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# Estimates of Hardy-Rellich constants for polyharmonic operators and their generalizations

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## Abstract

© Avkhadiev F.G. 2017. We prove the lower bounds for the functions introduced as the maximal constants in the Hardy and Rellich type inequalities for polyharmonic operator of order  $m$  in domains in a Euclidean space. In the proofs we employ essentially the known integral inequality by O.A. Ladyzhenskaya and its generalizations. For the convex domains we establish two generalizations of the known results obtained in the paper M.P. Owen, Proc. Royal Soc. Edinburgh, 1999 and in the book A.A. Balinsky, W.D. Evans, R.T. Lewis, The analysis and geometry of Hardy's inequality, Springer, 2015. In particular, we obtain a new proof of the theorem by M.P. Owen for polyharmonic operators in convex domains. For the case of arbitrary domains we prove universal lower bound for the constants in the inequalities for  $m$ th order polyharmonic operators by using the products of  $m$  different constants in Hardy type inequalities. This allows us to obtain explicit lower bounds for the constants in Rellich type inequalities for the dimension two and three. In the last section of the paper we discuss two open problems. One of them is similar to the problem by E.B. Davies on the upper bounds for the Hardy constants. The other problem concerns the comparison of the constants in Hardy and Rellich type inequalities for the operators defined in three-dimensional domains.

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## Keywords

Convex domain, Hardy inequality, Polyharmonic operator, Rellich inequality

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