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High-Order Accuracy Approximation for a Two-Point Boundary Value Problem of Fourth Order with Degenerate Coefficients

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Abstract—High-order accurate finite element schemes for a fourth-order ordinary differential equation with degenerate coefficients on the boundary are constructed. The method for solving the problem is based on multiplicative and additive-multiplicative separation of singularities. For right-hand sides of the given class of smoothness, an optimal convergence rate is proved.

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We consider the fourth-order two-point boundary value problem

$$Au \equiv D^2\big(x^\alpha a(x) D^2 u(x)\big) - D\big(a_1(x) Du(x)\big) + a_0(x)u(x) = f(x),$$
$$x \in \Omega = (0,1), \quad D = d/dx; \qquad (1)$$

$$u(0) = Du(0) = u(1) = Du(1) = 0 \quad \text{for } \alpha < 1; \qquad (2)$$

$$u(0) = x^\alpha D^2 u(x)\big|_{x=0} = u(1) = Du(1) = 0 \quad \text{for } 1 \le \alpha < 3. \qquad (3)$$

The coefficients of Eq. (1) are assumed to satisfy the conditions $a(x) \ge c_0 > 0$, $a_1(x) \ge 0$, and $a_0(x) \ge 0$. Additional conditions on these functions will be specified later.

Since the coefficient $x^\alpha a(x)$ degenerates in a neighborhood of the point $x = 0$, the solution of the problem has unbounded derivatives in a neighborhood of this singularity. For efficient numerical solution of problems with degeneration, such singularities have to be taken into account. High-order accurate finite element schemes for a second-order degenerate equation were proposed in [1]. For $\alpha < 1$, the Dirichlet boundary value problem for Eq. (1) with $a_1 \equiv 0$ was considered in [2].

In this paper, we show that, for $\alpha < 1$, the solution can be represented in the form $u(x) = x^{2-\alpha}\hat{u}(x)$, while, for $1 \le \alpha < 3$, it can be represented in the form $u(x) = u_0\varphi_0(x) + x^{3-\alpha}\hat{u}(x)$, where φ_0 is a fixed function, $u_0 \in R$, and $\hat{u}$ is a new unknown function, which is smooth, as follows from a priori estimates. According to these representations, an approximate solution is sought in the form $u_h = x^{2-\alpha}\hat{u}_h$ for $\alpha < 1$ (multiplicative separation of the singularity [3]) and $u_h = z_0\varphi_0 + x^{3-\alpha}\hat{u}_h$ for $1 \le \alpha < 3$ (additive-multiplicative separation of the singularity [1]), where $\hat{u}_h$ is a piecewise polynomial function. For the indicated approximations with singularity separation, we obtain error estimates that are optimal in the energy norm.

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