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Investigation of Lagrange–Galerkin Method for an Obstacle Parabolic Problem

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Abstract—The convergence and accuracy estimates are proved for Lagrange–Galerkin method, used for approximating the parabolic obstacle problem. The convergence analysis is based on the comparison of the solutions of Lagrange–Galerkin and backward Euler approximation schemes. First order in time step estimate for the difference of the solutions for above schemes in energy norm is proved under sufficiently weak requirements for the smoothness of the initial data. First order in time and space steps accuracy estimate for Lagrange–Galerkin method is derived in the case of discontinuous time derivative of the exact solution.

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1. INTRODUCTION

In this paper we consider parabolic variational inequality with convection-diffusion operator and unilateral condition in the domain: find $u(t): \ (0,T) \to K$ such that a.e. $t \in (0,T)$

$$\big(u_t(t) + v(t)\cdot\nabla u(t) + Au(t) - f(t), \eta - u(t)\big) \geqslant 0 \quad \forall \eta \in K, \tag{1}$$

with $u(x,0) \in K$, where $K = \{u(x) \geqslant \psi(x) \text{ in } \Omega\}$ with a given function $\psi(x)$ is a closed and convex subset of Sobolev space $H^1(\Omega)$. We assume that operator A is a linear elliptic operator of second order and that convective term $v(t)\cdot\nabla u(t)$ is not dominated: both the ellipticity constant of A and a corresponding norm of the convective coefficient $v(t)$ are of order $O(1)$.

A powerful numerical method for solving convection-diffusion PDEs is the method of characteristics (called also as Lagrange–Galerkin method or Galerkin-characteristics method). In this method the material derivative is discretized along the characteristic curve, that ensures the robustness of the method for convection-dominated problems. The combination of characteristics and finite elements has been introduced in the eighties (see, for instance, [1] or [2]) in the context of continuum mechanics. Applying this method to the equation, we obtain a discrete system with a symmetric and positive definite matrix, which makes it possible to use effective solvers. In the case of parabolic variational inequality (1) similar discretization leads to the need to solve a quadratic programming problem on each time level, which is much simpler than implementing an implicit scheme for (1).

Parabolic variational inequalities, an example of which is (1), were thoroughly investigated in [3]. In particular, the existence of weak solutions were proved in the supposition of low regularity for input data. It is worth to note that regularity of the solutions of parabolic variational inequalities both in the

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