ECONOPHYSICS Section

ZETA-MANAGEMENT: CATEGORICAL AND FRACTIONAL DIFFERENTIAL APPROACHES

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Abstract. *The aim of this note is to introduce and justify the reasons why the traditional differential approach of complex systems, and more specifically non-additive systems, must be recognized as an epistemological failure (e.g. in economy, finance or limited in social agent models). The categorical character of the context proper to any type of irreversible exchange is analyzed. This approach underlines and explains the weaknesses of the set theory generally utilized. Beyond the mathematical concepts their application in project management provides an illustrative example allowing an easy understanding of the statements of problems attached to non-additive and irreversible complex system. We will see why it is necessary to shift the analysis from the set theory toward the theory of categories and why this choice very naturally introduces the use of non-integer order Differential Equations.*

Keywords: *Categories Theory, Fractal and Hyperbolic Geometry, Fractional Differential Integral, Irreversibility, Management.*

1. Introduction

In his book "Time and Economy" Zeljko Rohatinski [1] severely criticizes the "Euclidean" models of economics an finances in a frame of an approach very convergent with the criticisms of Benoit Mandelbrot [2,3,4] Jean Philippe Bouchaud [5,6] and many others [7-11] concerning application of physical concepts in economy. The question of the status of the time considered in economics appears as a central question for the authors of this note [12] as well as for Z. Rohatinski who, -starting from large overview on finance and equilibrium economics [13-26]-, clarifies the bases of his premonitory works according to his following analysis:

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The literature specifically devoted to the time dimension of economic activity is relatively scarce, but there are several researchers that explore some segments of the problem presented here. In 1950, English economist George L. S. Shackle was one of the first economists to argue that economics that use equilibrium methods ignore the dimension of time and who questioned the validity of the mechanical time dynamics models. He focused on the dynamic movement in time: translation of the moment-in-being along the calendar axis (outside) and from one moment in to another (inside). A recently published analysis of Shackle's work (Madsen, 2015) additionally pointed to the specific nature of time in economics compared to other sciences.

In 1973, Branko Horvat, one of the most prominent Croatian economists, explored the problems of contraction of costs of fixed capital and dilation of "economic time" in evenly growing economic systems and showed that every economic system has its own, inherent "economic time", which is in line with the conclusion reached in this book. By contrasting the atemporal Marshallian model with an explicit time model with uncertainty about costs at the firm level, the American Nobel Laureate Peter A. Diamond in 1994 focused on modalities of establishing micro- and macroeconomic equilibria in periods of different durations (…) Although these approaches differ significantly, they all point to the essential problems related to the attempts to "fit" economic activity in the standard units of calendar time during the process of establishing partial or general equilibrium in the economic system. This problem is even more complex in the context of all causal, structural, and (dis)equilibrium relations between economic categories [1]. In practical terms interesting these issues, the management of the economics must consider at least two levels of analysis; (i) the static and local level, namely the issues relating to smart organizations that are flexible enough to adapt adequately to the constraints of the environment [27]; (ii) the dynamic and global level, in other words the questions relating to the "turns" able to alter the overall structure of the organizations themselves and their links by diffusion of the local constraints on the global behavior by involving the set-power, namely the equivalences of some classes of subsets. In addition to the work manages to link the distribution of multi temporalities to the creation of currency (credit) seen through investment strategy [12] and to Lorentzian Rohatinski approach [1], one could now add the works based on Project

Management extended to Zeta Management. The theoretical framework for this approach is Grothendieck topoi and sites theory [28], topos which is roughly *a category that serves as a place in which one can do mathematics* [29]. More precisely, a topos is category with certain extra properties that make it a kind of category of sets. There are many different topoi; we can do a lot of the same mathematics in all of them, but there are also many differences between them. For example, the axiom of choice cannot hold in a topos, and likewise the excluded middle principle. The reason is that the truth in a topos is not a yes-or-no affair: instead, we keep track of "how" true statements are, or more precisely where (in what topos) they are true. Some but not all topoi contain natural numbers as object. This approach is here analyzed from self-similar categorical point of view [30]. More precisely, it is by considering the notion of time within the framework of the conservation of categorical limits and co-limits that we shall extend the differential approach of the economy (equilibrium) to the non-integer orders and irreversible frameworks [12,31, 32]. We will show how in topoi, the time is both relativized and discretized by long-distance correlations especially for applications in which economic ends are considered. The econo-physics approach can be bound in category theory with the use of "division algebras" according to a deductive layout implementing the concepts and the morphism given in the diagram below. The purpose of this paper is to provide the mathematical underpinnings that justify among others analysis the Rohatinski's approach by linking it to incompleteness of fractional differentials processes. The main example of this note based on the (project) management and otherwise founded on self-similar structure of labor division, may easily be related to the creation of generalized currency already considered and shown as a generalized investments means [12]. This note will provide the tool of representation and illustration of these assertions. It will be noticed that to "economize" is theoretically to make choices, to rank them and to order them and index arbitrations on the basis of the set of integers seen as small categories. These aspects will be considered in the theoretical part of this note.

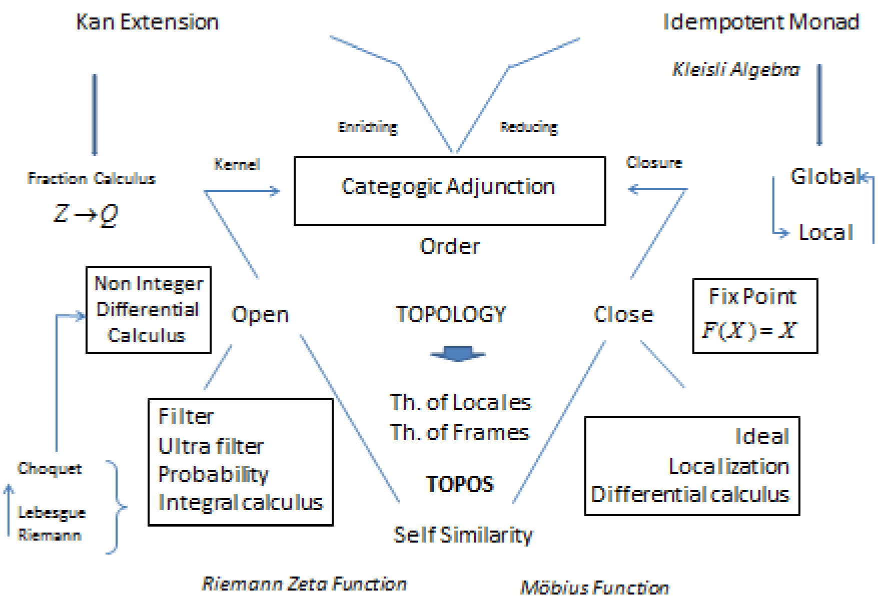

Figure 1. Schematic representation of the mathematical links that justify the developments given in the note, namely the use of category theory and of fractional differentiation operators in economics on the basis of non-additivity of integrals (Choquet integrals). This non-additivity underlines the specificity of the economy with respect to physics including quantum physics and field theory. The properties of self-similarity appear as naturally universal in category theory, the anthropic character of the theory allowing the emergence of reference laws of behavior in spite of the infinity of free factors related to humanity of the issues.

2. Categories, Points Logic and Topoi

In his work on *The truth of the beautiful in music* [33,34] Guerino Mazzola, among other authors, figures out that the concept of Euclidean point as an object without any subpart, disappears at the end of the 19th century especially with the progress of the algebraic analysis, soon transformed into algebraic geometry. It is in this rapidly evolving context that Saunders Mac Lane and Samuel Eilenberg develop the theory of categories [35-39] by first working on the natural transformations, namely morphism of morphisms.

A category C is first a collection of objects X, Y, ... as for any couple XY, a collection of new objects $C(X, Y)$ exists such as f, g, ... which are called morphisms of X into Y. These morphisms are denoted arrows; for instance, $f : X \rightarrow Y$. The sole axioms necessary to define the notion of categories are then the following ones: (i) the objects of the category are exclusively determined by their morphisms on the one hand as domain of definition, $X = dom(f)$ namely the source or empennage of the arrow and as codomain, $Y = codom(f)$ or goal, target or arrowhead; (ii) the laws of composition apply when a target of f becomes the empennage of g then $h : h = f \circ g$ exists and the law of composition is associative; finally (iii) there is a morphism called identity such that the arrow is a looping over itself with $Id_X : X \rightarrow X$. Due to this structure $f : X \rightarrow Y$ and $g : Y \rightarrow X$ assign a role to commutation with a distinction between the left $Id_X \circ g = g$ and the right $f \circ Id_X = f$, one of the major characteristics of the category theory with regard to set theory.

Among the categories we find for example the sets (Ens), the groups (Grp), the topological spaces (Top) for which the morphisms are the continuous functions; one finds also the oriented graphs for which the morphisms are the category of the schemas in geometry algebraic. The abstraction of the notion of categories is due to the fact that the only means of accessing objects are only the morphisms of which they are the domains and codomains. The theory of categories thus asserts that there is no object in itself, but in doing so it paradoxically confers a concrete character on all categorical abstractions. For example, a geometry is reconstructed by generalizing, as Grothendieck has shown, the notion of point [28].

The point can then be identified from a set function x: $1 \rightarrow X$ defined over $1 = \{\emptyset\}$ is such a way that $x(\emptyset) = x$ actually the function that sends the element $\emptyset$ of 1 into x. This morphism translates the concept set point into the categorical concept of domain of morphism 1. Grothendieck generalizes this approach by asserting that any morphism p: $A \rightarrow X$ given in the category C, is an A-valuated point of X. We call A the address of the point, namely $A@X$ [34]. The introduction of the arrow-point of Grothendieck creates a "mathematical subject" disassociated of the object [33]. The arrow has a target: the object X but also an empennage which is none other than the partial and subjective point of view given by the address A. The lemma of Yoneda, another great mathematician of the theory of the category, [34] asserts that the knowledge of the object X can be reduced to the set of the addresses $@X$ In other words, knowledge of an object X is reducible to the knowledge of the set of all its points $@X$ addresses (therefore the set of its morphisms p namely all the perspectives able to be considered with X as object). The space $@X$ is what mathematicians call the contravariant functor of X. If X is a topological space with a category of open object, then this contravariant functor is called the presheaf of X into any other category. A sheaf is a presheaf that can be defined locally. In geometry, the concept of a sheaf is a generalization of the notion of set of sections of vector bundles, hence its major interest for dealing with dynamic problems. In this context X becomes an algebraic manifold or a differential manifold in which the global concept of geodesic makes sense. In the wake of Yoneda we also locally recover all the logic that derives from the global functorial structure which leads quite naturally to the properties of self-similarity and fractality, the universality of which can thus be deeply understood from mathematics [30] and from physics of irreversible processes (arrows [12,31,32]). Geometry associated to the Grothendieck's points [28] matches logical structures (values of truths). The notion of topos of often emerges naturally as the conjunction of several categorical properties: (i) the existence of a sub-object classifier; (ii) a validity of the principles of functional division with relevance of the concepts of limit and colimit, and finally (iii) the existence of an exponential function Y^X guaranteeing the equivalence of the monoids $(C, +)$ namely $X \rightarrow X + Y \leftarrow Y$ and $(C, \times)$ namely $X \leftarrow X \times Y \rightarrow Y$; equivalence classes exist, in the frame set-power (all subsets of a set) the inclusion being the classifier.

3. Categorical Adjonction Order, Scaling and self Similarity

Given that, in the theory of categories, every object is always equipped with an arrow (called morphism) as well as an identity morphism, an ordered sets constitute an archetypal case of categories in which between any two objects there exists at most only one arrow. We will show that the notion of lattice becomes then a fundamental notion. Given P and Q a couple of ordered categories, associated with two morphisms ϱ and λ:

$$P \overset{\lambda}{\rightarrow} Q \text{ and } Q \overset{\rho}{\rightarrow} P.$$

P and Q are adjoint if and only if: $\forall\, p \in P, \forall\, q \in Q\; p \leq \rho(q) \Leftrightarrow \lambda(p) \leq q$. These properties figure out the following relations: ϱ and λ are "isotoneous" applications (namely preserve the order) and they verify: $p \leq \rho(\lambda(p))$ and $\lambda(\rho(q)) \leq q$. We can introduce the principal concept of ideal (Kernel like in a ring) and of filter, namely:

- Principal Ideal $(p] = \{x \in P: x \leq p\}$ as "small" elements,
- Principal Filter $[p) = \{x \in P: p \leq x\}$ as "large" elements.

Therefore: $P \overset{\lambda}{\rightarrow} Q$ has an adjoint at right if λ is residuated, that is, if the reverse (pull back) images of principal ideals by λ are also principal ideals. Its adjoint at right $Q \overset{\rho}{\to} P$ is then given by: $\rho(q) = \max\{p \in P: \lambda(p) \leq q\}$. Through duality, $Q \overset{\rho}{\to} P$ has an adjoint at left if and only if ϱ is residual, namely if the inverse images of the principal filters by ϱ are also principal filters. Its adjoint at left $P \overset{\lambda}{\to} Q$ is then given by: $\lambda(p) = \min\{q \in Q: p \leq \rho(q)\}$. Any residuated application between partially ordered sets preserves all existing joins (union); and any residual application between partially ordered sets preserves all existing meetings (intersections). Conversely, any application preserving the joints (respectively the meetings) between complete lattices is residuated (respectively residual). If $\lambda: P \overset{\lambda}{\to} Q$ application between complete lattice preserves the joins, then $Q \overset{\rho}{\to} P : q \mapsto \vee\{p \in P: \lambda(p) \leq q\}$ is adjoint at right with regard to λ. If $Q \overset{\rho}{\to} P$ application between complete lattice preserves the meets, then $P \overset{\lambda}{\to} Q: p \mapsto \wedge\{q \in Q: p \geq \rho(q)\}$ is adjoint at left with regard to ϱ [40,41].

There is a close link between adjunction and the notions of closure and kernel. A close operation on a partially ordered space P is application $\gamma : P \to P$ such as $x \leq y \Leftrightarrow \gamma(x) \leq \gamma(y)$. Thus the closing operations are in one-to-one correspondence, by surjective restriction, with the adjoint on the left of inclusions between ordered sets. A kernel operation is defined by duality; it is an adjoint at right of an inclusion. A subset C of P is called closing domain if whatever $\forall\, x \in P$, a smallest element exists $\bar{x} \in C$ with $x \leq \bar{x}$. Then $\gamma_C : P \to P : x \mapsto \bar{x}$ Is a closure operation. The association $C \mapsto \gamma_C$ defines an isomorphism between the partially ordered set of closure by inclusions. Closing domains in a complete lattice are the subsets closed by the meetings. Any addition (λ, ϱ) defines a closure $\varrho \circ \lambda$, written $\lambda\varrho$, but also an operation of kernel $\lambda \circ \varrho$, written $\varrho\lambda$, based on: $p \leq \rho(\lambda(p))$ et $\lambda(\rho(q)) \leq q$, and of resulting equations: $\lambda\varrho\lambda = \lambda$ et $\varrho\lambda\varrho = \varrho$. The images of λ and of ϱ are written $P\lambda$ and $Q\varrho$. Therefore the following diagram emerges of the relations:

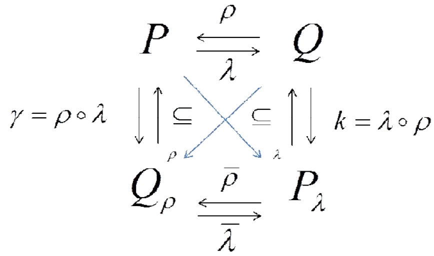

Figure 2. Schematic representation of Kernel and Closure morphisms.

By analogy with a lattice in *N*, the addition consists locally in forming the two-term line of a lattice and then reconstructing from this line all the relations which give the limits of the construction. A Galois connection [41], mapping posets, is distinguished from previous organization by inversion of the order of Q. This inversion figures out: $p \leq \rho(q) \Leftrightarrow q \leq \lambda(p)$, namely: ϱ et λ inverse the order and $p \leq \rho(\lambda(p))$ and $q \leq \lambda(\rho(q))$. The fixed points on both sides of a Galois connection are called Galois closure. For an adjunction given by (λ, ϱ) the fix points of $\lambda \circ \varrho$ are called Galoisian open (set), because $\lambda \circ \varrho$ is then a kernel operation. Consider a simple example. Either a subset Y of a complete lattice such as L : $\lambda_Y : L \to L : x \mapsto \wedge \{y \in Y : x \leq y\}$ is a closure operation; Then for any "isotoneous" operation γ of L: $\rho(\gamma) = \{x \in L : \gamma(x) \leq x\}$ which is a domain of closure. The pair (λ, ϱ) is then a Galois connection between the power set of L, written $\wp L$ (set of all subsets of L) is the complete lattice punctually ordered from "isotoneous" internal applications of L. Closed Galois subsets are exactly the closing domains; closed Galois applications are closing operations. For an addition, or a Galois connection, (λ, ϱ) between two partially ordered set *P* and Q, the following conditions are equivalent: (i) $\lambda \circ \varrho = id_Q$, (ii) $\lambda \circ \varrho$ injective, (iii) $\lambda \circ \varrho$ surjective, (iiv) ϱ injective, (iv) λ surjective, and mainly for the physicist:

- $\mu : P \to N$ exists such as $\mu \circ \rho$ is a scaling factor,
- $\tilde{\mu} : Q \to N$ exists such as $\tilde{\mu} = \mu \circ \lambda \circ \rho$ is also a scaling factor

A scaling factor is by definition a function on a partially ordered set *P* such that if $p < p'$ then $\mu(p) \neq \mu(p')$. In this case the following diagram is valid:

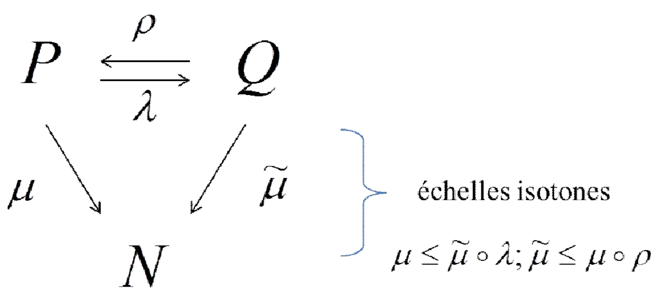

Figure 3. Schematic representation of the couple of scaling factors emerging of the connections between two ordered categories.

Actually we can use the following scales:

- $\mu : P \to P \times Q : p \mapsto (p, \lambda(p))$
- $\tilde{\mu} : Q \to P \times Q : q \mapsto (\rho(q), q)$.

This configuration corresponds precisely to the situation of self-similarity which thus naturally emerges from the conjunction of a relation of order and of adjunction. At this stage, it is useful to recall a major result established by Birkhoff: the polarities that are the Galois connections between set powers are unambiguously constructed from the relationships between the underlying sets. Given R a relation between sets A and B, it gives rise to a Galois connection$(R^{\rightarrow}, R^{\leftarrow})$ between $\wp A$ and $\wp B$; namely:

- $X \subseteq A$ sent onto $R^{\rightarrow}(X) = X^R = \{y \in B: x\ R\ y\ \forall\ x \in X\}$
- $Y \subseteq B$ sent onto $R^{\leftarrow}(Y) = Y_R = \{x \in A: x\ R\ y\ \forall\ y \in Y\}$.

The images of a couple of applications are dual isomorphic closure systems. Reciprocally, any connection of Galois between powers of sets, and therefore every dual isomorphism Φ between two systems of closure are based on a single relation R defined over the underlying sets such that $x\ R\ y$ means that y belongs to the image of the closure of $\{x\}$ by the isomorphism Φ. Thus an adjunction has a one to one, close link with the closure and kernel operators that form the basis of the mathematical topology with the concept of closed and open "support". Reciprocally we can therefore associate an adjunction with topology. These relationships serve as a starting point for the *theory of local and frames*. The concept of ideal, whose origin comes from arithmetic and goes back to Gauss with the decomposition of integers into products of prime numbers, leads to the notion of localization whom allows to introduce differential calculus, according to the path followed for example in algebraic geometry. The dual concept of filter, its reinforcement in terms of ultrafilter and monad of co-density, generalizes the concept of probability measures and thus the new integral calculus, the need of which is nowadays required for new formalization of the economy [12]. We will focus here on the particular case that corresponds to a specific situation of scaling and of fixed point, $T = \lambda \circ \varrho$ and $TX = X$, because these notions underlie a generalization of the notion of equilibrium or steady states. The analogy between arrow (diagrammatic representation of our environment) and action (will [42]) is mathematically analyzed here below.

4. Adjunctions, extensions and Fraction Calculus

The consideration of ordered categories appears to have to respond in a self-consistent manner to a triple "physical" constraint: (i) a closure that we called elsewhere dynamic closure in our physical models [32], (ii) an internal hierarchy which offers all the properties of a self-similarity, built from a kernel, nucleus whose monadic characteristics (iii) induce a global teleology of the system (existence of geodesics). However, to become differential, what may be important for economic models, this adjunction also requires a division of any functors; division which must be compatible with the preservation of categorical limits. We then suggest considering the concept of incompleteness as a central notion emerging from the duality of ordered structures for proving, among others in physics, that any fractional dynamics can be canonically extended by implementing the categorical concepts of limits and co-limits. It should be recalled at this step that, for the definition of its objects, category theory distinguishes the arrows on the right from those on the left as well as the domains (source, empennage) of the co-domains (target). This *distinguo* creates the generic dissymmetry suitable for implementing at least an one of elementary irreversibilities required in econo-physics. Let us also recall that, there is a close link between the adjunction and Kan extensions, extensions which figure out how this dissymmetry involves the incompleteness of some categories.

4.1. Kan extension and adjonction

Given the functor $F : \mathcal{A} \to \mathcal{B}$ between two small categories in other words here a set category. Using composition with F: F^* : Fonct ($\mathcal{B}$, Ens) → Fonct ($\mathcal{A}$, Ens). Both functors are co-complete and F^* preserve all the co-limits. According to the Yoneda's lemma F^* satisfied the dual condition of the analytical solution set, it follows that F^* has an adjoint at left (Source) which is by definition it's a Kan extension. This remarkable fact can be generalized by replacing the category Ens by any co-complete category. Kan's extension to the left of G (target) along F, if it exists, is a pair (K,α) where:

- $K : \mathcal{B} \to \mathcal{C}$ is a functor,
- $\alpha : G \Rightarrow K \circ F$ is a natural transform.

Given the following universal properties: if (H,β) is another pair, with:

- $H : \mathcal{B} \to \mathcal{C}$ is a functor,
- $\beta : G \Rightarrow H \circ F$ is a natural transform, there is a unique natural transformation $\gamma : K \Rightarrow H$ with $(\gamma * F) \circ \alpha = \beta$, namely.

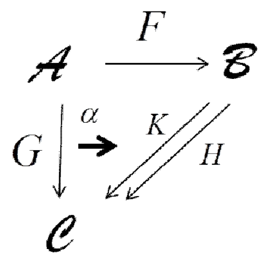

Figure 4. Schematic representation of Kan extension pointing out the role of Natural Transforms and of the associated morphisms.

When $\mathcal{A}$ and $\mathcal{B}$ are small categories, we can consider F^* : Fonct ($\mathcal{B}$, Ens) → Fonct ($\mathcal{A}$, Ens) Taking into account the composition by F, the Kan extension to the left of G along F exactly means that there is a reflection (K, α) of G along F^*. By applying the fundamental theorem of the adjunction for $\mathcal{A}$ small and $\mathcal{C}$ co-complete, Kan's extension at left of G along F exists. This extension is usually noted $Lan_F G$. For $F : \mathcal{A} \to \mathcal{B}$ functor between small categories, the following conditions are equivalents:

- F has an adjunct at right $F^* = Ran_F G$., which is reflection of G;
- $Lan_F 1_{\mathcal{A}}$ exists, and for any functor $G : \mathcal{A} \to \mathcal{C}$, the isomorphism $G \circ Lan_F 1_{\mathcal{A}} \cong Lan_F G$ is valid;
- $Lan_F 1_{\mathcal{A}}$ exists and the isomorphism $F \circ Lan_F 1_{\mathcal{A}} \cong Lan_F F$ is valid.

We will come back later to Kan extensions but at this step we would like to add some comments. It is possible to build an image borrowed from optics that intuitively accounts for the mechanism of implementing a Kan extension. Given an object B in $\mathcal{B}$, The aim is the canonical construction of an object C in $\mathcal{C}$ which extends G along F. For this we consider the set solution formed by all the objects in $\mathcal{A}$ who admit the object B as image by F. This set solution is then transferred to the category $\mathcal{C}$ through G. From this collection of image objects we create a canonical object that is defined as a co-cone. It is by definition characterized by the fact that any arrow starting from the previous collection of objects in $\mathcal{C}$ and targeting onto any object, say D, which is factorized into a composition of an arrow towards C followed by a second arrow going from C to D.

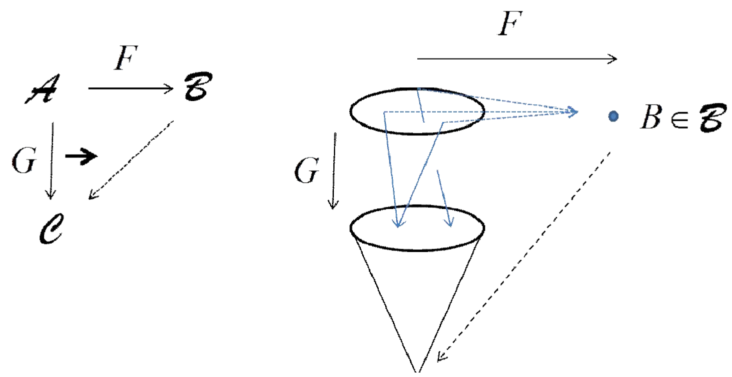

Figure 5. Schematic representation of Kan extension borrowed from optical analogy.

In other words C carries the different addresses of D. This results looks like an optical device, C thus concentrating all relevant information held by the previous collection of objects. Equivalence between the existence of a adjoint at right for the functor F and the properties relating to the transformation $Lan_F G$. shows clearly that a Kan extension generalizes the concept of adjunction since the property for F to admit a adjoint at right is equivalent to the property that $Lan_F 1_{\mathcal{A}}$ exists in addition with $F\circ Lan_F 1_{\mathcal{A}} \cong Lan_F F$. Therefore the existence of $Lan_F G$ for the functor G distinct of F appears as a stronger property than the mere existence of $Lan_F 1_{\mathcal{A}}$ corresponding to the case $G = 1_{\mathcal{A}}$. Observe also that the existence of a Kan extension for G along F amounts to somehow pulling back the functor F (by formally introducing F^{-1}), therefore Kan extension introduces a calculation of fractions. More generally, the multiplicity of the extensions of F can be associated with the covering of all universal bases built on a field of the auto-morphisms associable with a fiber, that is to say with the existence of a flow with first integral along this a flow, situation that involves the notion of unique parameter, the scale for flow along self-similar structure.

4.2. Monads and Associated Algebras

It is useful to introduce here the monoid structure which is the frame leading the definition of the monad, an self-ruling structure able lead any type of iteration. To do this, let consider a binary or quadratic operation defined on a set M: $M \times M \to M : (x, y) \mapsto xy$ (therefore the notation of an object in the form of "words"). It can be recalled in this connection that any self-similar structure can be written using an encoding based on letters and words, a remark that allows the reader to anticipate the reasoning on the links of the monadic approach with fractal structures. By iteration, to any finite sequence of elements $(x_1, x_2, \dots, x_n)$ we can then associate the composite word $x_1 x_2 \dots x_n$. By calling $T(M)$ the set of finite sequences or "words" of this kind; one can provide the set M using a monoid structure:

$$\xi : T(M) \to M : (x_1, x_2, \dots, x_n) \mapsto \xi(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n \in M.$$

The axioms verified by ξ respond to a composition law with:

$$\epsilon_M : x \mapsto \{x\} \text{ et } \mu_M : TT(M) \to T(M): ((a_1^1, \dots, a_{n_1}^1), \dots, (a_1^m, \dots, a_{n_m}^m)) \mapsto (a_1^1, \dots ; a_{n_1}^1, \dots ; a_{n_m}^m)$$

with associativity rule. In doing so, the concept of monoid is now "categorized" and leads the concept of monad on a category $\mathcal{C}$.

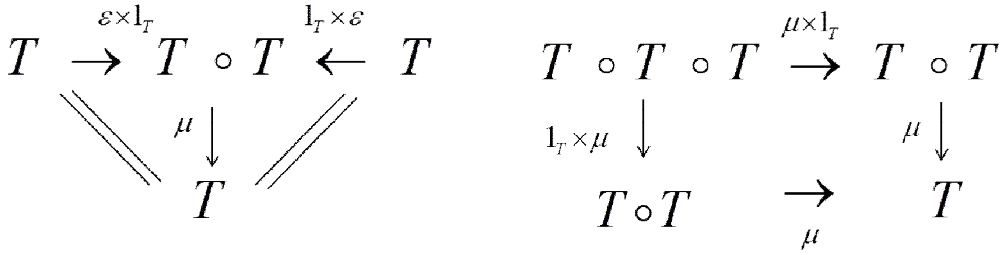

Figure 6. Representation of functorial diagrams giving the main relations associated to a monad.

This is a triplet based by a functor $T : \mathcal{C} \to \mathcal{C}$ and a couple of natural transforms $\varepsilon : 1_{\mathcal{C}} \to T$ and a product $\mu : T \circ T \to T$ such as the laws summarized in the above diagrams are verified.

The concept of monad is associated with the concept of algebra since it is recalled that an algebra on the monad is a pair (C, ξ) with C is an object of $\mathcal{C}$ and $\xi : T(C) \to C$ is a morphism over $\mathcal{C}$ agreeing with the following diagrammatic constraints:

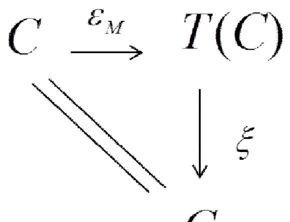

Figure 7. Determination of the morphisms required for building an algebra associated with a monad.

We thus find, as a special case, the concept of a monoid on a set. If (D,ζ) is another algebra, a morphism between algebras (C, ξ) → (D,ζ) is a morphism in $\mathcal{C}$: f : $C \to D$ such as

$$\begin{array}{ccc} T(C) & \xrightarrow{T(f)} & T(D) \\ \xi \downarrow & & \downarrow \zeta \\ C & \longrightarrow & D \end{array}$$

Figure 8. Constraints for a morphism between algebras.

The set of algebras able to be constructed with $\mathcal{C}$ and T constitutes itself a category, which is referred as $\mathcal{C}^T$, called d'Eilenberg-Moore algebra. A forgotten functor exists $U : \mathcal{C}^T \to \mathcal{C} : (C, \xi) \mapsto C$ with $f \mapsto f$. The functor U is faithful and reflects the isomorphisms. U admits an adjunction at left

$F : \mathcal{C} \to \mathcal{C}^T : C \mapsto (T(C), \mu_C)$;

$(f : C \to C') \mapsto (T(f) : (T(c), \mu_C) \to (T(C'), \mu_{C'}))$.

The unit of adjunction is $\varepsilon : 1_{\mathcal{C}} \to UF = T$, and the co-unit is $\eta : FU \to 1_{\mathcal{C}^T}$ such as $\eta_{(C,\xi)} = \xi$. Algebra of the form ($T(C)$, μ_C) is a special case of algebra which is then called free algebra. The complete subcategory of $\mathcal{C}^T$ generated by free algebras is called the Kleisli category and is referred as $\mathcal{C}_T$ with:

- Objects of $\mathcal{C}_T$ = objects of $\mathcal{C}$,
- Morphisms f : $C \to D$ in $\mathcal{C}_T$ is the morphism f : $C \to T(D)$ in $\mathcal{C}$, and the composition of the morphisms in $\mathcal{C}_T$ between $f : A \to B$ and $g : B \to C$ must be understood as a composition seen in $\mathcal{C} : A \xrightarrow{f} T(B) \xrightarrow{T(g)} TT(C) \xrightarrow{\mu_C} T(C)$.

Actually the identity in C in $\mathcal{C}_T$ is $\varepsilon_C : C \mapsto T(C)$, $(f : C \to D) \mapsto \mu_D \circ T(f)$. This functor is faithful, reflects isomorphisms and admits an adjoint on the left:

$\mathcal{C} \to \mathcal{C}_T : C \mapsto C; (f : C \to D) \mapsto \varepsilon_D \circ f$

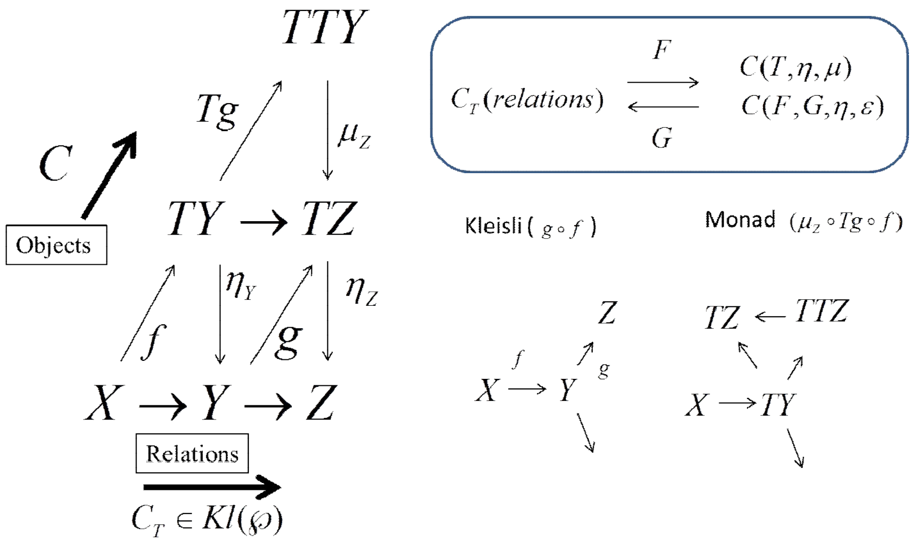

Figure 9. Schematic representation of the shift from T-monadic structure to the Kleisli initial algebra which can be associated to it. The vertical axis is the functorial axis T. The diagonal axis concerns the object of the categories. The horizontal axis concerns the relations, namely the Kleisli algebra, thus the initial algebra related to the monad. The monadic structure is equivalent to the F-G structure of adjunction.

It is appropriate to introduce here the notion of idempotent monad [39]. The idempotent property is associable with the notion of fixed point of an application whose result does not change that one applies it once or several times. An idempotent monad is a monad that "squares to the identity" in the category-theoretical meaning. An idempotent monad must consider henceforth as a "categorified" projection operator, in which are encoded the subcategories and any reflection / localization. In terms of type theory idempotent monads interpret (co-) modal operators. The major properties of the idempotent monads in relation to the monads are as follows: (i) $\mu : T \circ T \rightarrow T$ is a natural equivalence of functors; and in addition (ii) for any object X of $\mathcal{C}$, $\eta_X \in \mathcal{S}(F)$, as classes of morphisms and (iii) we have the symmetry $T\eta = \eta T$.

The existence of a canonical transformation that allows passing from a general monad to its idempotent subcategory, leads this note to elude this aspect, for focusing instead on understanding its links with the categorical foundations of the localization that one can obtain within posets, localization, localization which is justified by a functorial division at differential level.

4.3. Functorial Division

The first step in any analytical process is to construct indexed sequences of representation for any object; this indexation is provided by the use of integers, introducing hence the role of sets of numbers, relative integers, rational numbers, and so on in the problematic. For example a word can be constructed in a given order according to a basic algebra $\mathbb{N}$ as $(wo)_1(rd)_2 \equiv w_1o_2r_3d_4$. However the theory of the categories brings a lot more by allowing factorization on the set of the rational numbers $\mathbb{Q}$ for functors by means of the concept of Span. As T. Leinster [28] has shown, there is a generic link between the notion of adjunction and the notion of self-similarity so any adjunction can naturally be divided into batches in a canonical way. In particular, we can analyze the consequences of this factorization in this context. The link with the fractality is clearly drawn in particular if, for example, $P\ (X)$ seen as sum of polynomials on the objects X, is considered as an analytic functor [43,44]. It can then benefit from the well-known factorization of polynomial structures. Zeros become fixed points $P(X) = X$ of Newton's series. Differential calculus then emerges in the framework of categorical limits as the object of morphisms corresponding to the standard composition of arrows by using the rules of associativity. Yielded homogeneous in the set of real numbers these laws of associativity lead to functional equations. Axcel Theorems [45-47] allowing to associate a total order with a partial order, by means of an invertible function (Exponential for local point of view but also zeta of Riemann function for global point of view), - a duality between a linear addition (coproduct) an multiplication (product) -, with not any regard to scales. It is then, through the mathematical capacity Choquet theory and thus through the consideration about non-additive but ordered sets [48,49], that the differential calculus of fractional order appears as naturally associable with idempotent monadic structures.

Adjunction, Spans and functor divisions

We start from the fact that all the preceding categories can be enriched since we are able to conceive a splitting of the functors. Mathematically such fractions arise from the notion of Span. In a category $\mathcal{C}$ a Span is of object x to an object y is a diagram:

$$x \leftarrow s \rightarrow y$$

with s as a third object in $\mathcal{C}$. Let us consider immediately the two particular cases:

- $f \equiv 1$, the span is then the morphism $x \overset{g}{\rightarrow} y$,

- $g \equiv 1$, the span is then the morphism $y \overset{f}{\rightarrow} x$.

If the category $\mathcal{C}$ has reciprocal or pullback images, it becomes possible to compose the spans. Indeed, given an application of x to y and a second of y to z, or : $x \overset{f}{\leftarrow} s \overset{f}{\rightarrow} y \overset{h}{\leftarrow} t \overset{i}{\rightarrow} z$, the central pullback image can be built above $s \overset{f}{\rightarrow} y \overset{h}{\leftarrow} t$, such as $s \overset{p_s}{\leftarrow} s\times_y t \overset{p_t}{\rightarrow} t$, then by composition we end up with $x \overset{fp_s}{\longleftarrow} s\times_y t \overset{ip_y}{\longrightarrow} z$. Let remark that the notion of span generalizes the notion of relation, for example between two sets A and B. A relation R can be defined as a subset of the Cartesian product $A\times B$, equipped with two canonical projections: $A \overset{p_A}{\leftarrow} A\times B \overset{p_B}{\rightarrow} B$ and if $a \in A$, $b \in B$, we write aRb instead of $(a,b) \in R$. In the category of finite sets Ens_{fin}, the spans constitute a "categorization" of matrices whose coefficients are natural numbers, namely: $X_1 \leftarrow N \rightarrow X_2$, the cardinal of the fiber X_{x_1,x_2} above $x_1 \in X_1$, $x_2 \in X_2$ takes the place of the index matrix coefficient (x_1, x_2). The composition of the spans introduced previously is then interpreted as the matrix multiplication. A same composition of spans accounts for the composition of the relations: if R relation between A and B, S relation between B and C, then the composition $S\circ R$ between A et C : $(a,c) \in S\circ R \Leftrightarrow \exists\ b \in B$ such as $(a,b) \in R$ et $(b,c) \in S$. The analogy of the concept of span with the zeros of a polynomial equation used for its factorization logically connects it to the concept of localization and the calculation of fractions. This is fundamentally because a span generalizes morphism by breaking the asymmetry between source and goal. In general, localization is a process of adding formal inverses to an algebraic structure. Thus the localization of a category C associated with a collection W of singularized morphisms is a universal process by which all the morphisms of W become isomorphisms. The canonical example is that of the passage of $\mathbb{N}$ to $\mathbb{Q}$ by formal inversion of all prime numbers. A detour via algebraic geometry is useful here. In this particular context, to locate upon a given prime number consists in inverting any number which is not divisible by this one; on the other hand inverting the prime number p is called inversion outside of p. This way of seeing things lights up by considering the ring $\mathbb{R}[x]$ polynomial functions on the real line. Let a point such as $a \in \mathbb{R}$, located by introducing an inverse of $(x - a)$; this results is located in the ring of rational functions that are defined everywhere on $\mathbb{R}$ except on the point a. It is said localization outside of "a", or localization outside of the ideal I based on $(x - a)$. If now we introduce an inverse for any element of $\mathbb{R}[x]$ located in the ideal I, we obtain the ring of rational functions defined on the real line at least in a neighborhood of a. These are rational functions that do not have a factor $(x - a)$ in the denominator. We then say that we have located in "a", or localized in the ideal I. We see that we can then start a differential calculus from these considerations alone. Indeed: $f(x) = A_0 + A_1 (x - a) + A_2 (x - a)^2 + \ldots$ in a vicinity of "a". The coefficient A_1 is the derivative for f in a. This is how the localization becomes the canonical process of algebraic nature which makes it possible to give birth to the differential calculus, including obviously the non-integer fractional calculus.

Role of the none commutativity

The above approach can be generalized to the case of ordered structures and it can be associated with category theory since every polynomial is similar to an analytic functor [43,44]. We can show that this approach of the differential calculus passes in category theory through Kleisli categories already evoked (Fig. 9) that is to say, by the smallest associative algebraic category related to a monad. The class of morphisms W which makes it possible to define a calculation of fractions must indeed satisfy certain axioms which generalize in a categorical context, the notion of multiplicative system. It intervenes in the method of localization in any types of rings. The well-known case is the case of commutative rings. But the composition in above categories is precisely none commutative. Thus it is necessary to distinguish the left fraction calculus of the right fraction calculus. The practical idea is to introduce diagrams of the form $\overset{f}{\rightarrow}\overset{w}{\leftarrow}$ or $\overset{w}{\leftarrow}\overset{f}{\rightarrow}$ with $w \in W$ and to consider them as fractions: $w^{-1}f$ or fw^{-1}. We immediately underline the link with the spans $\overset{w}{\leftarrow}\overset{f}{\rightarrow}$ in $\mathcal{C}$ or $\overset{f}{\rightarrow}\overset{w}{\leftarrow}$ in $\mathcal{C}^{op}$. We can then make the following assumptions about the W class:

- W is a subcategory of $\mathcal{C}$, W contains all identities and is closed for composition,
- One condition: for $x \overset{v}{\rightarrow} z$ in W an $y \overset{f}{\rightarrow} z$ in $\mathcal{C}$, $w \overset{v'}{\rightarrow} y$ exists in W and $w \overset{f'}{\rightarrow} x$ in $\mathcal{C}$ such as the diagram is commutative, namely $v \circ f' = f \circ v'$,
- Cancellation at right: for the co-equalizer diagram $f,g : x \rightarrow y$, $v \in W$: $y \rightarrow z$ $v \circ f = v \circ g$, then $v' \in W : w \rightarrow x$ exists such as $f \circ v' = g \circ v'$.

The localization of the category $\mathcal{C}$ in W is a new category $\mathcal{C}[W^{-1}]$ where:

- The objects of $\mathcal{C}[W^{-1}]$ are objects of $\mathcal{C}$,

- The morphisms $a \to b$ in $\mathcal{C}[W^{-1}]$ are equivalence classes of spans of the form $a \overset{v}{\leftarrow} a' \overset{f}{\rightarrow} b$ where $v \in W$.

The equivalence is defined as followed: $a \overset{v}{\leftarrow} a' \overset{f}{\rightarrow} b \sim a \overset{w}{\leftarrow} a'' \overset{g}{\rightarrow} b$ if an only if an object $\bar{a}$ exists and as well the morphisms $\bar{a} \overset{s}{\rightarrow} a'$, $\bar{a} \overset{t}{\rightarrow} a''$ such as $f \circ s = g \circ t$, $v \circ s = w \circ t$ et $v \circ s = w \circ t$ in W. The class of equivalence $a \overset{v}{\leftarrow} a' \overset{f}{\rightarrow} b$ is referred as $f \circ v^{-1}$. With this notation it is easy to interpret the previous conditions in a formal way as similar to those which allow to define equivalence classes of pairs of relative integers leading to build $\mathbb{Q}$ from $\mathbb{Z}$. These recalls being given, we can go back to building a Kleisli category and verify that this type of category results precisely from a calculation of fractions.

Idempotent monads and orthogonality of the morphisms

The adjunction can be considered from a perspective that will allow us to better understand the place held by the idempotent monads. Let $\mathcal{A}$ and $\mathcal{B}$ a couple of categories and $F : \mathcal{A} \to \mathcal{B}$ a functor. Let consider the class $\mathcal{S}(F)$ of the morphisms of $\mathcal{A}$ equipped with an inversion via F. In parallel we write $\mathcal{D}(F)$ the class of objects in $\mathcal{B}$ isomorph to an FX for a certain object X in $\mathcal{A}$. Therefore:

$$\mathcal{A} \overset{F}{\rightarrow} \mathcal{D}(F) \overset{K}{\rightarrow} \mathcal{B} \; with \; K \text{ inclusion.}$$

Given $T = (T, \eta, \mu)$ a monad on the category $\mathcal{C}$. A pair of adjoint functors F and G with a unit η and a co unit ε gives birth to the monad T over $\mathcal{C}$ with $T = GF$ and $\mu = G\varepsilon F$ (Fig. 9). We have seen previously that reciprocally every monad is induced, although in a non-unique way, by a pair of adjoint functors. Among possible algebras there is one which is initial, namely the Kleisli monad, and another which is terminal, namely that of Eilenberg-Moore. We also have met though without going into detail, the notion of idempotent monad. In this case the factorization $\mathcal{C} \overset{T}{\rightarrow} \mathcal{D}(T) \overset{K}{\rightarrow} \mathcal{C}$ is an adjoint pair T. $\mathcal{D}(T)$ is then isomorphic to the category of Eilenberg-Moore, as well as to the category of Kleisli, the latter being the category of fractions of $\mathcal{C}$ with respect to the class $\mathcal{S}(T)$. In other words, the two adjunction solutions, initial and terminal, attached to the monad T, merge together. As we know that any monad can canonically be associated to an idempotent monad this situation becomes somehow canonical and universal; it corresponds to a situation of fixed point $F(X) = X$ which induces self-similar constructions.

We now would like to introduce the notion of orthogonality applied to morphisms. A morphism $f\colon A \to B$ is an object X in $\mathcal{C}$; there are orthogonal if $\mathcal{C}(f,X) : \mathcal{C}(B,X) \to \mathcal{C}(A,X)$ is a bijection, where the last two categories, source and target, are categories referred as comma respectively denoting the set of morphisms in $\mathcal{C}$ from A or B to X; Intuitively, this accounts for the fact that the object X is "seen" from A or B from an identical manner. For a class of morphisms $\mathcal{S}$, the class of orthogonal objects to any f of $\mathcal{S}$ is written $\mathcal{S}^{\perp}$. In the same way, for a class of objects $\mathcal{D}$, $\mathcal{D}^{\perp}$ denotes the class of morphisms orthogonal to any object X in $\mathcal{D}$. It is said that a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ create a pair of orthogonal forms if: $\mathcal{S}^{\perp} = \mathcal{D}$ et $\mathcal{D}^{\perp} = \mathcal{S}$.

If (T,η,μ) is an idempotent monad, both classes previously introduced, namely $\mathcal{S}(T)$ and $\mathcal{D}(T)$ form such an orthogonal pair. By construction $\mathcal{D}(T)$ coincides with the class of X objects in $\mathcal{C}$ such as $\eta_X : X \cong TX$. This relation underlines the self-similar character of the objects. Given an orthogonal pair $(\mathcal{S},\mathcal{D})$, an idempotent monad exists $(T,\eta,\mu) : \mathcal{S}(T) = \mathcal{S}$ and $\mathcal{D}(T) = \mathcal{D}$ if and only if for any object X, there is a morphism $\varphi : X \to Y$ in $\mathcal{S}$ with Y in $\mathcal{D}$. *In this case we say that T is a location functor associated with the pair* $(\mathcal{S},\mathcal{D})$; by continuing the examination of categorical constraints it is shown that $\mathcal{D}(T)^{\perp} \subseteq \mathcal{S}(T)^{\perp}$; this means that for any monad (T,η,μ) the objects having a TX, form or any isomorph objects, are orthogonal to all morphisms whose image by T is an isomorphism. On the other hand $\mathcal{S}(T)^{\perp}$ may be none included, and the relation of order can be questioned if the monad considered is not strictly idempotent. The concept of orthogonality, which has just been briefly described above, is well articulated with the notion of Kan extension by staging a particular important idempotence case which is at the heart of the interpretation of self-similar structures when they are part of the category theory [30].

From Kan extension to the location under an idempotent constraint

Given $K : \mathcal{A} \to \mathcal{C}$ and a pair of functors F et L. If $\mathcal{M}$ is a complete category and with additional quite trivial conditions, we can build the Kan extension at right $R = Ran_K\ F$ of F along K. R can be chosen in such way that $RK = F$. Assuming that $K : \mathcal{A} \to \mathcal{C}$ has an adjoint at left $L : \mathcal{C} \to \mathcal{A}$, then for any $F\colon \mathcal{A} \to \mathcal{M}$ and any $X \in \mathcal{C}$, with the same conditions: $Ran_K\ F = FL$. Therefore if (T,η,μ) is the monad involves by the pair of adjoint functors (L,K) then $T = Ran_K\, K$. Such an extension at right of a functor along itself – here in K – is called the co-density monad of this functor. Indeed, either $(\mathcal{C},\mathcal{D})$ an orthogonal pair $\mathcal{C}$ with $K\colon \mathcal{D} \to \mathcal{C}$

inclusion, $(\mathcal{S}, \mathcal{D})$ admits a location functor – as defined above – if and only if $Ran_K\ K$ exists locally. In this case $Ran_K\ K$ is the localisation functor associated to the pair $(\mathcal{S}, \mathcal{D})$. As we have previously announced, there is a canonical process for constructing a idempotent monad from any kind of monad under only some weak assumptions. We can assert on a monad $\mathcal{C} \sim (R, \eta, \mu)$, that an idempotent monad $(R_\infty, \eta_\infty, \mu_\infty)$ exists on $\mathcal{C}$ and as well, a monomorphism of monads $\lambda : R_\infty \rightarrow$ R such as: (i) λ is universal, if $\phi : \hat{R} \rightarrow R$ with $\hat{R}$ idempotente, an unique morphism $\phi_\infty : \hat{R} \rightarrow R_\infty$ exists with $\lambda\phi_\infty = \phi$; in addition (ii) $\eta_\infty R : R \rightarrow R_\infty R$ is a isomorphism and (iii) $f: X \rightarrow Y$ in $\mathcal{C}$ such as $R_\infty f$ is an isomorphism if an only if RF is an isomorphism. The construction of R_∞ is achieved by iteration from $(R_0, \eta_0, \mu_0) = (R, \eta, \mu)$. For $\alpha + 1$ ordinal successor, $R_{\alpha+1}$ is an equalizateur of $R_\alpha \eta_\alpha$ an of $\eta_\alpha R_\alpha$, for a limit ordinal $R_\omega = \lim_{\leftarrow} R_\alpha$. For any object X thus has an inverse system of monadique arrows stabilized at infinity. The result is the object $R_\infty X$ and then $R_\infty \eta_\infty = \eta_\infty R_\infty$, property that ensures that R_∞ is idempotent. By substituting the idempotent monad to a monad by constructing the process just described, we are brought back to the following canonical situation: $\mathcal{C}$ category asserting the right properties, $\mathcal{D}$ a full sub category of $\mathcal{C}$ an $K : \mathcal{D} \rightarrow \mathcal{C}$ the inclusion. If $Ran_K\ K$ exists locally, the orthogonal pair $(\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$ admits a location functor. Starting from the categorical basis already specified, what interests us is our capability of defining a differential system over an ordered discrete set as a power set compelled to inclusion functors that are able to be divided. It is at this point that the functions of Möbius and Riemann naturally appear [50-54]. Let show that the differential calculus that emerges then belongs mathematically to "capacities Choquet integral calculus" [48,49,56,57].

5. Set Derivation and Choquet Capacity

A set function, on a finite set is an application such that $\xi : 2^X \longrightarrow \mathbb{R}$, which associates a real number with any subset of *X*. The notion of partition is the first motivation for such a function. It must be:

- Additive if $\xi(A \cup B) = \xi(A) + \xi(B)$ for any A, B disjoint in 2^X
- monotonous if $\xi(A) \leq \xi(B)$ when $A \subseteq B$
- based if $\xi(\emptyset) = 0$ (*retrograde or initial horizon*)
- norm if $\xi(X) = 1$ (*Direct of final horizon*).

An additive set function is uniquely determined by the values taken on the elements of the set X, when $\xi(A) = \sum_{x\in A} \xi(\{x\})$. A game is a set based function. We can associate him with his conjugate $\bar{\xi}$, when: $\bar{\xi}$ (A) = $\xi(X) - \xi(A^c)$ for any $A \in 2^X$. A measure with a norm is called a probability measure. A capacity $\mu : 2^X \longrightarrow \mathbb{R}$ is a monotonous and based function $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ when $A \subseteq B$. The constant function 0 is a capacity. In addition, a capacity is a monotonous game and takes only non-negative values. A capacity is normalized if in addition $\mu(X) = 1$. An additive normalized capacity is a measure of probability. By analogy with what happens with real-valued functions, the derivative of a set function is by definition its variation when an element is added or subtracted from the set.

By pursuing the reasoning it is possible to define partial and multiple order derivatives we can define the derivatives with respect to any number of elements of X; in other words, we can define the derivative with respect to any subset $K \subseteq X$ and more specifically, for $A,K \subseteq X$ and forr ξ set function on X, the derivative of ξ in A with respect to K is defined via inductive approach: $\Delta_K \xi(A) = \Delta_{K\setminus\{i\}} \xi(\Delta_{\{i\}} \xi(A))$. With the conventions $\Delta_\emptyset \xi = \xi$ et $\Delta_{\{i\}} \xi = \Delta_i \xi$. Similarly $\Delta_{ij} \xi = \Delta_{\{i,j\}} \xi$. When $K \cap A = \emptyset$, we get: $\chi = S - A + F$

$$\Delta_K \xi(A) = \sum_{L\subseteq K}(-1)^{|K\setminus L|}\xi\big((A \setminus K) \cup L\big).$$

It is important here to understand the meaning of the power of the term minus one: “–1”. The origin of this term comes from homology algebras. A way for memorizing the appearance of the role of parity is to refer to the Euler constant χ about 3D polytopes. $\chi = S - A + F$ (where S, A, F are respectively the number of vertices, edges and faces of the polytope). The approach consists of generalizing this Euler's approach when he looks for a constant capable of classifying n-dimensional polytopes. χ Euler number helps to classify the objects by a relation between their “volume” or determinants and their "successive edges" thus setting up the germ of the cobordism theory. The fact that the edges are affected by the minus sign is due to that, two finite sets cannot be in cobordism if their cardinal has the same parity, which is in contrast the case of a point and a segment or a face and an edge for example. This type of treatment is obviously generalized when one is led to triangulate a compact manifold, or a polytope of any integer dimension, by considering it as a simplicial complex by means of a tiling by simplexes. In this case, the transformations between simplexes lead to the design of so-called cohomology groups or rings as a quotient of structures, graduated by inclusion, further associable by generalization to the Grothendieck's Topoi. It is in accounting all authorized morphisms that the transformation and the function of Mobius steps in. This function will behold associated with the differential equations of non-integer order via the concept of convolution.

5.1. Mobius transform and convolution

We can introduce the Möbius transform, or Möbius inverse, of the set function ξ defined as:

$$m^{\xi}(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \xi(B) \text{ pour tout } A \subseteq X.$$

It is itself set function m^{ξ} over X. Indeed $m^{\xi}(\emptyset) = \xi(\emptyset)$. An important property is that m^{ξ} allows to find ξ through inversion: $\xi(A) = \sum_{B \subseteq A} m^{\xi}(B)$ for $A \subseteq X$ (Eq 1). By induction about the size of A, therefore on $|A|$, it is possible to write the set-derivative in terms of the Möbius transform. More precisely:

$$\Delta_K \xi(A) = \sum_{L \in [K, A \cup K]} m^{\xi}(L)$$

with $m^{\xi}(A) = \Delta_A \xi(\emptyset)$. At this step, it is interesting to introduce a couple of capacities that frequently appears in physics:

- A 0 – 1– *capacity* a capacity with values in {0,1}. Besides the zero capacity 0, all others are normalized. Among this family of capacities we must mention the smallest μ_{min} and the biggest μ_{max} defined by:
  $\mu_{min}(A) = 0$ for any $A \subset X$ et $\mu_{max}(A) = 1$ for any $\emptyset \neq A \subset X$.
- A second important family consists of what is called the Unanimous Games Family (JU). For $\emptyset \neq A \subset X$, le JU focused on A is a game u_A defined by:

$$u_A(B) = \begin{cases} 1 \; si \; B \supseteq A \\ 0 \; sinon \end{cases}.$$

We can then introduce the notion of convolution. Let f and g be two functions with real values on a partially ordered set, a poset $(P, \leq)$ which is locally finite, in the sense that all intervals $[x, y]$ are finite, and which admits a smaller element 0. Consider the system of equations

$$f(x) = \sum_{y \leq x} g(y) \text{ pour } x \in P \text{ (Eq. 2).}$$

Our issues in physics and econophysics are to express g according to f. The question considered is therefore that of the identification of the process underlying a series of experimental data. We know how this identification is a complex problem and the models that compete together for the representation, are in infinite numbers. We recall here that the experimental work ultimately involving non-integer order derivation models, relies precisely on convolution algebras and that it is also an identification in terms of self-similar geometries, – above recognized as monadic based property –. These aspects have raised the most interesting controversies [57-63]

Mathematically the function *g* is called the inverse of Möbius of *f*. In the context presented, the solution exists but in addition is proved to be unique: $g(x) = \sum_{y \leq x} \mu(y,x) f(y)$ with μ, Möbius function. This function is inductively defined:

$$\mu(y,x) = \begin{cases} 1 \; si \; x = y \\ -\sum_{x \leq t < y} \mu(t,x) \; si \; x < y. \\ 0 \; sinon \end{cases}$$

We observe here how the Möbius function naturally carries in it the principles of convolution. Function μ depends only of the structure of the poset $(P, \leq)$. The Mobius transform can be analyzed from many cases: multiplicative monoid, inclusion hierarchies, discrete sets, unanimity game, 0 – 1 capacity etc. The fact that the Möbius transform of a set function is closely related to its derivatives is therefore quite obvious since equation 2 is a discretized version of an integral equation: $f(x) = \int_0^x g(y)dy$ which solution is $g(x) = f'(x)$ under the assumption $f(0) = 0$. The Möbius transform is a transformation that runs on the space of set functions *X*, space written as $\mathbb{R}^{2^X}$ namely the application $T: \mathbb{R}^{2^X} \longrightarrow \mathbb{R}^{2^X}$ associating with each set function ξ its transform $T(\xi)$. Such a transformation is linear if and only if we have the relation $T(\alpha\xi_1 + \xi_2) = \alpha \, T(\xi_1) + T(\xi_2)$ for any set function ξ_1, ξ_2 and $\alpha \in \mathbb{R}$. It is invertible if T^{-1} exists. The Möbius transform is precisely linear and invertible. Its inverse is called Zeta transform. At this step one can introduce an algebraic structure well adapted to these transforms. Let call for an operator of a set function with two arguments $\Phi : 2^X * 2^X \longrightarrow \mathbb{R}$. The multiplication" $*$ " between operators and set functions is here defined in accordance with the principles of convolution:

$$(\Phi * \Psi)(A,B) = \sum_{C \subseteq X} \Phi(A,C)\Psi(C,B);$$

$$(\Phi * \xi)(A) = \sum_{C \subseteq X} \Phi(A,C)\xi(C);$$

$$(\xi * \Psi)(B) = \sum_{C \subseteq X} \xi(C)\Psi(C,B).$$

For example, we define a linear order on 2^X which is an extension of the partial order induced by the inclusion $\subseteq$; we can identified 2^X with $\{1,2,\ldots,2^n\}$ and $*$ becomes the ordinary multiplication of square matrices or matrices and vectors. Kronecker delta function is the only neutral element on the right and on the left. If Φ is invertible, the inverse of Φ, written Φ^{-1}, verifies $\Phi * \Phi^{-1} = \Delta$ et $\Phi^{-1} * \Phi = \Delta$. All operators verifying $\Phi(A,A) = 1$ for any $A \subset X$ and $\Phi(A,B) = 0$ if $A \nsubseteq B$, corresponds to the set of triangular matrices with 1 on the diagonal. It forms a group and the inverse of an operator in this set is calculated recursively:

$$\Phi^{-1}(A,B) = \begin{cases} 1 \; si \; A = B \\ -\sum_{A \subseteq C \subset B} \Phi^{-1}(A,C)\Phi(C,B) si \; A \subset B \end{cases}.$$

We then introduce the Zeta operator $Z(A,B)$ defined through $A \subseteq B$, $Z(A,B)$=1 and 0 if not. Möbius operator is its inverse. Therefore : $\xi = m^{\xi} * Z$ and then $m^{\xi} = \xi * Z^{-1}$ with $Z^{-1} = \begin{cases} (-1)^{|B \setminus A|} \\ 0 \; sinon \end{cases}$.

In engineering terms, the Zeta operator leads to think of impedances or of transfer functions, binding the local to the global properties through Fourier transform scaling [50,54,55]. We recall that in this frame, a local (respectively global) measurement multiplied by the impedance (respectively admittance) leads to a global measurement (respectively local).

It is then useful to consider some particular classes of operators, for example: $\Phi(A,B) = \Phi(\emptyset, B \setminus A)$ pour $A \subseteq B$.

This is an ordinary set function $\varphi(A) = \Phi(\emptyset, A)$. This function is called the generating function. All these particular operators form an abelian group; it is the same for the generating functions associated:

$\{\varphi: 2^X \longrightarrow \mathbb{R};\ \varphi(\emptyset) = 1\}$ with: $\varphi * \psi\,(A) = \sum_{C \subseteq A} \varphi(C)\psi(A \setminus C)$ for $A \subseteq X$.

The neutral element is $\delta(A) = 1$ only if A= Ø. The inverse for φ with respect to this operation is written φ^{*-1}. The operator Z enter this class; we can therefore introduce the generating function Zeta which is: $(A) = 1$ for any $A \in 2^X$. The zeta function appears here as a generating function of a set transfer operator naturally associated with the Mobius transformation, thus with an accounting of the subsets of larger states. Completion is thus underlying the convolution which plays a central role in the implement-tation of the different measures on set partitions 2^X (set of subsets). The order is then the order obtained by the 'inclusion'. The Choquet integral and the notion of Capacity naturally find their importance in the physics of non-additive sets exactly at this step of the reasoning.

5.2. Choquet integral for non-negative functions

We will not develop here the general set theory of Choquet integrals involving measurable bounded non-negative functions. This theory will be easily found in "mathematical references" [48,49]. Note however that all functions considered $B(\mathcal{F})$, builds a lattice which induces a closure (max) and a kernel (min) for any function $f \in B(\mathcal{F})$.

Let consider: $G_{\mu,f}$ $(t) = \mu$ $(\{x \in X: f(x) \geq t\})$ the partition distribution with $t \in \mathbb{R}$ according to the hypothesis on f the function $G_{\mu,f}$ is well defined for $f \in B^{+}(\mathcal{F})$ and μ is a capacity over $(X, \mathcal{F})$, the Choquet integral of f with respect to μ is defined by: $\int f\, d\mu = \int_0^{\infty} G_{\mu,f}\,(t)dt$ where the right-hand integral is a Riemann integral. In physical terms, the relation constructs a transposition between the physical time used by the experimenter and a set variable associated with the support of integration. It is useful and interesting to consider the case of mere and measurable functions, when the image of f is a finite set $\{a_1, a_2, \dots, a_n\}$ storing the image values in ascending order, $0 \leq a_1 < a_2 < \dots < a_n$. By introducing the order relation: $A_i = \{x \in X: f(x) \geq a_i\}$ as elementary set. Therefore $A_1 = X$. The partition function is a descending stair function. It is easy to verify that $\int f\, d\mu = \sum_{i=1}^{n}(a_i - a_{i-1})\mu(A_i)$ with $a_0 = 0$ then we can find the following equation: $\int f\, d\mu = \sum_{i=1}^{n} a_i\big(\mu(A_i) - \mu(A_{i+1})\big)$ by writing $A_{n+1} = \emptyset$. In the case of X is a finite set, namely $X = \{x_1, x_2, \dots, x_n\}$ and $\mathcal{F} = 2^X$. A non-negative function f can be identified with the vector $(f_1, f_2, \dots, f_n)$ where $f_i = f(x_i)$. Given the permutation X σ such as for example $f_{\sigma(1)} \leq f_{\sigma(2)} \leq \dots \leq f_{\sigma(n)}$. Introducing set $(\mathbb{R}_+^X)_\sigma = \{f: X \longrightarrow \mathbb{R}_+: f_{\sigma(1)} \leq f_{\sigma(2)} \leq \dots \leq f_{\sigma(n)}\}$, we build in the same way the set $\mathbb{R}_\sigma^X$. With the given set $A_\sigma^{\uparrow}(i) = \{x_{\sigma(i)}, x_{\sigma(i+1)}, \dots, x_{\sigma(n)}\}$ for $i = 1,\dots,n$ then we get the following expression $\int f\, d\mu = \sum_{i=1}^{n} f_{\sigma(i)}\mu(\{\sigma(i)\}) = \sum_{i=1}^{n} f_i\mu(\{i\})$ for the Choquet integral. Such an integral satisfies a series of linear properties, but also: $\inf f = \int f\, d\mu_{min}$ et $\sup f = \int f\, d\mu_{max}$ and futher $|\int f\, dv - \int g\, dv| \leq \|v\|\|f - g\|$ *où* $\|v\|$ is the variational norm of v and we get in addition $\|f\| = sup_{x \in X}|f(x)|$. An important remark must be given at this stage; f and g are co-monotonous functions and in particular assuming that X is finite with $|X| = n$, and therefore if $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$, a permutation σ exists on X such as $f_{\sigma(1)} \leq \dots \leq f_{\sigma(n)}$ and $g_{\sigma(1)} \leq \dots \leq g_{\sigma(n)}$ Nevertheless the Choquet integral is non additive in the sense that $\int (f + g)\, dv$ is generally different of $\int f\, dv + \int g\, dv$. However, it never exists x and x' such as $f(x) < f(x')$ and $g(x) > g(x')$. So we have the following important result: $f,g \in B\ (\mathcal{F})$ comonotonous: $f + g \in B\ (\mathcal{F})$, then for any game in $\mathcal{BV}(\mathcal{F})$ the Choquet integral is then additive in a comonotonous way: $\int(f\ +\ g)\ dv\ =\ \int f\ dv\ +\ \int g\ dv$ In other words, the Choquet integral is linear with respect to the games and the set of games on a set X form a vector space. It is then possible to consider the bases of these spaces and to order it by permutation to return to a game. Both basis are (i) the Dirac game and unanimity game. This is where the transformation of Mobius deals with. Let σ be a permutation on X ordering of f in a non decreasing order. Noting j the leftmost index in the ordered sequence $\{\sigma(i), i \in A\}$. Then $\int f\ du_A = f_j = \wedge_{i\in A} f_i$. Then

$$\int f\ dv = \sum_{A\subseteq X} m^v(A) \wedge_{i\in A} f_i.$$

Schmeidler characterized the Choquet integral as follows [48,49] : I:$B(\mathcal{F}) \longrightarrow \mathbb{R}$ a functional. Given the set function $v(A) = I(1_A)$, $A \in \mathcal{F}$, then the following propositions are equivalent: I is monotonous and additive in a co-monotonous meaning, and V is a capacity, and for all such as $f \in B(\mathcal{F})$, $I(f) = \int f\ dv$. The set functions on X with $|X| = n$ can then be considered as functions on the vertices of the hypercube $[0,1]^n$ (the polytopes mentioned above appear herein) Using the characteristic function $1 : A \mapsto 1_A$ which is an isomorphism between 2^X and $\{0,1\}^n$. A pseudo-Boolean function being a function $f: \{0,1\}^n \longrightarrow \mathbb{R}$; $x \mapsto f(x)$, any set function ξ corresponds to a single pseudo-boolean function: $f_\xi\ =\ \xi\ (\{i \in X: x_i\ =\ 1\})$ for $x \in \{0,1\}^n$. Reciprocally to any pseudo-Boolean function f corresponds a unique set function such as $\xi_f\ = f(1_A)$ for $A \subseteq X$. Therefore $\xi_f\ = f \circ 1$ and $f_\xi\ =\ \xi_f \circ\ 1^{-1}$. A pseudo-Boolean function is written: $f(x) = \sum_{A\in[n]} f(1_A) \prod_{i\in A} x_i \prod_{i\in A^c}(1 - x_i)$ for any $x \in \{0,1\}^n$ with $\prod_{i\in\emptyset} x_i\ =\ 1$. Unanimity games form a base of games. In fact the unanimity games correspond to the monomials $\prod_{i\in A} x_i$. Therefore the pseudo-Boolean functions are represented in the specific form involving products: $f(x) = \sum_{T\subseteq[n]} a_T \prod_{i\in T} x_i$ for any $x \in \{0,1\}^n$ where the coefficients a_T base the Mobius transform of ξ_f. This representation is called the Mobius representation. Because for any game $v(\emptyset)\ =\ 0$, being a basis $\{b_S\}_{S\in 2^X}$ of the space of functions, we can build the basis of the game $\{b'_S\}_{S\in 2^X\setminus\{\emptyset\}}$ by writing:

$$b'_T = \begin{cases} b_S\ (T)\ si\ T\ \neq\ \emptyset \\ 0\ sinon \end{cases} \quad S \in 2^X \setminus \{\emptyset\}.$$

5.3. Choquet integral on the real straight line and non-integer order derivation

We shall consider now the fractional derivation by computing the Choquet integrals on the real non-negative line by connecting them to Lebesgue measures. Consider the Lebesgue measure λ on $\mathbb{R}_+$ with $\lambda([a,b]) = b - a$ for any interval $[a,b] \subset \mathbb{R}_+$; consider also a distortion function, that is to say a h function: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous, monotonously increasing and verifying $h(0) = 0$. The distorted Lebesgue measure is an application such as $\mu_h = h \circ \lambda$; it's clearly a continuous Choquet capacity on $\mathbb{R}_+$. Given f: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is a non decreasing function and admits a Choquet integral over a subdomain $[0, t]$ for a certain $t > 0$. If f: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is non decreasing, continuously differentiable, and if μ is a continuous capacity on $\mathbb{R}_+$ such as $\mu([\tau, t])$ is differentiable with respect to τ over $[0, t]$ for any $t > 0$, $\mu(\{t\}) = 0$ for any $t \geq 0$, then:

$$\int_{[0,t]} f \, d\mu = -\int_0^t \frac{\partial\mu}{\partial\tau}([\tau, t]) f(\tau) d\tau \text{ avec } t > 0 \; (*).$$

The integral at right is a Riemann integral. In the particular case of a distorted Lebesgue measure μ_h with h continuously differentiable, this equation becomes:

$$\int_{[0,t]} f \, d\mu = \int_0^t \frac{\partial h}{\partial\tau}(t - \tau) f(\tau) d\tau.$$

This theorem is valid when f is a constant function or an increasing function. We note to simplify $\mu' = \frac{\partial\mu}{\partial\tau}$. In the case of the physical models having initiated these researches, the TEISI model [55], the distorted measure was none other than the measure associated with the fractal interface, while the function was the flow of extensity across the interface. Thus, in a purely physical way, the model account for fractional transfers, implemented Choquet integrals. As we shall see below, the transposition of the measure of a temporal reference to a spatial reference through non-linear distortion (power law in TEISI model) expresses a so-called Radon-Nikodym derivative. Indeed given $f(\tau) = C \; \forall\tau \geq 0$ then

$$\int_0^\infty \mu(\{\tau: C \geq r\} \cap [0,t] dr) =$$

$$\int_0^C \mu([0,t]) dr = \mathrm{C}\, \mu([0,t]) = -\int_0^t C\, \mu'([\tau,t]). d\tau \text{ (Eq*)}.$$

The result is thus acquired in this particular case. Suppose now that the function f is increasing we show using integration by parts that under the hypothesis $\mu(\{t\}) = 0$, then:

$$G(t) = \int_{[0,t]} f\ d\mu_h = -\int_0^t \mu'([\tau, t])f(\tau)\ d\tau \text{ (Eq**)}.$$

which is the convolution expected result. It is interesting to recognize in these formulas (* and **) convolution products; we can therefore use the Laplace transform to express the result in a simpler form. Consider $G(s) = s.H(s).F(s)$ with $s \in \mathbb{C}$. If now a continuous and increasing function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is given with $g(0) = 0$, the distorted Lebesgue measure according to h being given, it becomes possible to find a function f as continuous and increasing which verifies (Eq.**), namely $F(t) = \mathcal{L}^{-1}\left(\frac{G(s)}{sH(s)}\right)$. This analysis agrees with the concept of the Radon-Nikodym derivative of a measurement in the classical sense. For both measures ν and μ on $(X,\mathcal{F})$, the measure ν is said absolutely continuous with respect to μ; this can be written $\nu \ll \mu$, if $\mu(A)$ involves $\nu(A)$ whatever $A \in \mathcal{F}$. The Radon-Nikodym theorem then ensures that $\nu(A) = \int_A f\ d\mu$ is unique. The function f is then called the derivative of Radon-Nikodym and is written $\frac{d\nu}{d\mu}$. The computation of Choquet integrals at this stage is restricted to monotonous functions. It is possible to overcome this restriction by using a rearrangement of the measure support function (Annex 1).

6. Choquet integrals and fractional operators

At this step it is appropriate to make some general comments about the categorical approach followed to achieve Fractional Differential Equations [64-66]. The introduction of non-additive measures, or capacities, authorizes to abandon the location hypothesis of finite sets – considered here to simplify the mathematical approach –, consists of defining functions not only in terms of values taken from singletons but in terms of values over any subsets of a given set. Order relations then hold a fundamental place as in category theory. It is instructive to embrace the general evolution from Riemann integral theory of measure to Choquet's capacities (knowing that there are still many other approaches to integration and measurement). With Riemann integral the localization is a basic hypothesis – this localization results in the additive nature of the measures – on the other hand the notion of order is missing. With the Lebesgue's measure the location hypothesis is preserved but, conversely to Riemann integral, the order plays an important role. Finally with Choquet integrals the location hypothesis is abandoned – this amounts to consider general set functions defined not only on the singletons but also on any subset. In addition, to account for non-localizable processes such as those attached to self-similar structures or in connection with fractional derivations, it is no longer possible to reduce the analysis to the Lebesgue integral. Given the fact that order relations are essential for computing when monotonous non-additive measures are involved, it is natural that the Möbius functions, and their inverse zeta functions, figure out a close links with these measures and therefore with the integrals associated with it. The examination carried out by introducing distorted Lebesgue measurements involves, on the one hand, Laplace transforms and, on the other hand, the equivalent of Radon-Nikodym derivatives [48,49]. It becomes quite clear that there is a close link between the Choquet integrals and the fractional derivations as they have been defined and studied from Liouville to Caputo [64-66]. On imagine in the above lineaments how traditional economic theories can be revisited and transformed by introducing the non commutativity of the categorical action and the universality of idempotence, in the models of economic irreversible dynamics; namely the universality of the action (arrow), within connection with the global representation of the market and the society (diagrams). This universality may firstly be based on the Riemann-Liouville fractional integral of the function g over the interval [0,1] defined by:

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau.$$

with α a real positive number. This is an extension of the continuous Cauchy formula whose prominence has been associated with the Yoneda lemma. $\Gamma(\alpha)$ is the classical gamma function that extends the factorial. There is none unique definition of fractional derivation. For a positive real number α, the derivation of order α of the f, function written $d^{\alpha}f(t)/dt^{\alpha}$.

Let us recall that $\mathcal{L}[t^{\alpha}]= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$; then the Riemann-Liouville fractional derivative of order α is defined by:

$$\left[\frac{d^{\alpha}f(t)}{dt^{\alpha}}\right]_{RL} = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-1+n}}d\tau \; for \; n-1 < \alpha < n \\ \frac{d^n f(t)}{dt^n} \; for \; \alpha = n \end{cases}.$$

For a constant $a \neq 0$, The Riemann-Liouville integral is given by:

$$\left[\frac{d^{\alpha}a}{dt^{\alpha}}\right]_{RL} = \frac{a}{\Gamma(1-\alpha)t^{\alpha}} \text{ si } \alpha \neq n$$

which is a non-zero value. The derivative of Grünwald-Letnikov is defined by:

$$\left[\frac{d^\alpha f(t)}{dt^\alpha}\right]_{GL} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - k\Delta t).$$

It should be noted that the right-hand side is the expression extending the n-order derivative. In fact, both definitions of Riemann-Liouville and Grünwald-Letnikov for a given function f, lead to the same result. However, these two definitions suffer from the failure to disregard the initial conditions involved in the calculation of Laplace transforms. To correct this flaw, Caputo proposed another definition, namely:

$$\left[\frac{d^\alpha f(t)}{dt^\alpha}\right]_{C} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{d^n f(\tau)/d\tau^n}{(t-\tau)^{\alpha+1-n}} d\tau \; pour \; n-1 < \alpha < n \\ \frac{d^n f(t)}{dt^n} \; pour \; \alpha = n \end{cases}.$$

In this case fractional derivative of a constant function is zero. Moreover, it is easy to verify that the Riemann-Liouville and Caputo derivatives are identical applied to a function satisfying the constraint $f^{(k)} = 0$ for $0 \leq k \leq n-1$.

Given $r_\alpha = t^\alpha/\alpha$ then $\mathcal{L}(t^\alpha/\alpha) = \Gamma(\alpha)/s^{\alpha+1}$. Applying the previously stated theorem on distorted Lebesgue measure, it comes, for $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a continuous, non-decreasing and measurable function, the Riemann-Liouville integral of g is computed as a Choquet integral with respect to the distorted measure μ_{r_α}, namely:

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha)} \int_0^t g(\tau) d\mu_{r_\alpha}(\tau).$$

The right-hand member is precisely a Choquet integral. Consider a function that is differentiable for the distorted measure μ_{r_α}. therefore

$$\frac{d^k f}{dt^k}(0) = 0 \text{ pour } 1 \leq k \leq n-1$$

$$\mathcal{L}\left[d^n f/dt^n\right] = s^n F - s^{n-1} f(0)$$

$$\mathcal{L}\left[\frac{1}{\Gamma(n-\alpha)} \int (t-\tau)^{n-1-\alpha} \frac{d^n f}{d\tau^n} d\tau\right] = \frac{1}{\Gamma(n-\alpha)} \frac{\Gamma(n-\alpha)}{s^{n-\alpha}} \left(s^\alpha F - s^{n-1} f(0)\right) = s^\alpha \mathrm{F} - s^{n-1} \mathrm{f}(0)$$

$$\mathcal{L}\left[\Gamma(\alpha) \frac{df}{d\mu_{r_\alpha}}\right] = \Gamma(\alpha) \frac{sF - f(0)}{s^2 \mathcal{L}\left[t^\alpha/\alpha\right]} = \Gamma(\alpha) \frac{sF - f(0)}{s^2 \Gamma(\alpha)/s^{\alpha+1}} = s^\alpha F - s^{\alpha-1} f(0).$$

Hence the equality announced. For a differentiable and strictly increasing function: $m: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such as $m(0) = 0$, and for a real positive number α, we introduce the order generator of α defined by: $m^{*\alpha} = \mathcal{L}^{-1}[s^{\alpha-1}M^{\alpha}]$ with M the Laplace transform of m. It is then obvious that for any $m = t$, we have:

$$m^{*\alpha} = \frac{t^{\alpha}}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha)} r_{\alpha} \text{ with } r_{\alpha} = \frac{t^{\alpha}}{\alpha}.$$

One defines the Choquet integral of order α for the g function with respect to μ_m as given by $\int g d\mu_{m^{*\alpha}}$. Therefore: $\mathcal{L}[\int g d\mu_{m^{*\alpha}}] = (sM)^{\alpha} G$.

So in the case where $\alpha = n$ is an integer, we find that $\mathcal{L}^{-1}[s^{n-1}M^{n}] = m^{*n}$. Therefore the fractional Choquet integral is the direct extension of an integer n to a positive real number α. For $m = t$, the fractional Choquet integral coincides with the fractional Riemann-Liouville integral. Likewise when $m = t$, the fractional derivative with respect to μ_m coincides with Caputo fractional derivative. The calculations which have just been briefly described show that to a large extent the concept of fractional derivation, in line with the works of Liouville, Riemann, Grünwald, Letnikov and more recently Caputo, is directly related to the concept of non-additive measure and Choquet integral. The link of this type of integral with the zeta functions of Riemann and Mobiüs, already highlighted [12,50-54], is related to the categorical foundations of the concept of measure in terms of division, thus in the physical processes, to the geometrical absence of commutativity, therefore due to Fractal geometry in the TEISI model [55,56].

7. A geometrical approach of "zeta management"

The standard analysis of the dynamics of economy is mainly based on the assumption that equilibria and steady states are to be seen as the solutions of partial differential equations characterized by integer order and by spatially and temporally set-state constants assumed to be well defined. The analysis is local under constraint of boundary conditions a priori also defined. Variational developments based on “variations of constants” make it possible to reach optima of multi local behaviors under external constraints. The analysis based on set theory is proved very largely reductive and not in conformity with the functioning of the economy which involves correlations, non-local exchanges and most of the time deferred in timing. The immanent dynamics, therefore monadic one, the self-organization etc., are not considered in spite of regularity, universality of none-convergent power laws and self-similar properties generally observed. The reason is nonetheless rather simple: economic systems as language systems, are not additive and there is a grammar and long range correlations can induce efficiency or butterfly-effects, out of proportion with the optima provided by local variational analysis. Thus, for example, we see small teams performing miracles of creativity and productivity and, conversely, large resources invested can be dissipated in redundant, though perfectly thought organizations. Financial bubbles are examples of the harmfulness of correlations that in other contexts can be positive. However, the problem of additivity versus non-additivity of integrals in economic approaches can very usefully be apprehended by using the example of the project management. It can indeed be asserted that the paradigm of additive integrals is based on the implicit use of Poincaré's automorphic groups, which represent, as Poincaré has shown with his work on Fuchsian functions, the set of solutions of the integer differential equations. The diagram below summarizes the properties of auto internal correlations associated with any differential equations whose solutions are developable over an exponential basis. We recall that the Fourier Transformation of an exponential is given by the inversion of a straight line with respect to a singularity out of this line, in the complex plane: $Z \sim 1/(1+ i\ \omega\ \tau)$. In Project Management, the global task is divided into sub-tasks (each with its time constant), which are grouped in batches that can be run in series or in parallel. The representation of their organization in the complex plane used below makes it possible to distinguish the part of the task (the semicircle) which has a cost (real axis) of a non-dissipative part, delayed in its the execution, carried by the imaginary axis. Any semi-circle represents the inversion of an action line, vertical conveying the temporality of the task, inversion with respect to a singular point whose distance to the right represents the entropy associated with the task according to a priority given in the global time table.

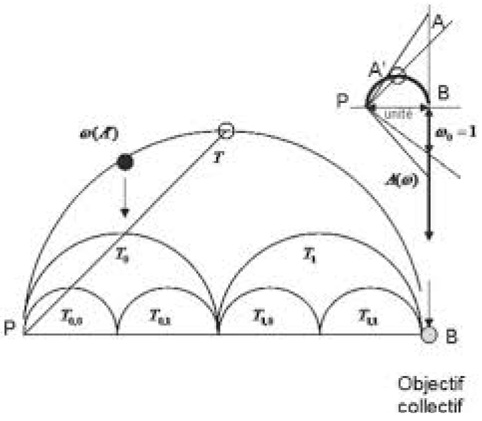

Figure 10. Schematic representation of the tiling of the complex plane underlining both (i) the Poincaré Fuschian like covering of an exponential dynamics and (ii) a representation of the sharing of any exponential action within a set of tasks characterized by the same dynamic properties. A usual mere self-similarity can be noticed herein. The link with standard Fourier transform of exponential is obvious.

Under its apparent objectivity this structuring of the project is however misleading. Indeed in the strict mathematical interpretation, to be perfectly accomplished every task requires infinite time. However, as the set-theoretic additivity hypothesis is applicable, the project manager considers at his level that each of the tasks must be achieved within a time given by its specific time constant of the task; a constant given by an exponential dynamics considered as convergent. The project manager thus legitimately assigns to the global process a schedule based on the sum of all these constants of time. The fact that each geodesic is none other than the Fourier transform of an exponential function with such a time constant merely justifies the managing hypothesis. However, the use of this hypothesis implies another hypothesis highly questionable: the strict local equivalence between the implementation of the task and partition up to the level of the agent, namely the completeness of the set covering of the tasks; the field of the actor is here considered as a hardware, a mechanical kit, extremely strong constraints on the human beings exclusively applicable for a Turing machine. But for a project, the machinist point of view is thinkable only at the global level not at a local level which is constraint by internal correlations between the tasks down to the level of the worker. Then the equivalence construction / partition (addition / multiplication) namely the duality applied on the tasks must take into account the none additivity and therefore the analysis has to introduce the role of universal functions according to the Voronin like approximation [67-69]. Among these universal function is the zeta Riemann function whose link with non-additive self-similar systems has been proved [50-54]. Returning to the so-called Poincaré representation above, the task to be performed is approximated on the basis of an action that locates its horizon on the real axis. The real axis thus represents the edge of the global task in its perfect fulfillment and the entropy (the cost of the task) is a real number. But it is no more the case for non-additive system. To understand the subtlety we have to point out that the cost of the task may be embedded in the field of complex number according to the diagram below, which leads to an interrelation of tasks and overlapping between them and makes singularities to emerge in the complex field. We have proved elsewhere that this interrelation is of a financial nature and obviously has an influence on the cost of the project [12].

Our analysis gives rise to a Poincaré-Grothendieck representation directly based on non-integer order differential equations, furthermore completed in a Galoisian way (economy / finance completion [12]). In this diagram, the standard Poincaré tiling of the complex plane by semicircles is transformed to make the place for interactions here represented by overlapping semi-circles. Exactly as it is possible to extend a straight line, any arc of a circle can be prolonged, but this time in a bounded manner by a singularity on zero. Thus the crossing of the semi-circle points out the presence of singularities at infinity (zero frequency) on dynamic tiling of the complex plane. These singularities are the mark of a suspension of the running of the physical time (the time of the experimenter different of the timing determined by the project manager clock). The suspension, clearly observed in the diagrams, is a physical representation of the Husserl's Epoché [70] which gives rise to the need of non-causal completion (motivation, knowledge, emergence of the currencies etc. [12]). The singular points of zero frequency are indeed located in a position out of the real entropic axis. The incompleteness and the necessity of extension of the non-integer order differential operator are here clearly perceived.

This diagram can be analyzed from equation of the causal arc $Z_\alpha \sim 1/[1+(i\omega\tau)^\alpha]$ which is precisely the integro-differentiation operator of non-integer order. It is obtained from a generalization of the exponential function called alpha-exponential. It has been shown that it is related to Riemann zeta function. The extension of the arc of representation pointing out the Epoché, introduces, an a-causal character because, according to a Galoisian way of thinking, it brings together in the same set (the arc for completion), all the possible discrete values of the time constants compatible with the dynamic equations. He can be shown that it is associated with the function of Möbius precisely dual of the zeta function, relevant for the causal part of the dynamics. The link between the two functions ultimately ensures an overall additivity justifying the timing of the complex plan, tiling required for the definition of a condition of possibility of the concept of project management in complex (self-similar) environment. As to remind, it has been shown elsewhere [12] that currency in relation to the real economy could be associated with this Möbius function. Therefore, in the same way that the complex plane can be "auto-morphically" paved with semicircles, it is natural to pave the complex plane by duopolies of any alpha value (the order of derivation).

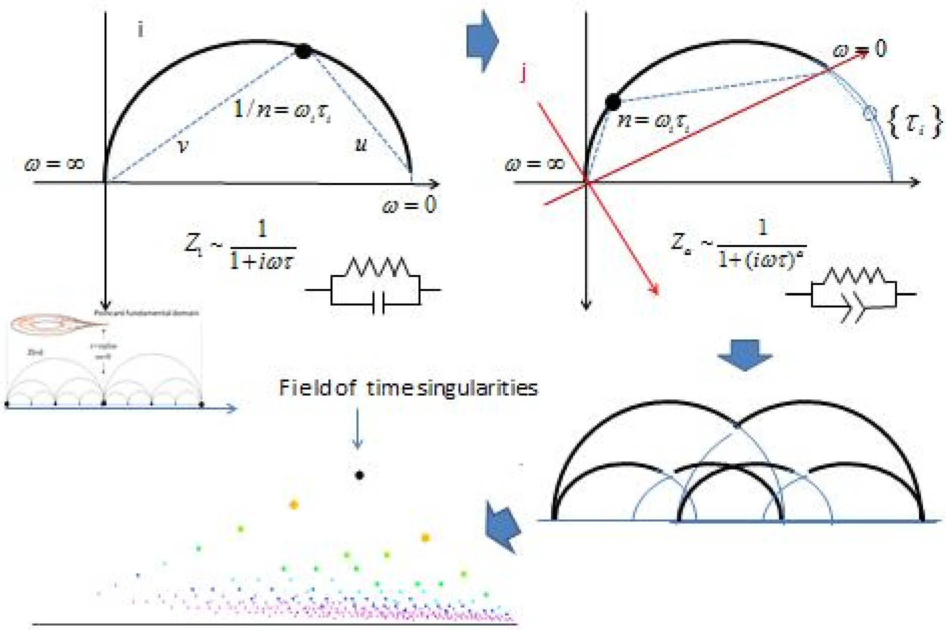

Figure 11. Extension of the figure 10 (Poincaré tiling) when fractional differentiation and completion are implemented. This extension is characterized by the overlapping of the semi-circles and the emergence of a field or singularities outside the real axis. This fields points out the role of the Husserl like époque, suspension of the action and complex extension of the unit of time required to consider the field of possible, namely the future or the completion of our representation of the world.

As shown in the diagram (Fig. 11), the representation gets complicated because the separable tasks in very elementary models of tasks management overlap systematically resulting that each of them possesses a singularity of whose anti-entropic character -concretized by the matching between the dual notions of entropy/anti-entropy- can precisely be quantified. As shown in the set of singularities, the link between field of complex numbers (the plane) seeded with singularities and the field of real numbers (the dynamics) is then characterized by auto-morphisms which builds a non-commutative geometry [71]. The auto-correlated system then appears systematically less dissipative than the strictly hierarchical system from tasks distinguished according to Poincaré tiling. But the model reveals features infinitely more paradoxical ...

The exponentiation alpha operator modifies our representation of time. The time of the hierarchical system according to Poincaré is none other than the time of the clock and the hierarchy operates by using time constant directly associated with a distance on the temporal line associated with the action seen as physical relaxation. This means that one can be assigned to each task a certain velocity. The time associated with the Poincaré Grothendieck representation proposed here is a radically different time because as complex number, it is composed of a usual temporality stricken by a singularity on board. The imaginary component of entropy, the anti-entropy component, carries the information associated with the irreducible collective share of the individual action and as well, a factor of anticipation, social investment, and rest. This element is perfectly quantifiable here as well as its dual complex component through the Mobius field. This field is the distribution of the whole possibilities offered in front of a sole managerial choice related to initial situation of choices which in fine cannot be many. To consider the Mobius component defines and expresses the trust within the link between the collective (global) and the individual (local) decision. By fixing the global and but also the distribution of the investment required by the correlations between tasks, the manager not only operates a pull back of the additive paradigms to be carried out but also determines the distribution of the time constants compatible with the global action. This entropic complex and represented in the figure 11 through an edge of the dynamics as a set of singular points located at the infinity. We call Zeta-Management the management classes that implement the operation that relies on a collective operation based on Poincaré Grothendieck type auto-morphisms here clearly opposed quantitative and machinist managerial strategies based on Poincaré automorphisms and on an additive set-theoretic vision, for example the military tactic, whose purpose can only be entropic (cost reduction without any consideration for intangible capital).

8. *Pro Tempore* Conclusion

The present work confirms Zeljko Rohatinski's hypothesis as regard the need to consider the concept of functional time in economics, a thesis explained in his recent book "Time and Economy" [1].

"The main purpose of this book is to introduce, explain, and demonstrate the importance of determining the time dimension of economic activity. This book employs an understanding of time as a relative concept against a background from modern physics and philosophy, and it shows that such an approach applied to economics enables better comprehension of the forms and modalities of economic activity".

"Definitions and understanding of time through the history have been presented through works of Aristotle, Descartes, Newton, Boscovich, Einstein, Bloch, and others who observe it as an objective category, as well as through the conclusions of Plotinus, Augustine, Leibnitz, Kant, Heidegger, Husserl, Merleau-Ponty, and others who perceive it as a subjective category. In questioning the problem of absolute (Newton) and relative time (Einstein), special focus is put on Whitehead's philosophical description of time relativity, which analyzed the space-time structure of an individual event, as well as the impact of that structure on relations of this event with other events.

Far from Einstein Relativity, the idea of writing the time variable in the field of complex numbers and of coupling this tactic with Riemann and Mobiüs functions logins very naturally to the framework of this thesis but, in addition, opens new issues even in physics, though this analytic solution is not the only one. Indeed, the time appears in the frame of category theory as an homological universal invariant of self-similar structures, namely to summary the Haudorff content. Therefore the theory of categories not only helps us to fully agree to what Rohatinski has written about the need of revisiting the concept of time, but also to give a categorical "measure" to this point of view. Virtually, his intuition anticipated this possibility:

"The economical phenomena observed in this model (the model of the Marx's Capital but herein the topos model of Poincaré Grothendieck) – *the appearance of dilation of time* (Hausdorff content) *and the contraction of costs in economic systems and sub-systems (*anti entropy*) that "move" relative to each other* (in scales) – *are analogous to the ones in physics-dilation of time and contraction of lengths of subjects in the inertial reference frames that move relative to each other that are described by modified Lorentz transformations (such as Einstein, 1950, and Ugarov, 1979). This implies different capital efficiency in these two systems, despite the equal initial conditions of their reproduction (*role of initial conditions*). Such an understanding of time has direct*

implications on modalities of economic activity (fractional differential equations) *and should be taken into account when considering the time dimension or maturity of economic activity, for instance in creating economic policy".*

We will simply assert here that the movements evoked and the spaces to which Rohatinski refers are, in above studies, extended to the drift within scaling and also to the spaces characterized by curvatures because the problems of dynamics appear in all hyperbolic spaces, moreover most of the time coupled with self-similarity. In project management these properties are likely to give rise to overlapping of the tasks, to some rules of homologies and internal project morphisms (categorization) whose most beautiful expression is none other than the social function of the enterprise that in fine justifies the concept of human project and the labor in a share common field. The categorical concepts taken into account to formalize this common field are synthesized in what we call the Zeta Management.

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Knowledgement: Materials Design SARL and Inc. and Federal University of Kazan for financial supports and Zeta Innovation for scientific exchanges.

ANNEX

For f: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a Lebegue measure λ, a reordering equi-mesurable of f with respect to λ is a function $f^{\sim}: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is non decreasing and verifies: $\lambda(f \geq t) = \lambda(f^{\sim} \geq t)$ $t \in \mathbb{R}_+$ If such a function exits therefore : $\int f \, d\mu_h = \int f^{\sim} \, d\mu_h$ for any distorted Lebesgue measure $\mu_h = h \circ \lambda$. If f: $[0, t] \longrightarrow \mathbb{R}_+$ is such as $max_{x \in [0,t]} f(x) = M$, if f is continuous and if λ is confirmed as a Lebesgue measure on $\mathbb{R}_+$ then we have $f^{\sim}$: $[0, t] \longrightarrow [0, M]$ given by: $f^{\sim}(\tau) = G_{\lambda,f}^{-1}(t - \tau)$ which is a non-decreasing reordering equi-measurable function. We know that $G_{\lambda,f}$ is decreasing and then inversible, since f is continuous and λ is a Lebesgue measure. If we have $G_{\lambda,f}(x) = \alpha$, then $x = f^{\sim}(t - \alpha)$. Therefore: $G_{\lambda,f^{\sim}}(x) = \lambda(f^{\sim} \geq x) = t - (t - \alpha) = \alpha = G_{\lambda,f}(x)$.