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On the Problem of Boundedness of a Signed Measure on Projections of a von Neumann Algebra

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Let M be a von Neumann algebra and $M^{\sqcap}$ be the set of all orthogonal projections in M. We call a mapping $\eta: M^{\sqcap} \to \mathbb{C}$ a signed measure on M if η is totally orthoadditive, that is, $\eta(\sum_{i \in I} P_i) = \sum_{i \in I} \eta(P_i)$ for $P_i \in M^{\sqcap}$, $P_i \perp P_j$ $(i \neq j)$. Here the condition of boundedness is usually required for the effective study and application of signed measures. So a natural problem of the existence of unbounded signed measures arises. In the present paper it is proved that any signed measure on the set of projections of a continuous von Neumann algebra is bounded. This fact is generalized also for vector-valued measures. © 1992 Academic Press, Inc.

Let M be a von Neumann algebra and $M^{\sqcap}$ be the set of all orthogonal projections in M. A function $\eta: M^{\sqcap} \to \mathbb{C}$ is said to be a signed measure on M if $\eta(\sum_{i \in I} P_i) = \sum_{i \in I} \eta(P_i)$ for $P_i \in M^{\sqcap}$, $P_i \perp P_j$ $(i \neq j)$.

Here the convergence of an uncountable family of summands means that there exists only a countable set of nonzero terms in the family and the usual series with these summands converges absolutely.

It is the condition of boundedness which is usually required in the definition of a signed measure. Bounded signed measures have been examined in detail already (see, for example, [3]). So a natural problem of the existence of unbounded signed measures arises. This problem has been solved for a type I von Neumann algebra [5, 6]. In the present paper this problem is solved for a continuous von Neumann algebra. Then the result obtained is generalized for a vector-valued measure.

Let M be a von Neumann algebra acting in a complex Hilbert space H and $S = \{f \in H: \|f\| = 1\}$.

DEFINITION 1. Let $P, Q \in M^{\sqcap}$. We say that the projection P is separated

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