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On Generalization of Sierpiński Gasket in Lobachevskii Plane

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Abstract—We construct an analogue of Sierpiński gasket in Lobachevskii plane by means of iterated function system with maps from a transformation group of this space. The investigation of a new family of attractors and a Mandelbrot set associated with it reveals higher capacity of Lobachevskii geometry compared to that of Euclid.

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1. INTRODUCTION

This paper comes from motivation to build fractal sets inside Lobachevskii space with respect to its geometry (and not merely as images of fractals from Euclidean space). For this purpose, we use iterated function systems with maps from some "affine" transformation group of Lobachevskii space. Due to computational problems in hyperbolic geometry (complex formulae with transcendental functions), in this paper we constrain ourselves to Beltrami—Klein model of dimension 2.

Our first touchstone in this direction is a construction of Sierpiński gasket, and yet this originally simple example reveals many computational and theoretical problems as well as rich properties of Lobachevskii geometry compared to Euclidean.

We have to recall here some basic facts we need about iterated function systems, Sierpiński gasket and Lobachevskii plane.

The notion of iterated function system (IFS) and related basic results were established in [1] and [2] (see also [3] for general reference on fractals). Let X be a complete metric space and $f = \{f_i\colon X \to X\}_{i=1}^N$ be a set of contractions ($N \in \mathbb{N}$). We call the pair (X, f) *iterated function system*. For such system, there is a unique non-empty compact invariant set $A = \cup_{i=1}^N f_i(A)$ called its *attractor*. This set, being composed of several images of itself, is often a fractal set, and can be obtained efficiently by some iterative algorithms [3, 4]. IFS is a powerful way to generate, analyse and utilize huge family of fractal sets.

Consider the following IFS in $\mathbb{R}^2$ with two parameters $\lambda \in (0, 1)$ and $c \in (0, \infty)$:

$$f_i(x) = \lambda x + ce_i, \quad i = \overline{1, 3}, \tag{1}$$

where $e_1 = \{-\frac{\sqrt{3}}{2}, -\frac{1}{2}\}$, $e_2 = \{\frac{\sqrt{3}}{2}, -\frac{1}{2}\}$, $e_3 = \{0, 1\}$. Let S_λ be its attractor. Topological properties of S_λ depend on λ, whereas c only controls its size (see Fig. 1):

- for $\lambda \in (0, \frac{1}{2})$, S_λ is totally disconnected (a Cantor set);
- for $\lambda = \frac{1}{2}$, S_λ is the (conventional) Sierpiński gasket;

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