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Comment on integrability in Dijkgraaf–Vafa β-ensembles

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A B S T R A C T

We briefly discuss the recent claims that the ordinary KP/Toda integrability, which is a characteristic property of ordinary eigenvalue matrix models, persists also for the Dijkgraaf–Vafa (DV) partition functions and for the refined topological vertex. We emphasize that in both cases what is meant is a particular representation of partition functions: a peculiar sum over all DV phases in the first case and hiding the deformation parameters in a sophisticated potential in the second case, i.e. essentially a reformulation of some questions in the new theory in the language of the old one. It is at best obscure if this treatment can be made consistent with the AGT relations and even with the quantization of the underlying integrable systems in the Nekrasov–Shatashvili limit, which seem to require a full-scale β-deformation of individual DV partition functions. Thus, it is unclear if the story of integrability is indeed closed by these recent considerations.

1. Introduction

Nowadays the abstract matrix model theory [1] is once again on the rise. One of the reasons for that is that the reformulation of the Virasoro constraints or loop equations [2] in terms of the AMM/EO topological recursion [3] allowed to reveal hidden matrix model structures in somewhat unexpected areas like Seiberg–Witten theory and conformal models (through the AGT relations [4]) [5] and knots [6]. This poses the natural questions of how the other properties of matrix models express themselves in these circumstances. The first in the line is, of course, integrability: a mysterious fact that exact (non-perturbative) partition functions in quantum field theory satisfy *bilinear* relations (while usual Ward identities, like Virasoro constraints, provide only *linear* relations) [8].

The ordinary partition functions of eigenvalue matrix models are typically the τ-functions of the KP/Toda type hierarchies [1,7]. Among other things, this fact is reflected in existence of the Harer–Zagier recursion [9], a much more powerful than the ordinary AMM/EO one. However, this property is lost (or, better, modified in a still unknown way) in the two important deviations: after the β-deformation [10] and in the Dijkgraaf–Vafa phases [11]. Recently there were claims to the opposite: that integrable structure survives, moreover, in both cases and presumably even in the combination of two. The goal of this Letter is to briefly comment on this kind of statements. We choose two particular examples: the papers [12] on β-deformation and [13] on the Dijkgraaf–Vafa phases. In both cases the claim seems to reduce just to the statement that deformed model can be considered as a particular case of the non-deformed one, thus, integrability of the ordinary Hermitian matrix model implies bilinear relations for the deformed ones. This is, of course, being a correct statement does not provide any new interesting implications. In particular, this does not help to construct any efficient Harer–Zagier recursion, which would not be just a series in powers of $(\beta-1)$ or a result of peculiar summation over all the Dijkgraaf–Vafa phases. We remind [14] that resolution of this problem could provide a constructive interpretation of the AGT relations as the Hubbard–Stratonovich duality [15] in the doubly-quantized Seiberg–Witten theory (i.e. that in the Ω-background with the both non-zero deformation parameters[1]).

2. Integrability of Hermitian matrix model

The old statement [18,1,7] is that the integral

$$Z_N=\frac{1}{N!}\prod_{i=1}^{N}\int d\mu_i\, e^{V(\mu_i)}\Delta^2(\mu)=\det_{ij} C_{i+j} \tag{1}$$

where Van-der-Monde determinant $\Delta(\mu)=\prod_{i<j}(\mu_i-\mu_j)=\det_{ij}\mu_i^{j-1}$ and the moment matrix

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[1] When only one ϵ is non-vanishing, this corresponds to an ordinary quantization [16] of the underlying integrable system [17].