}{w}, a, e, t\right) \\ -\underline{x}^{\times}(e)\,\delta(t-e) \\ -\underline{x}^{\partial_n}(e)\,\delta(t-e) \\ \underline{\psi}\left(\underline{x}^{\partial_r}(t), \frac{\gamma}{r}, a, e, t\right) \end{bmatrix} = \underline{A}(t) \begin{bmatrix} \underline{x}^{\varphi}(t) \\ \underline{x}^{\times}(t) \\ \underline{x}^{\partial_n}(t) \\ \underline{x}^{\partial_r}(t) \end{bmatrix} + \underline{b} u^*(t) \tag{30}$$

$$u^*(t) := {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t) \tag{31}$$

where $\underline{x}(t) : \mathbb{R} \to \mathbb{R}^{nw+\kappa+\lceil n\alpha \rceil + z}$ and $u^*(t) : \mathbb{R} \to \mathbb{R}$. A normalization factor w

$$w := \min \left\{ k \in \mathbb{N}^+ \left| \frac{\alpha}{k} < 1 \right. \right\} \tag{32}$$

is used to meet the requirement that all orders of a fractional pseudo state space have to fulfill $\underline{\alpha} \in (0, 1]$. The model based auxiliary system consists of four sub-systems marked by φ, $\times$, ∂_n and ∂_r. Before system matrix $\underline{A}(t)$ and input vector $\underline{b}$ are presented, each subsystem is described. System matrix $\underline{A}(t)$ and input vector $\underline{b}$ will be a combination of the matrices and vectors of the subsystems. First, the fractional pseudo state vector is defined.

Definition 6: Fractional Pseudo State Vector.

$$\underline{x}^{\varphi}(t) := \begin{bmatrix} \underline{x}_0^{\varphi}(t) \\ \vdots \\ \underline{x}_{n-1}^{\varphi}(t) \end{bmatrix}, \quad \underline{x}_k^{\varphi}(t) := \begin{bmatrix} {}_e^t\mathrm{d}RL_t^{k\alpha}\varphi(t) \\ {}_e^t\mathrm{d}RL_t^{\frac{\alpha}{w}+k\alpha}\varphi(t) \\ \vdots \\ {}_e^t\mathrm{d}RL_t^{\frac{(w-1)\alpha}{w}+k\alpha}\varphi(t) \end{bmatrix} \tag{33}$$

$$\underline{x}^{\times}(t) := \underline{m}^{\mathsf{T}}(t) \tag{34}$$

$$\underline{x}^{\partial_n}(t) := \begin{bmatrix} {}_e^t\mathrm{d}RL_t^{n\alpha-\lceil n\alpha \rceil}\varphi(t) \\ {}_e^t\mathrm{d}RL_t^{n\alpha+1-\lceil n\alpha \rceil}\varphi(t) \\ \vdots \\ {}_e^t\mathrm{d}RL_t^{n\alpha-1}\varphi(t) \end{bmatrix} \tag{35}$$

$$\underline{x}^{\gamma}(t) := {}_e^t\mathrm{d}RL_t^{\underline{\gamma}}\varphi(t) \tag{36}$$

$$\underline{x}^{\partial_r}(t) := \begin{bmatrix} \underline{x}_1^{\partial_r}(t) \\ \vdots \\ \underline{x}_G^{\partial_r}(t) \end{bmatrix}, \quad \underline{x}_k^{\partial_r}(t) := \begin{bmatrix} \{\underline{x}^{\gamma}\}_k \\ {}_e^t\mathrm{d}RL_t^{\frac{\{\underline{\gamma}\}_k}{r}}\{\underline{x}^{\gamma}\}_k \\ \vdots \\ {}_e^t\mathrm{d}RL_t^{\frac{(r-1)\{\underline{\gamma}\}_k}{r}}\{\underline{x}^{\gamma}\}_k \end{bmatrix} \tag{37}$$

where $G = \sum_{k=1}^{n} \lceil (n-k)\,\alpha \rceil$ and $\underline{x}^{\varphi}(t) : \mathbb{R} \to \mathbb{R}^{nw\times 1}$, $\underline{x}^{\times}(t) : \mathbb{R} \to \mathbb{R}^{\kappa\times 1}$, $\underline{x}^{\partial_n}(t) : \mathbb{R} \to \mathbb{R}^{\lceil n\alpha \rceil \times 1}$ and $\underline{x}^{\partial_r}(t) : \mathbb{R} \to \mathbb{R}^{z\times 1}$.

After defining the model based auxiliary system and the fractional pseudo state vector, the subsystems which connect the fractional pseudo states with each other are explained. Subsystem φ is representing the connection between the different derivatives of the modulating function which are needed in (21). The connection is a chain of integrators whereby each integrator is of order $\frac{\alpha}{w}$ and this chain can be written as a system with a Jordan matrix where all eigenvalues are zero:

$$\begin{aligned} {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\alpha}{w}}\,\underline{x}^{\varphi}(t) &:= \begin{bmatrix} \underline{0}_{nw-1\times 1} & \underline{I}_{nw-1\times nw-1} \\ 0 & \underline{0}_{1\times nw-1} \end{bmatrix} \underline{x}^{\varphi}(t) \\ &\quad + \begin{bmatrix} \underline{0}_{nw-1\times 1} \\ 1 \end{bmatrix} u^*(t) \end{aligned} \tag{38}$$

$$= \underline{A}^{\varphi}\underline{x}^{\varphi}(t) + \underline{b}^{\varphi} u^*(t) \tag{39}$$

where $\underline{A}^{\varphi} \in \mathbb{R}^{nw\times nw}$ and $\underline{b}^{\varphi} \in \mathbb{R}^{nw\times 1}$.

Due to the application of the modulating function method, subsystem $\times$ and subsystem φ are connected by the input and output signal

$$\dot{\underline{x}}^{\times}(t) := \begin{bmatrix} -\tilde{y}(t) \cdot \underline{H}_{n\times nw} \\ \tilde{u}(t) \cdot \underline{H}_{m+1\times nw} \end{bmatrix} \underline{x}^{\varphi}(t) \tag{40}$$

$$= \underline{A}^{\times}(t)\,\underline{x}^{\varphi}(t) \tag{41}$$

where $\underline{A}^{\times}(t) : \mathbb{R} \to \mathbb{R}^{\kappa\times nw}$ and

$$\underline{H} := \begin{bmatrix} \underline{h}_1^{\mathsf{T}} \\ \vdots \\ \underline{h}_{\lambda}^{\mathsf{T}} \end{bmatrix}, \quad \{\underline{h}_k\}_l = \begin{cases} 1 & \text{if } l = (k-1)w + 1 \\ 0 & \text{else} \end{cases} \tag{42}$$

and $\lambda = n$ or $\lambda = m + 1$ depending on whether the output $\tilde{y}(t)$ or input signal $\tilde{u}(t)$ is considered.

The third subsystem ∂_n connects the control input $u^*(t)$ with the boundary terms $(P2)$ for $i = n$ and is a chain of integrators whereby each integrator is of order 1

$$\begin{aligned} \dot{\underline{x}}^{\partial_n}(t) &:= \begin{bmatrix} \underline{0}_{\lfloor n\alpha \rfloor \times 1} & \underline{I}_{\lfloor n\alpha \rfloor \times \lfloor n\alpha \rfloor} \\ 0 & \underline{0}_{1\times \lfloor n\alpha \rfloor} \end{bmatrix} \underline{x}^{\partial_n}(t) \\ &\quad + \begin{bmatrix} \underline{0}_{\lfloor n\alpha \rfloor \times 1} \\ 1 \end{bmatrix} u^*(t) \end{aligned} \tag{43}$$

$$= \underline{A}^{\partial_n}\underline{x}^{\partial_n}(t) + \underline{b}^{\partial_n} u^*(t) \tag{44}$$

where $\underline{A}^{\partial_n} \in \mathbb{R}^{\lceil n\alpha \rceil \times \lceil n\alpha \rceil}$ and $\underline{b}^{\partial_n} \in \mathbb{R}^{\lceil n\alpha \rceil \times 1}$.

The last subsystem ∂_r connects the control input $u^*(t)$ with the remaining boundary terms $(P2)$. Each boundary term is represented by a chain of integrators whereby each integrator is of corresponding order given in $\underline{\gamma}$ and can be interpreted as a subsystem of the subsystem ∂_r. Considering the k-th subsystem

$$\begin{aligned} {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\{\gamma\}_k}{r}}\,\underline{x}_k^{\partial_r}(t) := &\begin{bmatrix} \underline{0}_{r-1\times 1} & \underline{I}_{r-1\times r-1} \\ 0 & \underline{0}_{1\times r-1} \end{bmatrix} \underline{x}_k^{\partial_r}(t) \\ &+ \begin{bmatrix} \underline{0}_{r-1\times 1} \\ 1 \end{bmatrix} u^*(t). \end{aligned} \tag{45}$$

Combining all subsystems of subsystem ∂_r yields (46). Regarding the structure, the model based auxiliary system has chains of integrators which are parallel, but which have the same input.

Finally, all matrices and vectors of the subsystems are combined to the system matrix

$$\underline{A}(t) := \begin{bmatrix} \underline{A}^{\varphi} & \underline{0}_{nw\times\kappa} & \underline{0}_{nw\times\lceil n\alpha\rceil} & \underline{0}_{nw\times z} \\ \underline{A}^{\times}(t) & \underline{0}_{\kappa\times\kappa} & \underline{0}_{\kappa\times\lceil n\alpha\rceil} & \underline{0}_{\kappa\times z} \\ \underline{0}_{\lceil n\alpha\rceil\times nw} & \underline{0}_{\lceil n\alpha\rceil\times\kappa} & \underline{A}^{\partial_n} & \underline{0}_{\lceil n\alpha\rceil\times z} \\ \underline{0}_{z\times nw} & \underline{0}_{z\times\kappa} & \underline{0}_{z\times\lceil n\alpha\rceil} & \underline{A}^{\partial_r} \end{bmatrix} \tag{48}$$

and input vector of the model based auxiliary system

$$\underline{b} := \begin{bmatrix} \underline{b}^{\varphi} \\ \underline{0}_{\kappa\times 1} \\ \underline{b}^{\partial_n} \\ \underline{b}^{\partial_r} \end{bmatrix}. \tag{49}$$

C. Control Based Identification

A control input of the model based replacement system is a parameter specific control input, if it is derived from the parameter specific modulating function $\varphi_s(t)$. To determine such a control input, the initialization pseudo state and the final pseudo state have to be stated.

Definition 7: Parameter Specific Control Input.
A control input $u^*(t)$ steering the model based replacement system (30) into the final pseudo state

$$\begin{bmatrix} \underline{x}^{\varphi}(g) \\ \underline{x}^{\times}(g) \\ \underline{x}^{\partial_n}(g) \\ \underline{x}^{\partial_r}(g) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{e}_s \\ \underline{0} \\ \underline{0} \end{bmatrix} \tag{50}$$

from the initialization pseudo state

$$\begin{bmatrix} \underline{\psi}\left(\underline{x}^{\varphi}, \frac{\alpha}{w}, a, e, t\right) \\ -\underline{x}^{\times}(e) \\ -\underline{x}^{\partial_n}(e) \\ \underline{\psi}\left(\underline{x}^{\partial_r}, \frac{\gamma}{r}, a, e, t\right) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} \tag{51}$$

in a given time is called parameter specific control input $u_s^*(t)$.

Assuming such a parameter specific control input $u_s^*(t)$ is applied and considering the definition of the control input (31), (22) results in

$$\int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t = p_s. \tag{52}$$

Because the initialization pseudo state of the model based replacement system is zero, the control input of the model based replacement system has to be zero for $t \to -\infty$. Therefore, the modulating function is zero by definition for $t \in [a, e]$ and, hence, all occurring derivatives. So, the parameter specific control input fulfills $(P3)$.

In the following, the identification error which can occur because of noisy observation is considered.

Lemma 1: Identification Error.
The identification error is

$$\Delta p_s = \int_e^g \tilde{y}_e(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t - \underline{m}_e^{\mathsf{T}}(g)\,\underline{p} \tag{53}$$

where

$$\begin{aligned} \underline{m}_e^{\mathsf{T}}(g) := \Bigg[&\int_e^g \tilde{y}_e(t)\,\varphi_s(t)\,\mathrm{d}t \quad \dots \\ &\int_e^g \tilde{u}_e(t)\, {}_e^t\mathrm{d}RL_t^{m\alpha}\varphi_s(t)\,\mathrm{d}t \Bigg]. \end{aligned} \tag{54}$$

considering additive noise.

Proof: Assuming the observation of the input and output signal is superposed by additive noise (15) and (16) and inserting the noisy observation in (22) yields

$$\int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t = \underline{m}^{\mathsf{T}}(g)\,\underline{p}, \tag{55}$$

$$\begin{aligned} \int_e^g \tilde{y}_m(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t - \int_e^g \tilde{y}_e(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t = \\ \left(\underline{m}_m^{\mathsf{T}}(g) - \underline{m}_e^{\mathsf{T}}(g)\right)\underline{p} \end{aligned} \tag{56}$$

where

$$\begin{aligned} \underline{m}_m^{\mathsf{T}}(g) := \Bigg[&\int_e^g \tilde{y}_o(t)\,\varphi_s(t)\,\mathrm{d}t \quad \dots \\ &\int_e^g \tilde{u}_o(t)\, {}_e^t\mathrm{d}RL_t^{m\alpha}\varphi_s(t)\,\mathrm{d}t \Bigg]. \end{aligned} \tag{57}$$

Considering (24) for the measured data $\tilde{y}_o(t)$ and $\tilde{u}_o(t)$, (56) results in

$$\hat{p}_s - \int_e^g \tilde{y}_e(t)\, {}_e^t\mathrm{d}RL_t^{n\alpha}\varphi(t)\,\mathrm{d}t = p_s - \underline{m}_e^{\mathsf{T}}(g)\,\underline{p}. \tag{58}$$

Calculating the difference $\Delta p_s := \hat{p}_s - p_s$ leads to the in identification error given in Lem. 1. ■

Lemma 2: Minimum of Identification Error.
Using the energy-optimal control (106) in appendix leads to a minimal upper limit of the identification error

$$\begin{aligned} |\Delta p_s| \le\ & P\left(e, g, \underline{p}, \tilde{y}_e(t), u_s^*(t)\right) \\ & \cdot \sqrt{\frac{2}{\Gamma(2\delta)}\, {}_e^t i_g^{2-2\delta} J(u_s^*(t), e, g)} \end{aligned} \tag{59}$$

where $P\left(e, g, \underline{p}, \tilde{y}_e(t), \tilde{u}_e(t)\right)$ is given in (60).

$$
{}_{e}^{t}\underline{\mathrm{d}}RL_{t}^{\frac{\gamma}{r}}\underline{x}^{\partial_r}(t) := \begin{bmatrix} \underline{0}_{r-1\times 1} & \underline{I}_{r-1\times r-1} & & \\ 0 & \underline{0}_{1\times r-1} & & \\ & & \ddots & \\ & & \underline{0}_{r-1\times 1} & \underline{I}_{r-1\times r-1} \\ & & 0 & \underline{0}_{1\times r-1} \end{bmatrix} \underline{x}^{\partial_r}(t) + \begin{bmatrix} \underline{0}_{r-1\times 1} \\ 1 \\ \vdots \\ \underline{0}_{r-1\times 1} \\ 1 \end{bmatrix} u^*(t) \tag{46}
$$

$$
= \underline{A}^{\partial_r}\underline{x}^{\partial_r}(t) + \underline{b}^{\partial_r}u^*(t) \tag{47}
$$

where $\underline{A}^{\partial_r} \in \mathbb{R}^{z\times z}$ and $\underline{b}^{\partial_r} \in \mathbb{R}^{z\times 1}$.

$$
\begin{aligned}
P\left(e,g,\underline{p},\tilde{y}_e(t),\tilde{u}_e(t)\right) = &\left(1+\sum_{i=1}^{n}\frac{(g-e)^{2i}}{\Gamma(i)(2i-1)2i}\left|p_{n-i+1}\right|\right)\sqrt{\int_e^g \left|\tilde{y}_e(t)\right|^2} \\
&+\left(\sum_{j=n-m}^{n}\frac{(g-e)^{2j}}{\Gamma(j)(2j-1)2j}\left|p_{2n-j+1}\right|\right)\sqrt{\int_e^g \left|\tilde{u}_e(t)\right|^2}
\end{aligned} \tag{60}
$$

Proof: Applying the triangle inequality on (53) yields

$$
\left|\Delta p_s\right| \le \left|\int_e^g \tilde{y}_e(t)\,{}_{e}^{t}\mathrm{d}RL_{t}^{n\alpha}\varphi(t)\,\mathrm{d}t\right| + \left|\underline{m}_e^{\top}(g)\,\underline{p}\right|. \tag{61}
$$

Applying Cauchy-Schwarz inequality results in

$$
\begin{aligned}
\left|\Delta p_s\right| \le &\sqrt{\int_e^g \left|\tilde{y}_e(t)\right|^2}\sqrt{\int_e^g \left|u_s^*(t)\right|^2} \\
&+\sum_{i=1}^{n}\sqrt{\int_e^g \left|\tilde{y}_e(t)\right|^2}\sqrt{\int_e^g \left|{}_{e}^{t}\mathrm{d}RL_{t}^{(i-1)\alpha}\varphi_s(t)\right|^2}\left|p_i\right| \\
&+\sum_{j=0}^{m}\sqrt{\int_e^g \left|\tilde{u}_e(t)\right|^2}\sqrt{\int_e^g \left|{}_{e}^{t}\mathrm{d}RL_{t}^{j\alpha}\varphi_s(t)\right|^2}\left|p_{n+j+1}\right|.
\end{aligned} \tag{62}
$$

Next, a bound of the integrals depending on $\varphi_s(t)$ is calculated

$$
\int_e^g \left|{}_{e}^{t}\mathrm{d}RL_{t}^{(n-k)\alpha}\varphi_s(t)\right|^2 \mathrm{d}t = \int_e^g \left|{}_{e}^{t}i_{t}^{k\alpha}\,{}_{e}^{t}\mathrm{d}RL_{t}^{n\alpha}\varphi_s(t)\right|^2 \mathrm{d}t \tag{63}
$$

First, the fractional derivative is separated in a fractional integration of order $k\alpha$ and a fractional derivative of order $n\alpha$. Second, the order of integration is switched and the absolute value is drawn in the integral

$$
\int_e^g \left|{}_{e}^{t}\mathrm{d}RL_{t}^{(n-k)\alpha}\varphi_s(t)\right|^2 \mathrm{d}t \le {}_{e}^{t}i_{t}^{k\alpha}\int_e^g \left|u_s^*(t)\right|^2 \mathrm{d}t. \tag{64}
$$

Third, integrals are evaluated for $\alpha = 1$ considering mean value theorem [19]

$$
\int_e^g \left|{}_{e}^{t}\mathrm{d}RL_{t}^{(n-k)\alpha}\varphi_s(t)\right|^2 \mathrm{d}t \le \frac{(g-e)^{2k}}{\Gamma(k)(2k-1)2k}\int_e^g \left|u_s^*(t)\right|^2 \mathrm{d}t. \tag{65}
$$

Inserting in (62) yields

$$
\left|\Delta p_s\right| \le P\left(e,g,\underline{p},\tilde{y}_e(t),\tilde{u}_e(t)\right)\sqrt{\int_e^g \left|u_s^*(t)\right|^2} \tag{66}
$$

where $P\left(e,g,\underline{p},\tilde{y}_e(t),\tilde{u}_e(t)\right)$ is given in (60).

The integral of the parameter specific control input is rewritten using fractional integrals

$$
\int_e^g \left|u_s^*(t)\right|^2 = \frac{2}{\Gamma(2\delta)}\,{}_{e}^{t}i_{g}^{2-2\delta}\,{}_{e}^{t}i_{t}^{2\delta}\frac{\Gamma(2\delta)}{2}\frac{\left|u_s^*(t)\right|^2}{(g-t)} \tag{67}
$$

$$
= \frac{2}{\Gamma(2\delta)}\,{}_{e}^{t}i_{g}^{2-2\delta}J(u_s^*(t),e,t) \tag{68}
$$

where $\delta = \min\{\{\alpha\}_k, k=1,...,l\,|\{\alpha\}_k<1\}$. Because $J(u_s^*(t),e,t)$ is positive on $[e,g]$ and increases with increasing time, the maximum is reached for $t=g$

$$
\int_e^g \left|u_s^*(t)\right|^2 \le \frac{2}{\Gamma(2\delta)}\,{}_{e}^{t}i_{g}^{2-2\delta}J(u_s^*(t),e,g) \tag{69}
$$

and, therefore, the identification error is bounded by

$$
\begin{aligned}
\left|\Delta p_s\right| \le &P\left(e,g,\underline{p},\tilde{y}_e(t),u_s^*(t)\right) \\
&\cdot\sqrt{\frac{2}{\Gamma(2\delta)}\,{}_{e}^{t}i_{g}^{2-2\delta}J(u_s^*(t),e,g)}.
\end{aligned} \tag{70}
$$

Comparing $J(u_s^*(t),e,g)$ in (70) and (108) completes the proof. ■

IV. PRACTICAL APPLICABILITY OF THE AUTOMATIC DETERMINING

In subsection III-C, it is shown that the energy-optimal control minimizes the upper limit of the identification error considering additional noise. In classical case, the parallel chains of integrators would be a reason of non-controllability. Because of different fractional orders of the model based auxiliary system, it is possible that the auxiliary system is complete controllable. In this section, it is shown using the

Gramian (107) defined in [18] that the model based auxiliary system is controllable under some condition.

Evaluating (107) for the system matrix (48) and the input vector (49), the scaled Gramian can be written as

$$\underline{K}(e,g) := \int_e^g \underline{k}(\tau,g,\underline{A})\,\underline{k}^{\mathrm{T}}\left(\tau,g,\underline{A}^{\mathrm{T}}\right)\mathrm{d}\tau \tag{71}$$

where

$$\begin{aligned}
&\underline{k}^{\mathrm{T}}\left(t,g,\underline{A}^{\mathrm{T}}\right) = \\
&\Bigg[\frac{(g-t)^{n\alpha-1}}{\Gamma(n\alpha)}, \frac{(g-t)^{(nw-1)\frac{\alpha}{w}-1}}{\Gamma((nw-1)\frac{\alpha}{w})}, \ldots, \frac{(g-t)^{\frac{\alpha}{w}-1}}{\Gamma(\frac{\alpha}{w})}, \\
&-{}_t^t\mathcal{I}_g^{n\alpha}\tilde{y}(t), -{}_t^t\mathcal{I}_g^{(n-1)\alpha}\tilde{y}(t), \ldots, -{}_t^t\mathcal{I}_g^{\alpha}\tilde{y}(t), \\
&-{}_t^t\mathcal{I}_g^{n\alpha}\tilde{u}(t), -{}_t^t\mathcal{I}_g^{(n-1)\alpha}\tilde{u}(t), \ldots, -{}_t^t\mathcal{I}_g^{(n-m)\alpha}\tilde{u}(t), \\
&\frac{(g-t)^{\lceil n\alpha\rceil-1}}{\Gamma(\lceil n\alpha\rceil)}, \frac{(g-t)^{\lceil n\alpha\rceil-2}}{\Gamma(\lceil n\alpha\rceil-1)}, \ldots, 1, \\
&\frac{(g-t)^{\{\underline{\gamma}\}_1-1}}{\Gamma(\{\underline{\gamma}\}_1)}, \frac{(g-t)^{(r-1)\frac{\{\underline{\gamma}\}_1}{r}-1}}{\Gamma((r-1)\frac{\{\underline{\gamma}\}_1}{r})}, \ldots, \frac{(g-t)^{\frac{\{\underline{\gamma}\}_1}{r}-1}}{\Gamma(\frac{\{\underline{\gamma}\}_1}{r})}, \\
&\ldots, \\
&\frac{(g-t)^{\{\underline{\gamma}\}_k-1}}{\Gamma(\{\underline{\gamma}\}_k)}, \frac{(g-t)^{(r-1)\frac{\{\underline{\gamma}\}_k}{r}-1}}{\Gamma((r-1)\frac{\{\underline{\gamma}\}_k}{r})}, \ldots, \frac{(g-t)^{\frac{\{\underline{\gamma}\}_k}{r}-1}}{\Gamma(\frac{\{\underline{\gamma}\}_k}{r})}\Bigg]
\end{aligned} \tag{72}$$

where k indicates the last element in $\underline{\gamma}$ and $\left(\underline{k}^{\mathrm{T}}\left(t,g,\underline{A}^{\mathrm{T}}\right)\right)^{\mathrm{T}} = \underline{k}(t,g,\underline{A})$.

In the following, sets depending of the fractional order α and the system order n are given as well as a Q-Function which describes all fractional orders occurring in the model based auxiliary system. But before, auxiliary sets have to be defined.

Definition 8: Auxiliary Sets.

$$\mathcal{K}_1 := \{k_1 \in \mathbb{N} \,|\, 1 \le k_1 \le nw\} \tag{73}$$

$$\mathcal{K}_2 := \{k_2 \in \mathbb{N} \,|\, 1 \le k_2 \le \lceil n\alpha\rceil\} \tag{74}$$

$$\mathcal{K}_3 := \{k_3 \in \mathbb{N} \,|\, 1 \le k_3 \le r\} \tag{75}$$

$$\mathcal{K}_{4,k} := \{k_4 \in \mathbb{N} \,|\, 0 \le k_4 \le \lfloor k\alpha\rfloor\} \tag{76}$$

Definition 9: Set of Fractional Order.

$$\mathcal{M}_1 := \left\{m_1 \in \mathbb{R} \,\middle|\, \exists k_1 \in \mathcal{K}_1 : m_1 = k_1\frac{\alpha}{w} - 1\right\}, \tag{77}$$

$$\mathcal{M}_2 := \{m_2 \in \mathbb{N}^+ \,|\, \exists k_2 \in \mathcal{K}_2 : m_2 = \lceil n\alpha\rceil - k_2\}, \tag{78}$$

$$\mathcal{M}_3 := \mathcal{M}_{3,0} \cap \cdots \cap \mathcal{M}_{3,n-1}, \tag{79}$$

$$\begin{aligned}
\mathcal{M}_{3,k} := \Bigg\{m_3 \in \mathbb{R} \,\Bigg|\, &\exists k_3 \in \mathcal{K}_3 \wedge \exists k_4 \in \mathcal{K}_{4,k} : \\
&m_4 = k_3\frac{\lceil k\alpha\rceil - k_4 + (n-k)\alpha}{r} - 1\Bigg\}.
\end{aligned} \tag{80}$$

Definition 10: Q-Function.

$$\begin{aligned}
Q(t) := &\sum_{k=1}^{nw} q_{nw-k+1}\frac{(g-t)^{k\frac{\alpha}{w}-1}}{\Gamma\left(k\frac{\alpha}{w}\right)} \\
&+ \sum_{k=0}^{\lfloor n\alpha\rfloor} q_{nw+\kappa+1-k+1+\lceil n\alpha\rceil}(g-t)^k \\
&+ \sum_{i=0}^{n-1}\sum_{k=0}^{\lfloor j\alpha\rfloor}\sum_{l=1}^{r} q_{i,k,l}\frac{(g-t)^{l\frac{\lceil i\alpha\rceil-k+(n-i)\alpha}{r}-1}}{\Gamma\left(l\frac{\lceil i\alpha\rceil-k+(n-i)\alpha}{r}\right)}
\end{aligned} \tag{81}$$

Lemma 3: Controllability of the Model Based Auxiliary System.

The model based auxiliary system is complete controllable if the intersection of the sets $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ depending on fractional order α and the system order n is empty

$$\mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3 = \varnothing \tag{82}$$

and the input signal $u^*(t)$ is chosen, such that

$${}_\tau^\tau d_{t_e}^{\frac{nw-1}{w}\alpha+1} Q(\tau) + \sum_{j=0}^{m} q_{nw+n+j+1}\,{}_\tau^\tau d_{t_e}^{j\alpha}\tilde{u}(\tau) \equiv 0 \tag{83}$$

is only fulfilled for the trivial solution.

Proof: If the Gramian (107) of the model based auxiliary system is regular resp. the time functions of $\underline{k}^{\mathrm{T}}\left(t,g,\underline{A}^{\mathrm{T}}\right)$ are linear independent, the model based auxiliary system will be complete controllable (see [20]).

So, it has to be shown that $\underline{k}^{\mathrm{T}}\left(t,g,\underline{A}^{\mathrm{T}}\right)\underline{q} \equiv 0$ holds only true for $\underline{q} = \underline{0}$. In the following, $\underline{q} \neq \underline{0}$ is assumed. Considering the entries without output or input signal in (72), it is equivalent to the first term of the sum in (83). Because it is of polynomial type, single terms can only eliminate each other if and only if the exponents are equal. So, considering non-vanishing coefficients, $Q(t) \equiv 0$ can never occur if $\mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3 = \varnothing$ holds true.

Assuming that all elements of $\underline{q}_k = \underline{0}$ for the polynomial type elements of (72), the second term of the sum in (83) can be transformed into the form (14) by right-sided differentiation of order $n\alpha$

$$\begin{aligned}
&-q_{nw+1}\tilde{y}(t) - \cdots - q_{nw+n}\,{}_t^t\mathrm{D}C_g^{(n-1)\alpha}\tilde{y}(t) \\
&+ q_{nw+n+1}\tilde{u}(t) + \cdots + q_{nw+n+m+1}\,{}_t^t\mathrm{D}C_g^{m\alpha}\tilde{u}(t) \equiv 0
\end{aligned} \tag{84}$$

which is a FDE of order $(n-1)\alpha$. Because of (18) and (19), (14) is explicit and $\tilde{y}(t)$ and $\tilde{u}(t)$ can only solve FDEs of order $n\alpha$. So, $\tilde{y}(t)$ can only vanish if $\tilde{u}(t)$ is a homogeneous FDE

$$q_{nw+n+1}\tilde{u}(t) + \cdots + q_{nw+n+m+1}\,{}_t^t\mathrm{D}C_g^{m\alpha}\tilde{u}(t) \equiv 0 \tag{85}$$

which is excluded by Ass. 1.

Therefore, if the input signal is chosen according to (83) and the intersection of the sets (77)-(79) depending on fractional order α and system order n is empty, the model based auxiliary system is complete controllable. ■

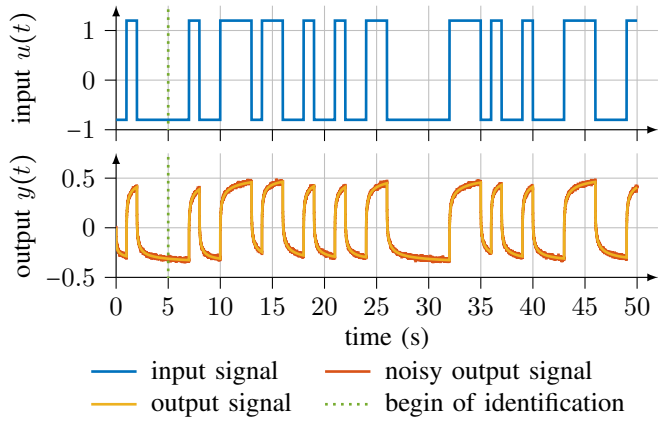

Fig. 1. Input and output signals for parameter identification.

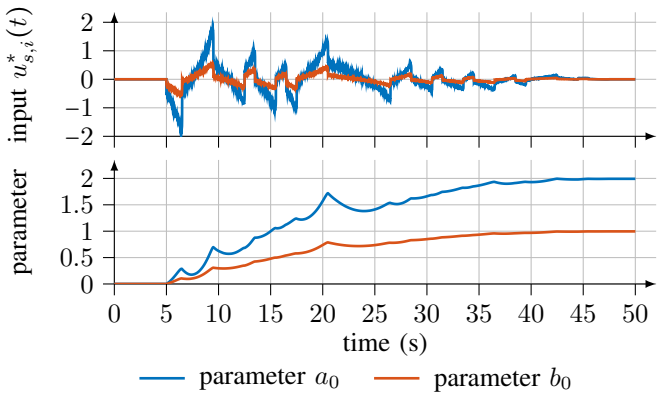

Fig. 2. Parameter specific control input and course of unknown parameter.

V. NUMERICAL EXAMPLE

In this subsection, the automatic determining of a modulating function and the parameter identification are illustrated. The following system is considered

$$ {}_t^{\mathrm{D}}C_g^{\alpha}\tilde{y}_m(t) + a_0\tilde{y}_m(t) = b_0\tilde{u}_m(t) \tag{86} $$

where $n = 1$, $m = 0$ and $\alpha = 0.35$ are assumed to be known and $a_0 = 2$ as well as $b_0 = 1$ are unknown. The simulation is started at $a = 0\,\mathrm{s}$ and the identification at $e = 5\,\mathrm{s}$. The duration of simulation is $50\,\mathrm{s}$ with a sampling time of $T_s = 0.01\,\mathrm{s}$ and the duration of identification is $42.5\,\mathrm{s}$. A pseudorandom binary sequence is used as an input signal. The input, output and noisy output (SNR = $30\,\mathrm{dB}$) signals are shown in Fig. 1. The starting time of identification is marked.

To state the model based replacement system, the fractional orders of the boundary terms have to be calculated, first. Evaluating (27) yields $\underline{\gamma} = 1.35$ and $r = 2$. Because $\lceil n\alpha \rceil = 1$, the following fractional pseudo state space yields

$$ \begin{bmatrix} {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\alpha}{w}}\underline{x}^{\varphi}(t) \\ \dot{\underline{x}}^{\times}(t) \\ \underline{\dot{x}}^{\partial_n}(t) \\ {}_e^t\underline{\mathrm{d}}RL_t^{\frac{\gamma}{r}}\underline{x}^{\partial_r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tilde{y}(t) & 0 & 0 & 0 & 0 & 0 \\ \tilde{u}(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}^{\varphi}(t) \\ \underline{x}^{\times}(t) \\ \underline{x}^{\partial_n}(t) \\ \underline{x}^{\partial_r}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u^*(t). \tag{87} $$

Depending on the requested parameter, here e.g. a_0, the final pseudo state is

$$ \underline{x}(g) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{88} $$

and, therefore, the parameter is identified by evaluating (22)

$$ \int_e^g \tilde{y}(t)\, {}_e^t\mathrm{d}RL_t^{\alpha}\varphi_1(t)\,\mathrm{d}t = a_0. \tag{89} $$

To calculate the parameter specific control input (72) has to be evaluated. Therefore, the matrix approach of [21] is used. Because of Lem. 6, the uninitialized integration can be used.

Because one measurement of the input and output signal can be used to identify all parameters, the parameter specific control input as well as the course of the parameters of a_0 and b_0 are illustrated in Fig. 2. At the end of the steering $t = 45\,\mathrm{s}$, parameter $a_0 = 2$ resp. $b_0 = 1$ in the absence of noise and $a_0 = 1.995$ resp. $b_0 = 0.996$ in the illustrated noisy case.

VI. CONCLUSIONS

The modulating function method is often used to identify the parameter of a fractional order model (see e.g. [4, 5, 6]). The main drawback is that a lot of parameters of the modulating function have to be adapted to the actual problem.

In this article, a model based auxiliary system is defined. This model based auxiliary system is used to determine a modulating function automatically. So, the drawback of heuristic parameterization of the modulating function is transferred into control of the model based auxiliary system. Another characteristic is that every parameter can be identified separately, although only one measurement is done. Considering additive noise, it is also shown that the energy optimal control leads to a minimal upper limit of the identification error regarding the used objective function. Another advantage using the model based auxiliary system is that no initialization function of the original system has to be considered as long as no initialization pseudo state of the model based auxiliary system is assumed. Considering the control of a system, the controllability of the system has to be shown. In the article, requirements are given which have to be fulfilled by the model based auxiliary system and its control input for complete controllability.

APPENDIX

A. SYSTEM THEORETICAL ASPECTS

1) Fractional Pseudo State Space:

The model based replacement system is a fractional pseudo state space. In [18], a fractional pseudo state space is defined as follows.

Definition 11: Fractional Pseudo State Space.

With k, $n_k \in \mathbb{N}$ and $n = \sum_{k=0}^{N} n_k$ follows

$$ {}_{e}^{t}\underline{\mathrm{d}}_{t}^{\underline{\alpha}}\underline{x}(t) + \underline{\psi}(\underline{x}(t), \underline{\alpha}, a, e, t) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t) \quad (90) $$

where $\alpha_k \in (0,1]$, $\underline{x}_k(t) : \mathbb{R} \to \mathbb{R}^{n_k}$ and $\underline{x}(t) : \mathbb{R} \to \mathbb{R}^{n}$ is denoted as fractional pseudo state vector, $\underline{\psi}_k(\cdot, t) : \mathbb{R} \to \mathbb{R}^{n_k}$ and $\underline{\psi}(\cdot, t) : \mathbb{R} \to \mathbb{R}^{n}$ is the initialization function vector, the system matrix $\underline{A}(t) : \mathbb{R} \to \mathbb{R}^{n \times n}$ as well as the input matrix $\underline{B}(t) : \mathbb{R} \to \mathbb{R}^{n \times p}$ are continuous and bounded matrix functions and input vector $\underline{u}(t) : \mathbb{R} \to \mathbb{R}^{p}$ is a piece-wise continuous vector function.

Remark 6: It should be noted that in Def. 11, depending whether the pseudo state space is based on (5) or (6), the specific expression has to be used.

Remark 7: The notation for the vectorial fractional derivative ${}_{e}^{t}\underline{\mathrm{d}}_{t}^{\underline{\alpha}}\underline{x}(t)$ in (90) is

$$ {}_{e}^{t}\underline{\mathrm{d}}_{t}^{\underline{\alpha}}\underline{x}(t) = \begin{bmatrix} {}_{e}^{t}\underline{\mathrm{d}}_{t}^{\alpha_1}\underline{x}_1(t) \\ {}_{e}^{t}\underline{\mathrm{d}}_{t}^{\alpha_2}\underline{x}_2(t) \end{bmatrix} = \begin{bmatrix} {}_{e}^{t}d_{t}^{\alpha_1} x_{1,1}(t) \\ {}_{e}^{t}d_{t}^{\alpha_1} x_{1,2}(t) \\ {}_{e}^{t}d_{t}^{\alpha_2} x_{2,1}(t) \\ {}_{e}^{t}d_{t}^{\alpha_2} x_{2,2}(t) \\ {}_{e}^{t}d_{t}^{\alpha_2} x_{2,3}(t) \end{bmatrix} \quad (91) $$

where $\underline{\alpha} \in \mathbb{R}^{2\times 1}$, $\underline{x}_1 \in \mathbb{R}^{2\times 1}$ and $\underline{x}_2 \in \mathbb{R}^{3\times 1}$, so $\underline{x} \in \mathbb{R}^{5\times 1}$.

2) Generalized Peano-Baker Series and R-Matrix Function:

In [22], the uninitialized generalized peano-baker series and the $\underline{R}$-matrix function are defined. These matrix functions are required to solve time-variant fractional pseudo state spaces of type (90). The drawback is that the integration of the system matrix $\underline{A}(t)$ can only be started at $t = a$ where $\forall \tilde{t} \le a : \underline{A}(\tilde{t}) = \underline{0}$ and $\forall \tilde{t} \le a : \underline{B}(\tilde{t}) = \underline{0}$ hold. Therefore, the both matrix functions are extended to the initialized case.

Definition 12: Initialized Generalized Peano-Baker Series

- Left-Sided F-Matrix

$$ \begin{aligned} \underline{F}_{\underline{\alpha}}(\tau, t, \underline{\Lambda}) &:= \underline{I} + {}_{\tau}^{\xi_1}\underline{\mathcal{I}}_{t}^{\underline{\alpha}}\underline{\Lambda}(\xi_1) \\ &\quad + {}_{\tau}^{\xi_1}\underline{\mathcal{I}}_{t}^{\underline{\alpha}}\underline{\Lambda}(\xi_1) \sum_{k=1}^{\infty}\prod_{j=1}^{k} {}_{\tau}^{\xi_{j+1}}\underline{\mathcal{I}}_{\xi_j}^{\underline{\alpha}}\underline{\Lambda}(\xi_{j+1}) \end{aligned} \quad (92) $$

  where $t \le \tau$

- Right-Sided F-Matrix

$$ \begin{aligned} \underline{F}_{\underline{\alpha},\mathrm{rs}}(t, \tau, \underline{\Lambda}) &:= \underline{I} + {}_{t}^{\xi_1}\underline{\mathcal{I}}_{\tau}^{\underline{\alpha}}\underline{\Lambda}(\xi_1) \\ &\quad + {}_{t}^{\xi_1}\underline{\mathcal{I}}_{\tau}^{\underline{\alpha}}\underline{\Lambda}(\xi_1) \sum_{k=1}^{\infty}\prod_{j=1}^{k} {}_{\xi_j}^{\xi_{j+1}}\underline{\mathcal{I}}_{\tau}^{\underline{\alpha}}\underline{\Lambda}(\xi_{j+1}) \end{aligned} \quad (93) $$

  where $\tau \le t$

Definition 13: Initialized R-Matrix

- Left-Sided R-Matrix

$$ \underline{R}_{\underline{\alpha}}(\tau, t, \underline{\Lambda}) := {}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(\underline{o}-\underline{\alpha})^{\top}}\underline{F}_{\underline{\alpha}}(\tau, t, \underline{\Lambda}) \quad (94) $$

  where $t \le \tau$ and $\underline{o}$ consists only of ones and has the dimension of $\underline{\alpha}$ (see Rem. 7). Here, the transpose operator $\top$ marks that the derivative is applied column-wise instead of row-wise.

- Right-Sided R-Matrix

$$ \underline{R}_{\underline{\alpha},\mathrm{rs}}(t, \tau, \underline{\Lambda}) := {}_{t}^{\tau}\underline{\mathrm{d}}RL_{\tau}^{(\underline{o}-\underline{\alpha})^{\top}}\underline{F}_{\underline{\alpha},\mathrm{rs}}(t, \tau, \underline{\Lambda}) \quad (95) $$

  where $\tau \le t$

The uninitialized RL-derivative in (94) and (95) is valid because the derivative with respect to the lower bound resp. upper bound is calculated. Therefore,

$$ \lim_{\tau \to t-} {}_{\tau}^{t}i_{t}^{\alpha} f(t) = 0 \quad (96) $$

and

$$ \lim_{\tau \to t+} {}_{t}^{t}i_{\tau}^{\alpha} h(t) = 0 \quad (97) $$

hold true by applying the definitions (see [12]). Because of the definition ${}_{\tau}^{t}\mathcal{I}_{t}^{\alpha} f(t) = 0, \forall \tau > t$ and ${}_{t}^{t}\mathcal{I}_{\tau}^{\alpha} f(t) = 0, \forall t > \tau$, it follows that no initialization in (94) and (95) is needed.

Lemma 4: Derivative Property of R-Matrix.

The derivative property of the left-sided

$$ {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\underline{R}_{\alpha}(\tau, t, \underline{\Lambda}) = \underline{\Lambda}(t)\,\underline{R}_{\alpha}(\tau, t, \underline{\Lambda}). \quad (98) $$

and right-sided R-Matrix

$$ {}_{t}^{t}\underline{\mathrm{d}}_{\tau}^{\alpha}\underline{R}_{\alpha,\mathrm{rs}}(t, \tau, \underline{\Lambda}) = \underline{\Lambda}(t)\,\underline{R}_{\alpha,\mathrm{rs}}(t, \tau, \underline{\Lambda}). \quad (99) $$

still holds true.

Proof: Because of the uninitialized RL-derivative in (98) and (94), the derivative order in (98) can be commuted (see [22]):

$$ \begin{aligned} {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\underline{R}_{\alpha}(\tau, t, \underline{\Lambda}) &= {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,{}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(1-\alpha)^{\top}}\underline{F}_{\alpha}(\tau, t, \underline{\Lambda}) && (100) \\ &= {}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(1-\alpha)^{\top}}\,{}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\underline{F}_{\alpha}(\tau, t, \underline{\Lambda}). && (101) \end{aligned} $$

Therefore, ${}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\underline{F}_{\alpha}(\tau, t, \underline{\Lambda})$ will be investigated first. Definitions (2) and (92) are applied in (102) and the uninitialized part is rearranged as described in [22].

Applying (103) in (101) yields (104). The result of the first summand is known (see [22]). Exchanging the derivative order and using that the lower bound is constant, the second summand becomes zero. Therefore, the derivative property of the R-Matrix

$$ {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\underline{R}_{\alpha}(\tau, t, \underline{\Lambda}) = \underline{\Lambda}(t)\,\underline{R}_{\alpha}(\tau, t, \underline{\Lambda}). \quad (105) $$

holds still true. The right-sided case follows these steps applying the right-sided definitions. ■

$$
\begin{aligned}
{}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,\underline{F}_{\alpha}\left(\tau,t,\underline{\Lambda}\right) &= {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,\underline{I} + {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\left[{}_{\tau}^{\xi_1}\underline{i}_{t}^{\alpha}\,\underline{\Lambda}\left(\xi_1\right) + \eta\left(\underline{\Lambda}\left(\xi_1\right),\alpha,a,\tau,t\right)\right] \\
&\quad + {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\left[{}_{\tau}^{\xi_1}\underline{i}_{t}^{\alpha}\,\underline{\Lambda}\left(\xi_1\right) + \eta\left(\underline{\Lambda}\left(\xi_1\right),\alpha,a,\tau,t\right)\right]\sum_{k=1}^{\infty}\prod_{j=1}^{k}{}_{\tau}^{\xi_{j+1}}\underline{\mathcal{I}}_{\xi_j}^{\alpha}\,\underline{\Lambda}\left(\xi_{j+1}\right) \qquad (102)\\
&= {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,\underline{I} + \underline{\Lambda}\left(t\right)\underline{F}_{\alpha}\left(\tau,t,\underline{\Lambda}\right) + {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\eta\left(\underline{\Lambda}\left(\xi_1\right),\alpha,a,\tau,t\right)\left(\underline{I} + \sum_{k=1}^{\infty}\prod_{j=1}^{k}{}_{\tau}^{\xi_{j+1}}\underline{\mathcal{I}}_{\xi_j}^{\alpha}\,\underline{\Lambda}\left(\xi_{j+1}\right)\right) \qquad (103)
\end{aligned}
$$

$$
\begin{aligned}
{}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(1-\alpha)^T}\,{}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,\underline{F}_{\alpha}\left(\tau,t,\underline{\Lambda}\right) &= {}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(1-\alpha)^T}\left({}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\,\underline{I} + \underline{\Lambda}\left(t\right)\underline{F}_{\alpha}\left(\tau,t,\underline{\Lambda}\right)\right) \\
&\quad + {}_{\tau}^{\tau}\underline{\mathrm{d}}RL_{t}^{(1-\alpha)^T}\left({}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\eta\left(\underline{\Lambda}\left(\xi_1\right),\alpha,a,\tau,t\right) + {}_{\tau}^{t}\underline{\mathrm{d}}_{t}^{\alpha}\eta\left(\underline{\Lambda}\left(\xi_1\right),\alpha,a,\tau,t\right)\sum_{k=1}^{\infty}\prod_{j=1}^{k}{}_{\tau}^{\xi_{j+1}}\underline{\mathcal{I}}_{\xi_j}^{\alpha}\,\underline{\Lambda}\left(\xi_{j+1}\right)\right) \qquad (104)
\end{aligned}
$$

3) Energy Optimal Control of a Fractional Pseudo State Space:

The energy optimal control of a pseudo state space (90) is given in [18]. Because the derivative properties (98) and (99) holds true in the initialized case, the energy optimal control remains as stated.

Definition 14: Energy Optimal Control.

$$
\begin{aligned}
\underline{u}\left(t\right) &= \left(g-t\right)^{2-2\delta}\underline{B}^{\mathsf{T}}\left(t\right)\underline{R}_{\underline{\alpha},\mathrm{rs}}\left(t,g,\underline{A}^{\mathsf{T}}\right)\underline{K}^{-1} \\
&\quad\cdot\left[\underline{x}\left(g\right) + \int_{e}^{g}\underline{R}_{\underline{\alpha}}\left(\tau,t,\underline{A}^{\mathsf{T}}\right)\underline{\psi}\left(\underline{x}\left(t\right),\underline{\alpha},e,g,t\right)\right]
\end{aligned} \qquad (106)
$$

where $\underline{\alpha}\in\mathbb{R}^{l\times 1}$, $\{\alpha\}_k\in(0,1], k=1,...,l$ and

$$
\begin{aligned}
\underline{K}\left(e,g\right) &= \int_{e}^{g}\left(g-t\right)^{2-2\delta}\underline{R}_{\underline{\alpha}}\left(\tau,g,\underline{A}\right)\underline{B}\left(t\right) \\
&\quad\cdot\underline{B}^{\mathsf{T}}\left(t\right)\underline{R}_{\underline{\alpha},\mathrm{rs}}\left(\tau,g,\underline{A}^{\mathsf{T}}\right)\mathrm{d}t
\end{aligned} \qquad (107)
$$

steers a time-variant fractional pseudo state space of type (90) from an initialization function vector $\underline{\psi}\left(\underline{x}\left(t\right),\underline{\alpha},e,g,t\right)$ to any given final pseudo state vector $\underline{x}\left(g\right)$ and minimizes the specific performance index regarding the control energy

$$
J\left(\underline{u}\left(t\right)\right) = \frac{1}{2}\int_{e}^{g}\left(g-\tau\right)^{2\delta-2}\underline{u}^{\mathsf{T}}\left(\tau\right)\underline{u}\left(\tau\right)\mathrm{d}\tau. \qquad (108)
$$

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