ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 3, pp. 352–362. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © M.F. Pavlova, E.V. Rung, 2018, published in Differentsial'nye Uravneniya, 2018, Vol. 54, No. 3, pp. 358–368.

PARTIAL DIFFERENTIAL EQUATIONS

On the Existence of a Generalized Solution of the Saturated-Unsaturated Filtration Problem

M. F. Pavlova* **and E. V. Rung****
Kazan Federal University, Kazan, 420008 Russia
*e-mail: *m.f.pavlova@mail.ru, **helenrung@mail.ru*

Received February 28, 2017

Abstract—We study the existence of a generalized solution of an initial–boundary value problem describing the process of unsteady filtration of a liquid in a bounded region of an n-dimensional space. We consider the case in which the Kirchhoff transformation used to determine the generalized solution takes the real axis into a semiaxis bounded below. An auxiliary problem is constructed. It is proved that any solution of the auxiliary problem is a solution of the problem under study. The solvability of the auxiliary problem is established by using the method of semidiscretization in time and the Galerkin method.

DOI: 10.1134/S0012266118030072

INTRODUCTION

In the present paper, we prove the existence of a generalized solution of a mixed initial–boundary value problem for the equation of nonlinear nonstationary filtration (see [1, p. 95 of the Russian translation])

$$m\frac{\partial s(p)}{\partial t} - \operatorname{div}\,(b(s(p))(\nabla p - \rho g)) = f(s(p)) \tag{1}$$

considered in a bounded domain Ω in the space $\mathbb{R}^n$, $n > 1$, where $p = p(x,t)$, $x \in \Omega \times [0,T]$, is the desired function, the functions s, b, and f and the positive constants m, ρ, and T are given, and g is the free fall acceleration vector. The variables occurring in Eq. (1) have the following physical meaning: $p(x,t)$ is the pressure, $s(p)$ is the saturation, $b(s(p))$ is the relative phase permeability, m is the medium porosity, ρ is the liquid density, and t is time.

The physical content of the saturated-unsaturated filtration problem ensures the following properties of the functions s and b, which are assumed to be satisfied in what follows.

1. The function $s = s(p) : \mathbb{R} \to [0,1]$ in the regions of saturated filtration (for $p \geq 0$ in the present paper) is equal to 1, and in the regions of incomplete saturation (for $p < 0$), $s(p)$ is a continuous nondecreasing function ranging from 0 to 1, with $\lim_{p\to-\infty} s(p) = 0$.

2. The function $b : [0,1] \to [0,1]$ is continuous and nonnegative, and $b(0) = 0$.

Properties (1) and (2) imply the degeneracy of the spatial operator in Eq. (1). Therefore, to study the solvability of the initial–boundary value problems for Eq. (1), one often uses the Kirchhoff transformation

$$\vartheta(p) = \int_0^p b(s(\xi))\,d\xi \tag{2}$$

to pass from the unknown function $p(x,t)$ to a function $u(x,t) = \vartheta(p(x,t))$ satisfying the equation

$$m\frac{\partial \widetilde{\varphi}(u)}{\partial t} - \operatorname{div}\,(\nabla u - b(\widetilde{\varphi}(u))\rho g) = f(\widetilde{\varphi}(u)), \qquad \widetilde{\varphi}(u) = s(\vartheta^{-1}(u)). \tag{3}$$