ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2018, Vol. 39, No. 7, pp. 949–956. © Pleiades Publishing, Ltd., 2018.

# Finite Element Approximation of the Minimal Eigenvalue of a Nonlinear Eigenvalue Problem

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(Submitted by A. V. Lapin)

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Received January 10, 2018

**Abstract**—The problem of finding the minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem for the ordinary differential equation with coefficients depending on a spectral parameter is investigated. This problem arises in modeling the plasma of radio-frequency discharge at reduced pressures. A necessary and sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem is established. The original differential eigenvalue problem is approximated by the finite element method on a uniform grid. The convergence of approximate eigenvalue and approximate positive eigenfunction to exact ones is proved. Investigations of this paper generalize well known results for eigenvalue problems with linear dependence on the spectral parameter.

**DOI:** 10.1134/S199508021807020X

Keywords and phrases: *Radio-frequency induction discharge, eigenvalue, positive eigenfunction, nonlinear eigenvalue problem, ordinary differential equation, finite element method.*

## 1. INTRODUCTION

In the present paper, we investigate the following differential nonlinear eigenvalue problem: find minimal eigenvalue $\lambda \in \Lambda$, $\Lambda = [0, \infty)$, corresponding to a positive eigenfunction $u(x)$, $x \in \Omega$, $\Omega = (0, \pi)$, $\overline{\Omega} = [0, \pi]$, satisfying the following equations

$$-(p(\lambda s(x))u')' = r(\lambda s(x))u, \quad x \in \Omega, \quad u(0) = u(\pi) = 0. \tag{1}$$

We assume that $p(\mu)$, $r(\mu)$, $\mu \in \Lambda$, and $s(x)$, $x \in \overline{\Omega}$ are continuous positive functions. We also assume that the function $p(\mu)$, $\mu \in \Lambda$ is bounded and the function $r(\mu)$, $\mu \in \Lambda$ is unbounded. Note that the differential equation of problem (1) is treated in the weak sense.

Nonlinear eigenvalue problems of the form (1) arise in modeling the plasma of radio-frequency discharge at reduced pressures. An inductive coupled radio-frequency discharge has found broad applications in diverse technological plasma processes, such as processing textiles and leather-fur half-finished products, metals, hydrogen accumulation by silicon powders, synthesis of oxygen-free ceramic materials, and obtaining carbide and boride materials for nuclear and processing industry [1–5]. A more effective and qualitative choice of constructive solutions in designing inductive coupled radio-frequency devices requires mathematical models, because some technological characteristics of the plasma cannot be measured.

In the present paper, a necessary and sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem is established. The original nonlinear differential eigenvalue problem is approximated by the finite element method with numerical integration on a uniform grid. The convergence of approximate minimal eigenvalue and

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