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# Avkhadiev—Becker Type Univalence Conditions for Biharmonic Mappings

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(Submitted by F. G. Avkhadiev)

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**Abstract**—In this paper we consider complex-valued biharmonic functions that are locally univalent. We construct families of biharmonic univalent mappings of the unite disc similar to Avkhadiev-Becker type conditions for analytic functions. Also, we investigate the case where biharmonic functions are defined on the exterior of the unit disc. In this case we obtain three analogs of Avkhadiev—Becker type conditions of univalence.

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*To 70th anniversary of my teacher,*
*Professor F.G. Avkhadiev*

## 1. INTRODUCTION

Let $f$ be a complex-valued biharmonic function on a simply connected domain $\Omega$. More precisely, we suppose that $f \in C^4(\Omega)$ and that $f$ satisfies the equation $\triangle^2 f = 0$ on $\Omega$, where

$$\triangle = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (z = x + iy)$$

is the Laplacian. It is known that every biharmonic function $f$ has the representation of the form

$$f(z) = h(z) + \overline{g(z)} + |z|^2 \left( h_1(z) + \overline{g_1(z)} \right),$$

where $h, g, h_1, g_1$ are functions holomorphic on $\Omega$. A planar harmonic mapping $f$ of a simply connected domain $\Omega \subset \mathbb{C}$ has a canonical decomposition $f = h + \overline{g}$, where $h$ and $g$ are holomorphic on $\Omega$ (see [15]). Clearly, every harmonic function is biharmonic.

Avkhadiev—Becker type univalence conditions for analytic functions are well known. These conditions have wide applications (see [8, 17]). In [10, 14, 18] the authors establish univalence conditions for harmonic functions. For example, in [10] Avkhadiev, Nasibullin, and Shafigullin, using the methods of Ahlfors and Weill [1], obtained Becker type univalence conditions for harmonic mappings of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and of the exterior of the unit disc $\mathbb{D}^- = \{z \in \overline{\mathbb{C}} : |z| > 1\}$. Namely, in [10] it is proved the following

**Theorem A** (see [10]).*Suppose that $h$ and $g$ are functions holomorphic on the unit disc $\mathbb{D}$ and that $h'(z) \neq 0$, $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, where $\omega(z) := g'(z)/h'(z)$. If the inequality*

$$|\omega(z)| + (1 - |z|^2) \left| z \frac{h''(z)}{h'(z)} \right| \leq 1, \quad \forall z \in \mathbb{D},$$

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