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Decentralised Bilateral Trading in a Market with Incomplete Information

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Abstract

We study a model of decentralised bilateral interactions in a small market where one of the sellers has private information about her value. There are two identical buyers and another seller, whose valuation is commonly known to be in between the two possible valuations of the informed seller. We consider two infinite horizon games, with public and private simultaneous one-sided offers respectively and simultaneous responses. We show that there is a stationary perfect Bayes' equilibrium for both models such that prices in all transactions converge to the same value as the discount factor goes to 1.

JEL Classification Numbers: $C78$, $D82$

Keywords: Bilateral Bargaining, Incomplete information, Outside options, Coase conjecture.

1 Introduction

This paper studies a small market in which one of the players has private information about her valuation. As such, it is a first step in combining the literature on incomplete information with that on market outcomes obtained through decentralised bilateral bargaining.

We shall discuss the relevant literature in detail later on in the introduction. Here we summarise the motivation for studying this problem.

One of the most important features in the study of bargaining is the role of outside options in determining the bargaining solution. There have been several different approaches to this issue, starting with treating alternatives to the current bargaining game as exogenously given and always available. Accounts of negotiation directed towards practitioners and policy-oriented academics, like Raiffa's masterly "The Art and Science of Negotiation",([32]) have emphasised the key role of the "Best Alternative to the Negotiated Agreement" and mentioned the role of searching for such alternatives in preparing for negotiations. Search for outside options has also been considered, as well as search for bargaining partners in a general coalition formation context.

Real world examples of such search for outside options are abound. For example, firms that receive (public) takeover bids seek to generate other (also public) offers in order to improve their bargaining position. Takeovers are an instance also of public one-sided offers, where all the offers are by the buyer. The housing market is another example; there is a given (at any time) supply of sellers and buyers who are interested in a particular kind of house make (private) offers to the sellers of the houses they are interested in, one at a time. (This is, for instance, the example used in [25].)

Private targeted offers are prevalent in industry as well, for joint ventures and mergers. For example, the book [1] is concerned with the joint venture negotiations in the 1980s, in which Air Products, Air Liquide and British Oxygen were buyers and DuPont, Dow Chemical and Monsanto were sellers (of a particular kind of membrane technology). The final outcome of these negotiations were two joint ventures and one acquisition.

Proceeding more or less in parallel, there has been considerable work on bargaining with incomplete information. The major success of this work has been the complete analysis of the bargaining game in which the seller has private information about the minimum offer she is willing to accept and the buyer, with only the common knowledge of the probability distribution from which the seller's reservation price is drawn, makes repeated offers which the seller can accept or reject; each rejection takes the game to another period and time is discounted at a common rate by both parties. With the roles of the seller and buyer reversed, this has also been part of the development of the foundations of dynamic monopoly and the

Coase conjecture. Other, more complicated, models of bargaining have also been formulated (including by one of us), with two-sided offers and two-sided incomplete information, but these have not usually yielded the clean results of the game with one-sided offers and one-sided incomplete information.

Whilst this need not necessarily be a reason for studying this particular game, it does suggest that if we desire to embed bargaining in a more complex market setting with private information, it is rational for us, the modellers, to minimise the extent of complexity associated with the bargaining to focus on the changes introduced by adding endogenous outside options, as we intend to do here.

Our model therefore takes the basic problem of a seller with private information and an uninformed buyer and adds another buyer-seller pair; here the new seller's valuation is different from the informed seller's and commonly known and the buyers' valuations are identical. Each seller has one good and each buyer wants at most one good. This is the simplest extension of the basic model that gives rise to outside options for each player, though unlike the literature on exogenous outside options, only *one* buyer can deviate from the incomplete information bargaining to take his outside option with the other seller (if this other seller accepts the offer), since each seller only has one good to sell.

In our model, buyers make offers *simultaneously*, each buyer choosing only one seller.[1] Sellers also respond simultaneously, accepting at most one offer. A buyer whose offer is accepted by a seller leaves the market with the seller and the remaining players play the one-sided offers game with or without asymmetric information. We consider both the cases where buyers' offers are public, so the continuation strategies can condition on both offers in a given period, and private, when only the proposer and the recipient of an offer know what it is and the only public information is the set of players remaining in the game. Our analysis explores whether a Perfect Bayes Equilibrium similar to that found in the two-player asymmetric information game continues to hold with alternative partners on both sides of the market and with different conditions on observability of offers.

The equilibrium we describe is in (non-degenerate) randomized behavioral strategies (as in the two-player game). As agents become patient enough, in equilibrium competition always takes place for the seller whose valuation is commonly known. The *equilibrium* behavior of beliefs is similar to the two-player asymmetric information game and the same across public and private offers. However, the off-path behaviour sustaining this equilibrium is different and has to take into account many more possible deviations. The path of beliefs also differs once an out-of-equilibrium choice occurs. The case of private offers is quite interesting. For example a buyer who offers to the informed seller might see his offer rejected

[1] Simultaneous offers extensive forms probably capture the essence of competition best.

but his expectation that the other offer has been accepted is belied when he observes all players remain in the market. He is then unsure of whether the other buyer has deviated and made an offer to the informed seller, which the informed seller has rejected, or an offer to the seller with commonly known valuation. The beliefs have to be constructed with some care to make sure the play gets back to the equilibrium path (and to be plausible).

The interesting asymptotic characterisation obtained by taking the limit of the equilibrium prices, as the discount factor goes to 1, is that, despite the asymmetric information and two heterogeneous sellers, the different distributions of prices collapse to a single price that is consistent with an extended Coase conjecture.[2]

In the two-player game, the Perfect Bayesian Equilibrium is unique in the "gap" case. In our competitive setting, this is not true, at least for public offers. We include an example.

The intuition behind these results can be explained in the following way. In the benchmark case, when one of the sellers' valuations is known to be H and the other M, in the Walrasian setting there will be excess demand at any prices $p \in (M, H)$. This would suggest that the prices should move towards H. However, this depends on the nature of the market interactions. (In an alternating offers extensive form, [7], this does not happen either for public or private offers, and the two give different results.) We model an explicit trading protocol with simultaneous, one-sided offers made by the buyers. As $\delta \to 1$, the offers converge to H and the trade takes place immediately. It is generally supposed that simultaneous offers capture the essence of competition (a la Bertrand) but here the result is true in conjunction with one-sided offers and the earlier cited work shows it does not hold if sellers also make offers.

The private information case builds on the known results for the two-player game as well as using the intuition of the previous paragraph. For the two-player game with the privately informed seller, we know that as $\delta \to 1$, the price converges to H and trade takes place almost immediately. Thus, in the limit, the reservation price of the informed seller is H, regardless of her type. This suggests that, for high enough δ,we can use the results of the benchmark case to construct equilibrium strategies in which the offers to both the sellers tend to H. This turns out to be true, though constructing such an equilibrium involves careful analysis of beliefs off the equilibrium path. Incidentally, [8] show in an environment similar to this that, if buyers made offers sequentially rather than simultaneously, there would be an equilibrium

[2] The "Coase conjecture" relevant here is the bargaining version of the dynamic monopoly problem, namely that if an uninformed seller (who is the only player making offers) has a valuation strictly below the informed buyer's lowest possible valuation, the unique sequential equilibrium as the seller is allowed to make offers frequently, has a price that converges as the frequency of offers becomes infinite to the lowest buyer valuation. Here we show that even if one adds endogenous outside options for both players, a similar conclusion holds for an equilibrium that is common to both public and private offers-hence an extended Coase conjecture holds.

in which the second buyer to move would get a strictly higher payoff than the first and that the limiting behaviour would be different from the simultaneous offers extensive form of this paper. ([8] also has one-sided offers by buyers only.)

The result of this paper is not confined to uncertainty described by two types of seller. Even if the informed seller's valuation is drawn from a continuous distribution on $[L, H]$, we show that the asymptotic convergence to H still holds.

There could be other equilibria where essentially the buyers tacitly collude on offers.

The previous analysis assumed that the goods themselves were of identical quality. In this paper, we also consider a simple two-period game to show what happens when we have quality-differentiated goods, that is, the buyer's value is a function of the seller's type. Here the general convergence result does not necessarily hold, though if the probability of a H type of seller is sufficiently low, the offer made to the informed seller goes to H as the discount factor goes to 1.

Related literature: The modern interest in this approach dates back to the seminal work of Rubinstein and Wolinsky ([33], [34]), Binmore and Herrero ([5])and Gale ([16]),[17]). These papers, under complete information, mostly deal with random matching in large anonymous markets, though Rubinstein and Wolinsky (1990) is an exception. Chatterjee and Dutta ([7]) consider strategic matching in an infinite horizon model with two buyers and two sellers and Rubinstein bargaining, with complete information. In a companion paper (Chatterjee-Das 2011 [6]), we analyse markets under complete information where the bargaining is with one-sided offers.

There are several papers on searching for outside options, for example, Chikte and Deshmukh ([12]), Muthoo ([27]), Lee ([26]), Chatterjee and Lee ([11]). Chatterjee and Dutta ([8]) study a similar setting as this paper but with sequential offers by buyers.

A rare paper analysing outside options in asymmetric information bargaining is that by Gantner([21]), who considers such outside options in the Chatterjee-Samuelson ([10]) model. Our model differs from hers in the choice of the basic bargaining model and in the explicit analysis of a small market with both public and private targeted offers. (There is competition for outside options too, in our model but not in hers.)

Some of the main papers in one-sided asymmetric information bargaining are the well-known ones of Sobel and Takahashi([36]), Fudenberg, Levine and Tirole ([14]), Ausubel and Deneckere ([2]). The dynamic monopoly papers mentioned before are the ones by Gul and Sonnenschein ([22]) and Gul, Sonnenschein and Wilson([23]). See also the review paper of Ausubel, Cramton and Deneckere ([3]).

There are papers in very different contexts that have some of the features of this model. For example, Swinkels [38] considers a discriminatory auction with multiple goods, private values (and one seller) and shows convergence to a competitive equilibrium price for fixed supply as the number of bidders and objects becomes large. We keep the numbers small, at two on each side of the market. Hörner and Vieille [25] consider a model with one informed seller, two buyers with correlated values who are the only proposers and both public and private offers. They show that, in their model unlike ours, public and private offers give very different equilibria; in fact, public offers could lead to no trade.

Outline of rest of the paper. The rest of the paper is organised as follows. Section 2 discusses the model in detail. The qualitative nature of the equilibrium and its detailed derivation is given in section 3. The asymptotic characteristics of the equilibrium are obtained in Section 4. Section 5 analyses a model where the informed seller's valuation is drawn from a continuous distribution on $[L, H]$ and Section 6 discusses the possibility of other equilibria. An analysis with a simple two period game to show what happens when we have quality-differentiated goods is done in Section 7 and finally Section 8 concludes the paper.

2 The Model

2.1 Players and payoffs

The setup we consider has two uninformed homogeneous buyers and two heterogeneous sellers. Buyers (B_1 and B_2) have a common valuation of v for the good (the maximum willingness to pay for a unit of the indivisible good). There are two sellers. Each of the sellers owns one unit of the indivisible good. Sellers differ in their valuations. The first seller (S_M) has a reservation value of M which is commonly known. The other seller (S_I) has a reservation value that is private information to her. S_I's valuation is either L or H, where,

$$v > H > M > L$$

We assume that $L = 0$, for purposes of reducing notation. It is commonly known by all players that the probability that S_I has a reservation value of L is $\pi \in (0, 1)$. It is worthwhile to mention that $M \in [L, H]$ constitutes the only interesting case. If $M < L$ (or $M > H$) then one has no ambiguity about which seller has the lowest reservation value. Although our model analyses the case of $M \in (L, H)$, the same asymptotic result will be true for $M \in [L, H]$ (even though the analytical characteristics of the equilibrium for $\delta < 1$ are different).

Players have a common discount factor $\delta \in (0, 1)$. If a buyer agrees on a price p^j with seller S_j at a time point t, then the buyer has an expected discounted payoff of $\delta^{t-1}(v - p^j)$.

The seller's discounted payoff is $\delta^{t-1}(p^j - u_j)$, where u_j is the valuation of seller S_j.

2.2 The extensive form

This is an infinite horizon, multi-player bargaining game with one sided offers and discounting. The extensive form is as follows:

At each time point $t = 1, 2, ..$, offers are made simultaneously by the buyers. The offers are targeted. This means an offer by a buyer consists of a seller's name (that is S_I or S_M) and a price at which the buyer is willing to buy the object from the seller he has chosen. Each buyer can make only one offer per period. Two informational structures will be considered; one in which each seller observes all offers made (*public targeted offers*) and one (*private targeted offers*) in which each seller observes only the offers she gets. (Similarly for the buyers after the offers have been made-in the private offers case each buyer knows his own offer and can observe who leaves the market.) A seller can accept at most one of the offers she receives. Acceptances or rejections are simultaneous. Once an offer is accepted, the trade is concluded and the trading pair leave the game. Leaving the game is publicly observable (irrespective of *public* or *private* offers). The remaining players proceed to the next period in which buyers again make price offers to the sellers. As is standard in these games, time elapses between rejections and new offers.

3 Equilibrium

We will look for *Perfect Bayes Equilibrium*[15] of the above described extensive form. This requires sequential rationality at every stage of the game given beliefs and the beliefs being compatible with Bayes' rule whenever possible, on and off the equilibrium path. The PBE obtained is *stationary* in the sense that the strategies depend on the history only to the extent to which it is reflected in the updated value of π (the probability that S_I's valuation is L). Thus at each time point buyers' offers depend only on the number of players remaining and the value of π. The sellers' responses depend on the number of players remaining, the value of π and the offers made by the buyers.

3.1 The Benchmark Case: Complete information

Before we proceed to the analysis of the incomplete information framework we state the results of the above extensive form with complete information. A formal analysis of the complete information framework has been done in a companion paper.

Suppose the valuation of S_I is commonly known to be H. In that case there exists a stationary equilibrium (an equilibrium in which buyers' offers depend only on the set of players present and the sellers' responses depend on the set of players present and the offers made by the buyers) in which one of the buyers (say B_1) makes offers to both the sellers with positive probability and the other buyer (B_2) makes offers to S_M only. Suppose $E(p)$ represents the expected maximum price offer to S_M in equilibrium. Assuming that there exists a unique $p_l \in (M, H)$ such that,

$$p_l - M = \delta(E(p) - M)^3$$

, the equilibrium is as follows:

1. B_1 offers H to S_I with probability q. With the complementary probability he makes offers to S_M. While offering to S_M, B_1 randomises his offers using an absolutely continuous distribution function $F_1(.)$ with $[p_l, H]$ as the support. F_1 is such that $F_1(H) = 1$ and $F_1(p_l) > 0$. This implies that B_1 puts a mass point at p_l.

2. B_2 offers M to S_M with probability q'. With the complementary probability his offers to S_M are randomised using an absolutely continuous distribution function $F_2(.)$ with $[p_l, H]$ as the support. $F_2(.)$ is such that $F_2(p_l) = 0$ and $F_2(H) = 1$.

Appendix(A) establishes the existence and uniqueness of p_l.

The outcome implied by the above equilibrium play constitutes the unique stationary equilibrium outcome. It is shown in [6] that, as $\delta \to 1$,

$$q \to 0 \text{ , } q' \to 0 \text{ and } p_l \to H$$

This means that as market frictions go away, we tend to get a uniform price in different buyer-seller matches. In this paper, we show a similar asymptotic result even with incomplete information, with somewhat different analysis.

3.2 Equilibrium of the one-sided incomplete information game with two players

The equilibrium of the whole game contains the analyses of the different two-player games as essential ingredients. If a buyer-seller pair leaves the market after an agreement and the other pair remains, we have a continuation game that is of this kind. We therefore first review the features of the two-player game with one-sided private information and one-sided

[3] Given the nature of the equilibrium it is evident that $M(p_l)$ is the minimum acceptable price for S_M when she gets one(two) offer(s).

offers.

The setting is as follows: There is a buyer with valuation v, which is common knowledge. The seller's valuation can either be H or L where $v > H > L = 0$. At each period, the remaining buyer makes the offer and the remaining (informed) seller responds to it by accepting or rejecting. If the offer is rejected then the value of π is updated using Bayes' rule and the game moves on to the next period when the buyer again makes an offer. This process continues until an agreement is reached. The equilibrium of this game(as described in, for example, [13]) is as follows.

For a given δ we can construct an increasing sequence of probabilities, $d(\delta) = \{0, d_1, ..., d_t, ...\}$ so that for any $\tilde{\pi} \in (0, 1)$ there exists a $t \geq 0$ such that $\tilde{\pi} \in [d_t, d_{t+1})$. Suppose at a particular time point the play of the game so far and Bayes' Rule implies that the updated belief is π. Thus there exists a $t \geq 0$ such that $\pi \in [d_t, d_{t+1})$. The buyer then offers $p_t = \delta^t H$. The H type seller rejects this offer with probability 1. The L type seller rejects this offer with a probability that implies, through Bayes' Rule, that the updated value of the belief $\pi^u = d_{t-1}$. The cutoff points d_t's are such that the buyer is indifferent between offering $\delta^t H$ and continuing the game for a maximum of t periods from now or offering $\delta^{t-1} H$ and continuing the game for a maximum of $t - 1$ periods from now. Thus here t means that the game will last for at most t periods from now. The maximum number of periods for which the game can last is given by $N(\delta)$. It is already shown in [13] that this $N(\delta)$ is uniformly bounded by a finite number N^* as $\delta \to 1$.

Since we are describing a PBE for the game it is important that we specify the off-path behavior of the players. First, the off-path behavior should be such that it sustains the equilibrium play in the sense of making deviations by the other player unprofitable and second, if the other player has deviated, the behavior should be equilibrium play in the continuation game, given beliefs. We relegate the discussion of these beliefs to appendix (B).

Given a π, the expected payoff to the buyer $v_B(\pi)$ is calculated as follows:

For $\pi \in [0, d_1)$, the two-player game with one-sided asymmetric information involves the same offer and response as the complete information game between a buyer of valuation v and a seller of valuation H. Thus we have

$$v_B(\pi) = v - H \text{ for } \pi \in [0, d_1)$$

For $\pi \in [d_t, d_{t+1})$, ($t \geq 1$), we have,

$$v_B(\pi) = (v - \delta^t H)a(\pi) + (1 - a(\pi))\delta(v_B(d_{t-1})) \tag{1}$$

where $a(\pi)$ is the equilibrium acceptance probability of the offer $\delta^t H$.

These values will be crucial for the analysis of the four-player game.

3.3 Equilibrium of the four-player game with incomplete information.

We now consider the four-player game. The complete-information benchmark case suggests that there will be competition among the buyers for the more attractive seller, in the sense that that seller will receive two offers with positive probability in equilibrium, whilst the other seller will obtain at most one. However, the difference arises here because of the private information of one of the sellers. Even if one pair of players has left the market, a seller with private information has some power arising from the private information. In fact, for δ high enough, this residual power of the informed seller leads, in equilibrium, to competition taking place for the other seller (whose value is common knowledge), even if π is relatively high. The main result of this paper is described in the following proposition.

Proposition 1 *There exists a $\delta^* \in (0,1)$ such that if $\delta > \delta^*$, then for all $\pi \in [0,1)$ there exists a stationary equilibrium as follows (both public and private offers:):*

(i) One of the buyers (say B_1) will make offers to both S_I and S_M with positive probability. The other buyer B_2 will make offers to S_M only.

(ii) B_2 while making offers to S_M will put a mass point at $p_l^{'}(\pi)$ and will have an absolutely continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$ where $p_l^{'}(\pi)$ $(p_l(\pi))$ is the minimum acceptable price to S_M when she gets one(two) offer(s). For a given π, $\bar{p}(\pi)$ is the upper bound of the price offer S_M can get in the described equilibrium $(p_l^{'}(\pi) < p_l(\pi) < \bar{p}(\pi))$. B_1 while making offers to S_M will have an absolutely continuous (conditional) distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$, putting a mass point at $p_l(\pi)$.

(iii) B_1 while making offers to S_I on the equilibrium path behaves exactly in the same manner as in the two player game with one-sided asymmetric information.

(iv) S_I's behavior is identical to that in the two-player game. S_M accepts the largest offer with a payoff at least as large as the expected continuation payoff from rejecting all offers.

(v) Each buyer in equilibrium obtains a payoff of $v_B(\pi)$.

Remark 1 *The mass points and the distribution of buyers' offers will depend upon π though we show that these distributions will collapse in the limit. Off the path, the analysis is different from the two-player game because the buyers have more options to consider when choosing actions. For the description of off-path behavior refer to Appendix(C) and Appendix(D) for public and private offers respectively.*

Remark 2 *A "road map" of the proof: We construct the equilibrium by starting from the benchmark complete information case and showing that the complete information strategies essentially carry over to the game where π is in a range near 0. This includes, through the competition lemma, showing the nature of the competition among the sellers. Once π is outside this range, the mass points and support of the randomised strategies in the candidate equilibrium will depend upon π and these are characterised for all values of π. The equilibrium is then extended beyond the initial range (apart from the initial range, these are functions of δ) for sufficiently high values of δ by recursion. Finally, checking that the candidate equilibrium is immune to unilateral deviation at any stage involves specifying out-of-equilibrium beliefs. This is done in the two appendices.*

Proof. We prove this proposition in steps. (Not all of these steps are given here in order to reduce unwieldy notation-see also the appendices.) First we derive the equilibrium for a given value of π by assuming that there exists a threshold δ^*, such that if δ exceeds this threshold then for each value of π, a stationary equilibrium as described above exists. Later on we will prove this existence result.

To formally construct the equilibrium for different values of π, we need the following lemma which we label as the *competition lemma*, following the terminology of [8], though they proved it for a different model.

Consider the following sequences for $t \geq 1$:

$$\bar{p}_t = v - [(v - \delta^t H)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1})] \tag{2}$$

$$p_t^{'} = M + \delta(1-\alpha)(\bar{p}_{t-1} - M) \tag{3}$$

where $\alpha \in (0,1)$ and $\bar{p}_0 = H$.

Lemma 1 *There exists a $\delta^{'} \in (0,1)$, such that for $\delta > \delta^{'}$ and for all $t \in \{1, ...N(\delta)\}$, we have,*

$$\bar{p}_t > p_t^{'}$$

Proof.

$$\begin{aligned}\bar{p}_t - p_t^{'} &= v - [(v-\delta^t H)\alpha + (1-\alpha)\delta(v-\bar{p}_{t-1}))] - M \\ &\quad -\delta(1-\alpha)(\bar{p}_{t-1}-M)\end{aligned}$$

$$= (v-M)(1-\delta+\delta\alpha) - \alpha(v-\delta^t H)$$

$$= (1-\delta)(v-M) + \alpha(\delta v - \delta M - v + \delta^t H)$$

$$= (1-\delta)(v-M) + \alpha(\delta^t H - \delta M - (1-\delta)v)$$

If we show that the second term is always positive then we are done. Note that the coefficient of α is increasing in delta and is positive at $\delta = 1$. Take $t = N^*$, where N^* is the upper bound on the number of periods up to which the two player game with one sided asymmetric information (as described earlier) can continue. For $t = N^*$, $\exists\ \delta^{'} < 1$ such that the term is positive whenever $\delta > \delta^{'}$. Since this is true for $t = N^*$, it will be true for all lower values of t.

As $N(\delta) \leq N^*$ for any $\delta < 1$ and for all $t \in \{1, ...N(\delta)\}$,

$$\bar{p}_t > p_t^{'}$$

whenever $\delta > \delta^{'}$. ■

For both *public* and *private* targeted offers, the equilibrium path is the same. However the off-path behavior differs (to be specified later).

Fix a $\delta > \delta^*$. Suppose we are given a $\pi \in (0,1)$[4]. There exists a $t \geq 0$ (it is easy to see that this $t \leq N^*$) such that $\pi \in [d_t, d_{t+1})$. The sequence $d_\tau(\delta) = \{0, d_1, d_2, ...d_t..\}$ is derived from and is identical with the same sequence in the two-player game. Next, we evaluate $v_B(\pi)$ (from the two player game). Define $\bar{p}(\pi)$ as,

$$\bar{p}(\pi) = v - v_B(\pi)$$

Define $p_l^{'}(\pi)$ as,

$$p_l^{'}(\pi) = M + \delta(1-a(\pi))[E_{d_{t-1}}(p) - M] \tag{4}$$

where $E_{d_{t-1}}(p)$ represents the expected price offer to S_M in equilibrium when the probability that S_I is of the low type is d_{t-1}. From (4) we can posit that, in equilibrium, $p_l^{'}(\pi)$ is the minimum acceptable price for S_M if she gets only one offer.

[4] $\pi = 0$ is the complete information case with a H seller.

Lemma 2 *For a given $\pi > d_1$, the acceptance probability $a(\pi)$ of an equilibrium offer is increasing in δ and has a limit $\bar{a}(\pi)$ which is less than 1.*

Proof. The acceptance probability $a(\pi)$ of an equilibrium offer is equal to $\pi\beta(\pi)$, where $\beta(\pi)$ is the probability with which the L-type S_I accepts an equilibrium offer. From the updating rule we know that $\beta(\pi)$ is such that the following relation is satisfied:

$$\frac{\pi(1-\beta(\pi))}{\pi(1-\beta(\pi))+(1-\pi)} = d_{t-1}$$

From the above expression, we get

$$\beta(\pi) = \frac{\pi - d_{t-1}}{\pi(1-d_{t-1})}$$

Therefore, $\beta(\pi)$ is increasing in π and decreasing in d_{t-1}. From [13] the d_t are decreasing in δ and have a limit. Hence $\beta(\pi)$ (and also $a(\pi)$) is increasing in δ. Since the d_t have a limit as δ goes to 1, so does $\beta(\pi)$. Therefore, $a(\pi)$ also has a limit $\bar{a}(\pi)$ which is less than 1 for $\pi \in (0,1)$. ■

For $\pi = d_{t-1}$, the maximum price offer to S_M (according to the conjectured equilibrium) is $\bar{p}(d_{t-1})$. This implies that $E_{d_{t-1}}(p) < \bar{p}(d_{t-1})$ (this will be clear from the description below). Since $a(\pi) \in (0,1)$, from lemma (1) we can infer that $\bar{p}(\pi) > p_l^{'}(\pi)$. Suppose there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that,

$$p_l(\pi) = (1-\delta)M + \delta E_\pi(p)$$

We can see that p_l represents the minimum acceptable price offer for S_M in the event that he gets two offers. (Note that if S_M rejects both offers, the game goes to the next period with π remaining the same.)

From the conjectured equilibrium behavior, we derive the following[5] :

1. B_1 makes offers to S_I with probability $q(\pi)$, where

$$q(\pi) = \frac{v_B(\pi)(1-\delta)}{(v - p_l^{'}(\pi)) - \delta v_B(\pi)} \tag{5}$$

B_1 offers $\delta^t H$ to S_I. With probability $(1-q(\pi))$ he makes offers to S_M. The conditional distribution of offers to S_M, given B_1 makes an offer to this seller when the relevant probability

[5] We obtain these by using the indifference relations of the players when they are using randomized behavioral strategies.

is π, is

$$F_1^{\pi}(s) = \frac{v_B(\pi)[1-\delta(1-q(\pi))] - q(\pi)(v-s)}{(1-q(\pi))[v-s-\delta v_B(\pi)]} \tag{6}$$

We can check that $F_1^{\pi}(p_l(\pi)) > 0$ and $F_1^{\pi}(\bar{p}(\pi)) = 1$. This confirms that B_1 puts a mass point at $p_l(\pi)$.

2. B_2 offers $p_l^{'}(\pi)$ to S_M with probability $q^{'}(\pi)$, where

$$q^{'}(\pi) = \frac{v_B(\pi)(1-\delta)}{(v-p_l(\pi))) - \delta v_B(\pi)} \tag{7}$$

With probability $(1-q^{'}(\pi))$ he makes offers to S_M by randomizing his offers in the support $[p_l(\pi), \bar{p}(\pi)]$. The conditional distribution of offers is given by

$$F_2^{\pi}(s) = \frac{v_B(\pi)[1-\delta(1-q^{'}(\pi))] - q^{'}(\pi)(v-s)}{(1-q^{'}(\pi))[v-s-\delta v_B(\pi)]} \tag{8}$$

This completes the derivation. Appendix(C) and Appendix(D)(for public and private offers respectively) describes the off-path play and show that it sustains the equilibrium play in each of the cases.

Next, we show that there exists a δ^* such that $\delta^{'} < \delta^* < 1$ and for $\delta > \delta^*$ an equilibrium as described above exists for all values of $\pi \in [0, 1)$. To do these we need the following lemmas:

Lemma 3 *If $\pi \in [0, d_1)$, then the equilibrium of the game is identical to that of the benchmark case.*

Proof. From the equilibrium of the two player game with one sided asymmetric information, we know that for $\pi \in [0, d_1)$, buyer always offers H to the seller and the seller accepts this with probability one. Hence this game is identical to the game between a buyer of valuation v and a seller of valuation H, with the buyer making the offers. Thus, in the four-player game, we will have an equilibrium identical to the one described in the benchmark case. We conclude the proof by assigning the following values:

$$p_l^{'}(\pi) = M \text{ and } \bar{p}(\pi) = H \text{ for } \pi \in [0, d_1)$$

■

Lemma 4 [6]*If there exists a $\bar{\bar{\delta}} \in (\delta^{'}, 1)$ such that for $\delta \geq \bar{\bar{\delta}}$ and for all $t \in \{1, ..., N^*\}$ an*

[6]We use the following notation, from the appendix. For any $x \in (M, H)$ $E^x(p)$ be the expressions obtained from $F_1(.)$, $F_2(.)$, q, $q^{'}$ and $E(p)$ respectively by replacing p_l by x.

equilibrium exists for $\pi \in [0, d_t(\delta))$*, then there exists a* $\delta_t^* \geq \bar{\delta}$ *such that, for all* $\delta \in (\delta_t^*, 1)$ *an equilibrium also exists for* $\pi \in [d_t(\delta), d_{t+1}(\delta))$*.*

Proof. We only need to show that there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$ and for all $\pi \in [d_t(\delta), d_{t+1}(\delta))$,there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ with

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

From now on we will write d_t instead of $d_t(\delta)$. For each $\delta \in (\delta^{'}, 1)$ we can construct $d(\delta)$ and the equilibrium strategies as above (assuming existence). Construct the function $G(x)$ as

$$G(x) = x - [\delta E_\pi^x(p) + (1 - \delta)M]$$

We can infer from Appendix (A) that the function $G(.)$ is monotonically increasing in x. Since $E_\pi^x(p) < \bar{p}(\pi)$,

$$\lim_{x \to \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l^{'}(\pi)) = p_l^{'}(\pi) - [\delta E_\pi^{p_l^{'}(\pi)}(p) + (1 - \delta)M]$$

By definition $E_\pi^{p_l^{'}(\pi)}(p) > p_l^{'}(\pi)$. So for $\delta = 1$, $G(p_l^{'}(\pi))) < 0$. Since $G(.)$ is a continuous function, there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$, $G(p_l^{'}(\pi))) < 0$. By invoking the *Intermediate Value Theorem* we can say that there is a unique $x^* \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $G(x^*) = 0$. This x^* is our required $p_l(\pi)$.

This concludes the proof. ■

From lemma (3) we know that for any $\delta \in (0, 1)$ an equilibrium exists for $\pi \in [0, d_1)$.[7] Using lemma (4) we can obtain δ_t^* for all $t \in \{1, 2, ..., N^*\}$. Define δ^* as:

$$\delta^* = \max_{1,..,N^*} \delta_t^*$$

We can do this because N^* is finite. Lemma (3) and (4) now guarantee that whenever $\delta > \delta^*$ an equilibrium as described above exists for all $\pi \in [0, 1)$.

This concludes the proof of the proposition. ■

[7] Note that d_1 is independent of δ

4 Asymptotic characterization

It has been argued earlier that as $\delta \to 1$, $p_l^{'}(\pi)$ reaches a limit which is less than $\bar{p}(\pi)$. From (5) we then have,

$$q(\pi) \to 0 \text{ as } \delta \to 1$$

Then from (6) we have,

$$1 - F_1^{\pi}(s) = \frac{\bar{p}(\pi) - s}{(1 - q(\pi))[v - s - \delta v_B(\pi)]}$$

We have shown that $q(\pi) \to 0$ as $\delta \to 1$. Hence as $\delta \to 1$, for s arbitrarily close to $\bar{p}(\pi)$, we have

$$1 - F_1^{\pi}(s) \approx \frac{\bar{p}(\pi) - s}{\bar{p}(\pi) - s} = 1$$

Hence the distribution collapses and $p_l(\pi) \to \bar{p}(\pi)$. From the expression of $p_l(\pi)$ we know that $p_l(\pi) \to E_{\pi}(p)$ as δ goes to 1. Thus we can conclude that $E_{\pi}(p)$ approaches $\bar{p}(\pi)$. From the two-player game with one-sided asymmetric information we know that as δ goes to 1, $\bar{p}(\pi) \to H$, (since $v_B(\pi)$ goes to $v - H$) for any value of π. This leads us to conclude that as δ goes to 1, $E_{\pi}(p) \to H$ for all values of π. This in turn provides the justification of having $E_{d_{t-1}}(p) \approx E_{\pi}^{x}(p)$ for high values of δ(used in the proof of lemma (4)).

From the proof of lemma (4) we know that $G(\bar{p}(\pi)) > 0$. Hence there will be a threshold of δ such that for all δ higher than that threshold we have $G(\delta\bar{p}(\pi)) > 0$. Thus $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$. (7) implies that

$$q^{'}(\pi) = \frac{1}{\frac{v}{v_B(\pi)} + \frac{\delta\bar{p}(\pi) - p_l(\pi)}{(1-\delta)v_B(\pi)}}$$

Since $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$, $q^{'}(\pi) \to 0$ as δ goes to 1.

Thus we conclude that as δ goes to 1, prices in all transactions go to H. We state this (informally) as a result.

Main result: *With either public or private offers there exists a stationary Perfect-Bayes equilibrium, such that, as $\delta \to 1$, the prices in both transactions go to H. The bargaining ends "almost" immediately and both sellers, the one with private information and L type and the one whose valuation is common knowledge, obtain strictly positive expected profits.*

Comment : *It should be mentioned that we would expect the same result to be true, if, instead of a two-point distribution, the informed type's reservation value s is continuously distributed in $[L, H]$ according to some cdf $G(s)$. The following section describes this.*

5 Informed seller's reservation value is continuously distributed in $[L, H]$

Suppose the informed seller's valuation is continuously distributed on $[L, H]$ according to some cdf $G(s)$. As before, we first consider the two player game with a buyer and a seller, where the seller is informed.

5.1 Two-player Game

There is one buyer, whose valuation is commonly known to be v.

There is one seller, whose valuation is private information to her. Her valuation is distributed according to a continuous distribution function $G(.)$, over the interval $[L, H]$.

Let $g(.)$ be the density function which is assumed to be bounded:

$$0 < \mathrm{g} \leq g(s) \leq \bar{g}$$

Players discount the future using a common discount factor $\delta \in (0, 1)$.

We now state the equilibrium of the infinitely repeated bargaining game where the buyer makes offers in each period. The seller either accepts or rejects it. Rejection takes the game to the next period, when the buyer again makes an offer.

The result re-stated below (for completeness) is from [14] .

One can show that at any instant, the buyer's posterior distribution about the seller's valuation can be characterised by a unique number s^e, which is the lowest possible valuation of the seller. With a slight abuse of terminology, we will call s^e the buyer's posterior.

The Equilibrium: Given a $\delta \in (0, 1)$, we can obtain thresholds s^t's, such that $L < s^t < H$ and

$$s^t < s^{t-1} < .... < s^2 < s^1$$

If at a time point t, the posterior $s_t \in (s^{t+1}, s^t]$, then the buyer offers p^t. A seller with valuation less than s^{t-1} accepts the offer. Rejection takes the posterior to s^{t-1}.

The p^t's are such that the seller with a valuation s^{t-1} is indifferent between accepting the offer now or waiting until the next period. The off-path behavior of players is outlined in appendix (E).

It can be shown that as $\delta \to 1$, for all t, $p^t \to H$. Also the maximum number of periods for which the game would last is bounded above by N^*.

5.2 Four-player game

We now analyse the four player game. There are two buyers, each with a valuation v. There are two sellers. One of them has a valuation which is commonly known to be M. The other seller's valuation is private information to her. It is continuously distributed in $[L, H]$, according to some cdf $G(.)$ as discussed above.

First we prove an analogue of the *competition lemma*. From the two-player game, we know that the number of periods for which the game with one-sided asymmetric information would last is bounded above by N^*.

Lemma 5 *For $t \geq 1, ..., N^*$, define $\bar{p}_t$ and $p_t^{'}$ as*

$$\bar{p}_t = v - [(v - p^t)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1})]$$

$$p_t^{'} = M + \delta(1-\alpha)(\bar{p}_{t-1} - M)$$

where $\alpha \in (0,1)$ and $\bar{p}_0 = H$.

Then there exists $\delta^{'} \in (0,1)$ such that for $\delta > \delta^{'}$ and for all $t \in \{1, 2, 3, ..., N^*\}$ we have

$$\bar{p}_t > p_t^{'}$$

Proof.

$$\bar{p}_t - p_t^{'} = v - [(v - p^t)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1})]$$

$$-M - \delta(1-\alpha)(\bar{p}_{t-1} - M)$$

$$= (1-\delta)(v - M) + \alpha[p^t - \delta M - (1-\delta)v]$$

The first term is always positive. Let us consider the second term. Consider $t = N^*$. The coefficient of α is positive for $\delta = 1$. This is because $p^t \to H$ as $\delta \to 1$. Since this is true for $t = N^*$, it will be true for all lower values of t.

This concludes the proof. ■

For each $\delta \in (0,1)$ we can find a t such that $s \in (s^{t+1}, s^t]$. The sequence $\{s^{t+1}, s^t, ..., s^3, s^2\}$ is derived from and is identical with the same sequence in the two player game. Given these, we can evaluate $v_B(s)$ as

$$(v - p^t)\frac{[G(s^{t-1}) - G(s)]}{1 - G(s)} + \frac{1 - G(s^{t-1})}{1 - G(s)}\delta(v_B(s^{t-1}))$$

For $s > s^2$, $v_B(s) = v - H$.

Define $\bar{p}(s)$ as,

$$\bar{p}(s) = v - v_B(s)$$

As before, we first conjecture an equilibirum and derive it and then prove existence. We refer to the seller with known valuation as S_M and the one with private information as S_I.

The following proposition describes the equilibrium.

Proposition 2 *There exists a $\delta^* \in (0, 1)$ such that if $\delta > \delta^*$, then for all $s \in (L, H)$ there exists a stationary perfect Bayes' equilibrium as follows:*

(i) One of the buyers (say B_1) will make offers to both S_I and S_M with positive probability. The other buyer B_2 will make offers to S_M only.

(ii) B_2 while making offers to S_M will put a mass point at $p_l^{'}(s)$ and will have an abosolutely continuous distribution of offers from $p_l^{'}(s)$ to $\bar{p}(s)$ where $p_l^{'}(s)$ ($p_l(s)$) is the minimum acceptable price to S_M when she gets one (two) offer(s). For a given s, $\bar{p}(s)$ is the upper bound of the price offer S_M can get in the described equilibrium ($p_l^{'}(s) < p_l(s) < \bar{p}(s)$). B_1 while making offers to S_M will have an absolutely continuous (conditional) distribution of offers from $p_l(s)$ to $\bar{p}(s)$, putting a mass point at $p_l(s)$.

(iii) B_1 while making offers to S_I on the equilibrium path behaves exactly in the same manner as in the two player game with one-sided asymmetric information.

(iv) Each buyer obtains a payoff of $v_B(s)$.

(Out-of-equilibrium analysis is contained in appendix (F) and (G) for public and private offers respectively.)

Proof. Suppose $\delta > \delta^*$. Then assuming existence, we first derive the equilibrium.

Define $p_l^{'}(s)$ as,

$$p_l^{'}(s) = M + \delta(1 - \alpha(s))[E_{s^{t-1}}(p) - M]$$

where $\alpha(s) = \frac{1-F(s^{t-1})}{1-F(s)}$.

This is the minimum acceptable price for S_M, when she gets only one offer. Since $E_{s^{t-1}}(p) \leq \bar{p}_{t-1}$, from lemma (5) we can say that $\bar{p}(s) > p_l^{'}(s)$.

Suppose there exists a $p_l(s) \in (p_l^{'}(s), \bar{p}(s))$ such that

$$p_l(s) = M + \delta(E_s(p) - M)$$

We can now derive the equilibrium as conjectured.

Now we shall prove existence with the help of the following two lemmas.

Lemma 6 *If $s \in (s^2, 1]$, then the equilibrium is identical to that of the benchmark case*

Proof. From the equilibrium of the two player game with one-sided asymmetric information we know that the buyer always offers H to the seller, who accepts it with probability 1. Thus, in the four player game, we will have an equilibrium identical to the one described in the benchmark case. ■

Lemma 7 *If there exists a $\bar{\delta} \in (\delta', 1)$ such that for $\delta \geq \bar{\delta}$ and for all $t \in \{1, ..., N^*\}$ an equilibrium exists for $s \in (s^t, 1]$, then there exists a $\delta_t^* \geq \bar{\delta}$ such that, for all $\delta \in (\delta_t^*, 1)$ an equilibrium also exists for $s \in (s^{t+1}, s^t]$.*

We relegate the proof of this lemma to appendix (H).

The proof of the proposition now follows from lemma (6) and lemma (7). ■

6 A non-stationary equilibrium

We show that with public offers we can have a non-stationary equilibrium, so that the equilibrium constructed in the previous sections is not unique. This is based on using the stationary equilibrium as a punishment (the essence is similar to the *pooling equilibrium with positive profits* in [28]). The strategies sustaining this are described below. The strategies will constitute an equilibrium for sufficiently high δ, as is also the case for the stationary equilibrium.

Suppose for a given π, both the buyers offer M to S_M. S_M accepts this offer by selecting each seller with probability $\frac{1}{2}$. If any buyer deviates, for example by offering to S_I or making a higher offer to M, then all players revert to the stationary equilibrium strategies described above. If S_M gets the equilibrium offer of M from the buyers and rejects both of them then the buyers make the same offers in the next period and the seller S_M makes the same responses as in the current period.

Given the buyers adhere to their equilibrium strategies, the continuation payoff to S_M from rejecting all offers she gets is zero. So she has no incentive to deviate. Next, if one of the buyers offers slightly higher than M to S_M then it is optimal for her to reject both the offers. This is because on rejection next period players will revert to the stationary equilibrium play described above. Hence her continuation payoff is $\delta(E_\pi(p) - M)$,which is higher than the payoff from accepting.

Finally each buyer obtains an equilibrium payoff of $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. If a buyer deviates then, according to the strategies specified, S_M should reject the higher offer if the payoff from accepting it is strictly less than the continuation payoff from rejecting(which is the one period discounted value of the payoff from stationary equilibrium). Hence if a buyer wants S_M to accept an offer higher than M then his offer $p^{'}$ should satisfy,

$$p^{'} = \delta E_\pi(p) + (1-\delta)M$$

The payoff of the deviating buyer will then be $\delta(v - E_\pi(p)) + (1-\delta)(v-M)$. As $\delta \to 1$, $\delta(v - E_\pi(p)) + (1-\delta)(v-M) \approx \delta(v - \bar{p}(\pi) + (1-\delta)(v-M)$
$= \delta v_B(\pi) + (1-\delta)(v-M)$.

For $\delta = 1$ this expression is strictly less than $\frac{1}{2}(v-M) + \frac{1}{2}\delta v_B(\pi)$, as $(v-M) > \delta v_B(\pi)$. Hence for sufficiently high values of δ this will also be true. Also if a buyer deviates and makes an offer in the range $(M, p^{'})$ then it will be rejected by S_M. The continuation payoff of the buyer will then be $\delta v_B(\pi) < \frac{1}{2}(v-M) + \frac{1}{2}\delta v_B(\pi)$. Hence we show that neither buyer has any incentive to deviate.

We conclude this section by noting that this is not an equilibrium for private offers. This is because we have different continuation play for buyer's and seller's deviations. For public offers these deviations are part of the public history. However for private offers they are not.

7 Quality Differentiated Goods

Throughout our analysis this far, we have assumed that each seller possess a non-differentiated good. In other words, a buyer's valuation for the good does not depend on the identity of the seller selling it. We now allow for quality-differentiated goods. This implies that a buyer's valuation for a good depends on the seller's type. If the valuation of a seller is j, then the buyers' valuation for this seller's good is denoted by v_j.

7.1 The Environment

There are two buyers and two sellers. One of the sellers' valuations is common knowledge and is equal to M. The other seller's valuation is private information to her. It is known that with probability π, S_I's valuation is H and with the complementary probability it is L, such that,

$$H > M > L$$

Buyers are homogeneous and their valuation for the good depends on the seller's valuation. It is v_j for the good sold by the seller with valuation j. We have,

$$v_H > v_M > v_L > H \text{ and}$$

$$v_H - H > v_M - M > v_L - L$$

[8]

As before, we consider a bargaining game where the buyers simultaneously make offers. Offers are targeted.

7.2 One-period game

First we try to determine the equilibrium of the one-period game. Offers to the informed seller can be of two types. One is the *pooling type*, which is accepted by both types of S_I. The other is the one that is only targeted to the L-type S_I.

To begin with, we try to see if there is an equilibrium where one of the buyers (B_1 say) makes a pooling offer to S_I and the other buyer B_2 makes an offer only to S_M. It is easy to observe that any pooling offer $p^{'} \geq H$. Also, in an equilibrium as conjectured above, we must have $p^{'} = H$, B_2's offer to S_M equal to M and,

$$\pi v_H + (1-\pi)v_L - H = v_M - M$$

$$\Rightarrow \pi = \frac{H - M + v_M - v_L}{v_H - v_L} = \pi^*$$

Hence only for $\pi = \pi^*$, can we have an equilibrium as conjectured above.

If $\pi > \pi^*$, then

$$\pi v_H + (1-\pi)v_L - H > v_M - M$$

This shows that there will be competition for the informed seller.

Hence B_2 with some probability would like to make offers to S_I. This implies that the offers to S_I will be randomised. The lower bound of the support is H and the upper bound is p^0, such that

$$\pi v_H + (1-\pi)v_L - p^0 = v_M - M$$

Let $F_1(.)$ be the distribution of offers by B_1 to S_I, and $F_2(.)$ be the distribution of offers by B_2 to S_I, conditional on B_2 making offers to S_I. Let q be the probability with which B_2 makes offers to S_M. Then for $s \in (H, p^0]$, B_1's indifference condition gives us

$$[\pi v_H + (1-\pi)v_L - s][q + (1-q)F_2(s)] = v_M - M$$

$$\Rightarrow F_2(s) = \frac{(v_M - N) - q[\pi v_H + (1-\pi)v_L - s]}{(1-q)[\pi v_H + (1-\pi)v_L]}$$

[8] Clearly, some assumption has to be made about the relative sizes of the surplus. We have focused on one.

B_2's indifference condition gives us

$$[\pi v_H + (1-\pi)v_L - s]F_1(s) = v_M - M$$

$$\Rightarrow F_1(s) = \frac{v_M - M}{[\pi v_H + (1-\pi)v_L - s]}$$

We obtain the value of q by putting $s = H$ in B_1's indifference condition. It can be verified that $F_1(H) > 0$; $F_1(p^0) = 1$ and $F_2(H) = 0$; $F_2(p^0) = 1$.

Hence B_1 puts a mass point at H.

Now consider the case when $\pi < \pi^*$. We have

$$\pi v_H + (1-\pi)v_L - H < v_M - M$$

Also for $\pi > \frac{H-L}{v_H-L} = \pi^{**}$,

$$\pi v_H + (1-\pi)v_L - H > (1-\pi)(v_L - L)$$

Since $\pi^{**} < \pi^*$, for $\pi \in [\pi^{**}, \pi^*)$, competition will be for seller S_M. B_1 will make offers to both S_M and S_I. B_2 will make offers to S_M only. Offers to S_M will be randomised in the range $[M, p^m]$ where

$$\pi v_H + (1-\pi)v_L - H = v_M - p^m$$

B_1, while making offers to S_I, makes the pooling offer of H. The analytical characteristics of this equilibrium can be found in the same way as done for $\pi > \pi^*$.

If $\pi < \pi^{**}$, then again competition is for the seller S_M. However now, B_1 while making offers to S_I offers L, i.e targets the L-type S_I. The support of offers to S_M will be $[M, p_l^m]$ where

$$(1-\pi)[v_L - L] = v_M - p_l^m$$

7.3 Two-period game

Let us now consider the two-period game. Players discount the future using a common discount factor δ. We now find a perfect Bayes' equilibrium of this two-period game. We will show that in this case, prices in all transactions do not go to the same value. However we do show that when competition is for the known-valued seller, as people become patient enough, the price in the transaction with the informed seller goes to H, irrespective of the likelihood of the informed seller being the H-type.

First of all consider $\pi > \pi^*$. As explained above, competition will be for the informed seller. However, now the continuation payoff to S_I by rejecting an offer, when she gets one offer differs from that by rejecting all offers, when she gets two offers. If $E_I(p)$ is the expected equilibrium price to S_I from the one period game when competition is for S_I, then in period 1, if she gets two offers, her minimum acceptable price is:

$$p^i = \delta E_I(p) + (1-\delta)H$$

Hence B_1 makes offers to S_I only. He puts a mass point at H and randomises his offers in the range $[p^i, p^0]$ according to a continuous distribution $F_1(.)$. B_2 with probability q, makes offers to S_M. With probability $(1-q)$, he randomises his offers in the range $[p^i, p^0]$ according to a continuous distribution $F_2(.)$. $F_2(.)$ puts a mass point at p^i. The expressions for $F_1(.)$, $F_2(.)$ and q can be obtained from the indifference conditions of the buyers.

Now consider $\pi < \pi^*$, i.e when competition is for the seller S_M. In the two-period game, the nature of putting mass points while making offers to S_M in period 1 remains the same as in the previous paragraph.

In the one period game we have seen that for low values of π, offers to S_I are targeted to the L-type. The reservation value of the L-type was L. However, in the two-period game, when the competition is for S_M, the reservation value of the L-type in the first period game is higher than L. This is because if the L-type rejects an offer in the first period then next period she would face a two-player game with a buyer and the buyer would perceive her as a H-type seller. Thus next period her price offer will be H. This implies that if p^l is the minimum acceptable price, then we have

$$p^l - L = \delta(H - L)$$

$$\Rightarrow p^l = \delta H + (1-\delta)L$$

Thus when competition is for S_M, it is optimal to target the L-type only if

$$\pi v_H + (1-\pi)v_L - H < (1-\pi)[v_L - (\delta H + (1-\delta)L)]$$

$$\Rightarrow \pi > \frac{(1-\delta)[H-L]}{v_H - (\delta H + (1-\delta)L)} = \pi_l(say)$$

Observe that as $\delta \to 1$, $p^l \to H$ and $\pi_l \to 0$. Hence for high values of δ, the price offered to the informed seller goes to H, irrespective of the value of π.

Let us interpret the result we obtain in this two-period game. Consider the case when competition is for the known-valued seller. Then we see that as δ goes to 1, competition is always for the seller S_M and the price offered to the informed seller goes to H. Also competition for S_M tends to raise the upper bound of the price offered to her. However, it does not necessarily go to H. This depends on π. This is because the expected valuation by making offers to S_I depends on the value of π, because of quality differentiation. Hence the upper bound of the price offers to S_M is determined by π and it is not necessarily equal to H. This dispersion is attributable to quality differentiation.

8 Conclusion

In the model we described above we have shown that with either *public* or *private* offers there exists a stationary PBE, such that, as $\delta \to 1$, the prices in both transactions go to H. The bargaining ends within the first two periods and both sellers, the one with private information and L type and the one whose valuation is common knowledge, obtain strictly positive expected profits. This equilibrium is reminiscent of the "Coase Conjecture" on the rents from private information dominating the rents from having the sole right to make offers, if the offers can be made more and more quickly. However, the setting is different, in that there is an endogenous outside option for which buyers compete, and the model contains a potential interaction between this competition and the private information bargaining. This interaction comes through, at least in the equilibrium we study, mainly in the analysis of out-of-equilibrium behavior. It is interesting that the *equilibrium path* behavior is almost, though not quite, separable along these two dimensions.

It is also interesting that the equilibrium path in our model is essentially the same with the two different observability structures of public offers and private offers. We were somewhat hesitant to use the name PBE for the private offers case, since this is not a multistage game with observable actions and private information, in the sense of Fudenberg and Tirole, but the spirit of the analysis is very similar to theirs, so we have retained their name.

One question that might arise is how robust is our conclusion to different bargaining extensive forms. Clearly, simultaneous offers is best to represent competition and one-sided offers to represent the power to make offers. If we go to alternating offers, previous results in the complete information setting indicate that we cannot expect the same results. This is also true in the two-player setting, so the market element in the current model is not the driver for this difference.

We have shown that there could be non-stationary equilibria in this model. However, we have not been able to demonstrate an analogue to the uniqueness result for two-person bargaining with one-sided offers and one-sided private information, even for stationary equilibria.

We have also considered extensions to a continuum of seller types and quality differentiation (or buyer values being determined by informed seller types for that seller's good). The continuum of types does not affect the result. The example with correlation does.

In our future research we intend to address the issue of having two privately informed sellers and to extend this model to more agents on both sides of the market.

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Appendix

A Existence and uniqueness of p_l

For any $x \in (M, H)$, let $F_1^x(.)$, $F_2^x(.)$, q^x , $q^{'x}$, and $E^x(p)$ be the expressions obtained from $F_1(.)$, $F_2(.)$, q, $q^{'}$ and $E(p)$ respectively by replacing p_l by x. Thus all we need to show is that there exists a unique $x^* \in (M, H)$ such that,

$$x^* - M = \delta(E^{x^*}(p) - M)$$

Note that,

$$\begin{aligned} E^x(p) &= q^x[q^{'x}M + (1 - q^{'x})E_2^x(p)] \\ &\quad +(1 - q^x)[q^{'x}E_1^x(p) + (1 - q^{'x})E(\text{highest offer})] \end{aligned}$$

where, $E_i^x(p)$ is derived from $F_i^x(.)$, $i = 1, 2$.

The following lemma shows that as x increases by 1 unit, increase in $E^x(p)$ is by less than 1 unit.

Lemma 8

$$\frac{\partial E^x(p)}{\partial x} < 1$$

Proof.

We prove this using the following steps:

(i) From the expression obtained for $q^{'}$ we can say that $q^{'x}$[9] is increasing in x.

(ii) Next we show that as we raise x by 1 unit, there is an increase in $E_2^x(p)$ by less than 1 unit.

Increasing x by 1 unit means raising the lower bound of support of $F_2^x(.)$ by 1 unit. Thus we need to show that

$$E_2^{x+1}(p) < E_2^x(p) + 1$$

Consider the distribution $\tilde{F}_2^x(.)$ with $[x + 1, H + 1]$ as the support such that,

$$\tilde{F}_2^x(s) = F_2^x(s - 1)$$

Let $\widetilde{E_2^x(p)}$ be the expectation obtained under $\tilde{F}_2^x(s)$. Thus,

[9] This is done in the companion paper. $q^{'}$ is equal to $\frac{[v-H](1-\delta)}{(v-p_l)-\delta(v-H)}$

$$\widetilde{E_2^x(p)} = \int_{x+1}^{H+1} s\, d\tilde{F}_2^x(s)$$

$$\Rightarrow \widetilde{E_2^x(p)} = [\int_{x+1}^{H+1} (s-1)\, d\tilde{F}_2^x(s)] + 1$$

$$= [\int_{x+1}^{H+1} (s-1)\, dF_2^x(s-1)] + 1$$

$$= [\int_{x}^{H} (s)\, dF_2^x(s)] + 1$$

$$= E_2^x(p) + 1$$

$F_2^{x+1}(p)$ is obtained from $\tilde{F}_2^x(s)$ by transferring the mass from the interval $(H, H+1]$ to $[x+1, H]$, i.e transferring mass from higher quantities to lower quantities.. Thus it is clear that,

$$E_2^{x+1}(p) < \widetilde{E_2^x(p)} = E_2^x(p) + 1$$

By similar reasoning we can say that ,

$$E_1^{x+1}(p) < E_1^x(p) + 1$$

Hence, from the above arguments, it follows that,

$$\frac{\partial E^x(p)}{\partial x} < 1$$

■

Now we define the function $G(.)$ as,

$$G(x) = x - [\delta E^x(p) + (1-\delta)M]$$

Differentiating $G(.)$ w.r.t x we get,

$$G^{'}(x) = 1 - \delta \frac{\partial E^x(p)}{\partial x}$$

>From Lemma 8 we have,

$$G^{'}(x) > 0$$

From the equilibrium strategies, we know that $M < E^x(p) < H$ for any $x \in (M, H)$. Since $\delta \in (0, 1)$ we have,

$$\lim_{x \to M} G(x) < 0 \text{ and } \lim_{x \to H} G(x) > 0$$

Since $G(.)$ is a continuous and monotonically increasing function, using the *Intermediate Value Theorem,* we can say that there exists a unique $x^* \in (M, H)$ such that,

$$G(x^*) = 0$$

$$\Rightarrow x^* = \delta E^{x^*}(p) + (1 - \delta)M$$

This x^* is our required p_l.

Thus we have,

$$G(p_l) = 0$$

$$\Rightarrow p_l = (1 - \delta)M + \delta E(p)$$

B Off-path behavior of the 2 player game with incomplete information

We recapitulate here the off-path beliefs that sustain the equilibrium we have discussed for the two-player game. Suppose, for a given δ and π, the equilibrium offer is $\delta^t H$(i.e $\pi \in [d_t, d_{t+1})$) .We need to consider the following off-path contingencies.

(a) The buyer offers p^o to the seller such that $p^o < \delta^t H$: If $p^0 < \delta^{t+1} H$ then both the L-type and H-type seller reject this offer with probability 1. If $p^o \in [\delta^{t+1} H, \delta^t H)$ then the L-type seller rejects this with a probability, which, through Bayes' rule, implies that the updated belief is d_t. Let this probability be $\beta^{''}(p)$. Hence the acceptance probability of this offer is $a^{''}(p) = \pi\beta^{''}(p)$. The H-type seller always rejects this offer. Since $p^o \in [\delta^{t+1} H, \delta^t H)$, there exists a $k \in (0, 1]$ such that $p^o = k\delta^{t+1} H + (1-k)\delta^t H$. Next period (if the seller rejects now) the buyer offers $\delta^t H$ with probability k and $\delta^{t-1} H$ with probability $(1 - k)$. This is optimal from the point of view of the buyer because at $\pi = d_t$, the buyer is indifferent between offering $\delta^t H$ and $\delta^{t-1} H$. Also the expected continuation payoff to the L-type seller from rejection is equal to $\delta(k\delta^t H + (1 - k)\delta^{t-1} H) = p^o$. Thus the L-type seller is indifferent between accepting and rejecting the offer of p^o.

The way the cutoffs d_t's are derived ensures that the buyer has no incentive to deviate and offer something less than $\delta^t H$.

(b) Next, consider the case when the buyer offers p^o to the seller such that $p^o > \delta^t H$. If $p^o \in (\delta^t H, \delta^{t-1} H]$, the L-type seller rejects this offer with a probability that takes the updated belief to d_{t-1}. Since $p^o \in (\delta^t H, \delta^{t-1} H]$, there exists a $k \in (0, 1]$, such that $p^o = k\delta^{t-1} H + (1 - k)\delta^t H$. If the seller rejects then next period the buyer offers $\delta^{t-2} H$ with probability k and $\delta^{t-1} H$ with probability $1 - k$. This is optimal from the buyer's point of view since at $\pi = d_{t-1}$, the buyer is indifferent between offering $\delta^{t-1} H$ and $\delta^{t-2} H$. Since the expected payoff to the L-type seller from rejection is $\delta(k\delta^{t-2} H + (1 - k)\delta^{t-1} H) = p^o$, he is indifferent between accepting and rejecting an offer of p^o. As p^o is strictly greater than $\delta^t H$ and the acceptance probability is the same as that of the equilibrium offer, the buyer has no incentive to deviate and offer p^o to the seller where $p^o \in (\delta^t H, \delta^{t-1} H]$.

If $p^o \in (\delta^\tau, \delta^{\tau-1}]$ (for $\tau \leq t - 1$) then the L-type seller rejects this with a probability which through Bayes' rule implies that the updated belief is $d_{\tau-1}$. If the seller rejects then next period the buyer randomises between offering $\delta^{\tau-1} H$ and $\delta^{\tau-2} H$ such that the expected continuation payoff to the L-type seller from rejection is p^o. It can be checked that the buyer has no incentive to deviate and offer p^o where $p^o \in (\delta^\tau, \delta^{\tau-1}]$ ($\tau \leq t - 1$).

C Off-path behavior of the 4 player game with incomplete information(public offers)

Suppose B_2 adheres to his equilibrium strategy. Then the off-path behavior of B_1 and that of L-type S_I, while B_1 makes an offer greater than $\delta^t H$ to S_I, are the same as in the 2-player game with incomplete information. If B_1's offer to S_I is less than $\delta^t H$ then the off-path behavior of the L-type S_I is described in the following manner. If B_2's offer to S_M is in the range $[p_l(\pi), \bar{p}(\pi)]$, then the L-type S_I behaves in the same way as in the 2-player game. If B_2 offers $p_l^{'}(\pi)$ to S_M then the L-type S_I accepts the offer with the equilibrium probability so that rejection takes the posterior to d_{t-1}. Next period, B_1 randomises between d_{t-1} and d_{t-2} so that the L-type S_I is indifferent between accepting or rejecting the offer now. For high values of δ, B_1 has no incentive to deviate.

Next, suppose B_2 makes an unacceptable offer to S_M, (which is observable to S_I) and B_1 makes an equilibrium offer to S_I. The L-type S_I rejects this offer with a probability that takes the updated belief to d_{t-1}. If S_I rejects this equilibrium offer and next period both the buyers make offers to S_M, then two periods from now, the remaining buyer offers $\delta^{t-2} H$ (the buyer is indifferent between offering $\delta^{t-1} H$ and $\delta^{t-2} H$ at $\pi = d_{t-1}$) to S_I. Thus the expected continuation payoff to S_I from rejection is $\delta(q(d_{t-1})\delta^{t-1} H + \delta(1 - q(d_{t-1}))\delta^{t-2} H) = \delta^t H$. This implies that the L-type S_I is indifferent between accepting and rejecting an offer of $\delta^t H$

if he observes S_M to get an unacceptable offer.

Now consider the case when B_2 deviates and makes an offer to S_I. It is assumed that if S_I gets two offers then she disregards the lower offer.

Suppose B_1 makes an equilibrium offer to S_I and B_2 deviates and offers something less than $\delta^t H$ to S_I. S_I's probability of accepting the equilibrium offer (which is the higher offer in this case) remains the same. If S_I rejects the higher offer (which in this case is the offer of $\delta^t H$ from B_1) and next period both the buyers make offers to S_M, then two periods from now, the remaining buyer offers $\delta^{t-2}H$ to S_I.

If B_2 deviates and offers $p^o \in (\delta^t H, \delta^{t-1}H]$ to S_I, then S_I rejects this with a probability that takes the updated belief to d_{t-2}. If S_I rejects this offer then next period if B_1 offers to S_I, he offers $\delta^{t-2}H$. If both B_1 and B_2 make offers to S_M then two periods from now the remaining buyer randomises between offering $\delta^{t-2}H$ and $\delta^{t-3}H$ to S_I (conditional on S_I being present). Randomisations are done in a manner to ensure that the expected continuation payoff to S_I from rejection is p^o. It is easy to check that for high values of δ, this can always be done. Lastly, if B_2 deviates and offers to S_I and B_1 offers to S_M (according to his equilibrium strategy), then the off-path specifications are the same as in the 2-player game with incomplete information.

We will now show that B_2 has no incentive to deviate. Suppose he makes an unacceptable offer to S_M. His expected discounted payoff from deviation is given by,

$$\mathcal{D} = q(\pi)[\delta\{a(\pi)(v-M) + (1-a(\pi))v_B(d_{t-1})\}] + (1-q(\pi))\delta v_B(\pi) \tag{9}$$

From (4) we know that,

$$p_l^{'}(\pi) < M + \delta(1-a(\pi))[\bar{p}(d_{t-1}) - M]$$

as $E_{d_{t-1}} < \bar{p}(d_{t-1})$. Hence we have,

$$p_l^{'}(\pi) < M + \delta(1-a(\pi))[(v-M) - (v-\bar{p}(d_{t-1}))]$$

Rearranging the terms above we get,

$$(v - p_l^{'}(\pi)) > \delta\{a(\pi)(v-M) + (1-a(\pi))v_B(d_{t-1})\} + (1-\delta)(v-M) \tag{10}$$

By comparing (9) and (10) we have,

$$q(\pi)(v-p_l^{'}(\pi)) + (1-q(\pi))\delta v_B(\pi) > \mathcal{D}$$

The L.H.S of the above relation is B_2's equilibrium payoff, as he puts a mass point at $p_l^{'}(\pi)$. Hence he has no incentive to make an unacceptable offer to S_M.

Next, suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o \in (\delta^t H, \delta^{t-1} H]$. B_2's payoff from deviation is:

$$\Gamma_H = q(\pi)[(v-p^o)a^{'}(\pi)+(1-a^{'}(\pi))\delta v_B(d_{t-2})]+(1-q(\pi))[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})]$$

where $a^{'}(\pi)$ is the probability with which B_2's offer is accepted by S_I in the event when both B_1 and B_2 make offers to S_I and B_2's offer is in the range $(\delta^t H, \delta^{t-1} H]$. From our above specification it is clear that $a^{'}(\pi) > a(\pi)$, where $a(\pi)$ is the acceptance probability of an equilibrium offer to S_I. This is also very intuitive. In the contingency when B_1 makes an equilibrium offer to S_M and B_2's out of the equilibrium offer to S_I is in the range $(\delta^t H, \delta^{t-1} H]$, the acceptance probability is equal to $a(\pi)$, the equilibrium acceptance probability. In this case if the L-type S_I rejects an offer then next period he will get an offer with probability 1. However if both B_1 and B_2 make offers to S_I and B_2's offer is in the range $(\delta^t H, \delta^{t-1} H]$ then the L-type S_I accepts this offer with a higher probability. This is because, on rejection, there is a positive probability that S_I might not get an offer in the next period. This explains why $a^{'}(\pi) > a(\pi)$.

Since $p^o > p_l^{'}(\pi)$[10] and $\bar{p}(d_{t-2}) > p_l^{'}(\pi)$[11], we have

$$v - p_l^{'}(\pi) > (v - p^o)a^{'}(\pi) + (1 - a^{'}(\pi))\delta v_B(d_{t-2}) \tag{11}$$

Also, since $p^o > \delta^t H$, we have

$$(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < v_B(\pi)$$

The expression $[(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1})-\delta v_B(\pi)]$ is strictly negative for $\delta = 1$. From continuity, we can say that for sufficiently high values of δ, $(v-p^o)a(\pi)+(1-a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. This implies that,

$$(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_H$$

The L.H.S of the above inequality is the equilibrium payoff of B_2. Similarly if B_2 deviates and make an offer to S_I such that his offer p^0 is in the range $[\delta^{t+1} H, \delta^t H)$, the payoff from

[10] For sufficiently high values of δ this will always be the case.

[11] Since $\bar{p}(d_{t-2}) > \bar{p}(\pi) > p_l^{'}(\pi)$.

deviation is

$$\begin{aligned}\Gamma_L &= q(\pi)[\delta\{a(\pi)(v-M)+(1-a(\pi))v_B(d_{t-1})\}] \\ &+(1-q(\pi))[(v-p^0)a^{''}(\pi)+(1-a^{''}(\pi))\delta v_B(d_t)]\end{aligned}$$

>From the 2-player game we know that $[(v-p^0)a^{''}(\pi)+(1-a^{''}(\pi))\delta v_B(d_t)] < v_B(\pi)$. Also from the previous analysis we can posit that $(v-p_l^{'}(\pi)) > \delta\{a(\pi)(v-M)+(1-a(\pi))v_B(d_{t-1})\}$. Thus for sufficiently high values of δ, $(v-p_l^{'}(\pi))q(\pi)+(1-q(\pi))\delta v_B(\pi) > \Gamma_L$.

Hence B_2 has no incentive to deviate and make an offer to S_I.

D Off-path behavior with private offers

The off-path behavior described in the preceding appendix is not applicable to the case of private offers. This is because it requires the offers made by both the buyers to be publicly observable. The off-path behavior of the players in the case of private offers is described as follows.

Specifically we need to describe the behavior of the players in the following three contingencies.

(i) B_2 makes an unacceptable offer to S_M.

(ii) B_2 makes an offer of p^o to S_I such that $p^o < \delta^t H$.

(iii) B_2 makes an offer of p^o to S_I such that $p^o > \delta^t H$.

We denote the above three contingencies by E_1, E_2 and E_3 respectively. We now construct a particular belief system that sustains the equilibrium described in the text.

Suppose B_1 attaches probabilities λ,λ^2 and λ^3 ($0 < \lambda < 1$) to E_1, E_2 and E_3 respectively. Thus he thinks that B_2 is going to stick to his equilibrium behavior with probability $[1-(\lambda+\lambda^2+\lambda^3)]$.

If E_1 or E_2 occurs and B_1 makes an equilibrium offer to S_I, then S_I's probability of accepting the equilibrium offer remains the same and two periods from now (conditional on the fact that the game continues until then), if B_2 is the remaining buyer he offers $\delta^{t-2}H$ to S_I. If E_3 occurs and all players are observed to be present, then next period B_2 offers $\bar{p}(d_{t-1})$ to S_M. In any off-path contingency, if B_1 is the last buyer remaining (two periods from now) then he offers $\delta^{t-2}H$ to S_I.

The L-type S_I accepts an offer higher than $\delta^t H$ with probability 1 if she gets two offers. If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

We will now argue that the off-path behavior constitutes a sequentially optimal response by the players to the limiting beliefs as $\lambda \to 0$.

Suppose B_1 makes an equilibrium offer to S_I and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if B_1 finds that all four players are present he infers that this is due to an out-of-equilibrium play by B_2. Using Bayes' rule he attaches the following probabilities to E_1, E_2 and E_3 respectively.

$$\frac{1}{1+\lambda+\lambda^2} \text{ to } E_1$$
$$\frac{\lambda}{1+\lambda+\lambda^2} \text{ to } E_2$$
$$\frac{\lambda^2}{1+\lambda+\lambda^2} \text{ to } E_3$$

As $\lambda \to 0$, the probability attached to E_1 goes to 1. Thus B_1 believes that his equilibrium offer of $\delta^t H$ to S_I was rejected and the updated belief is d_{t-1}. In the case of E_1 or E_2 the beliefs of B_1 and B_2 coincide. However, in the case of E_3 they differ. Suppose E_3 occurs and B_1's equilibrium offer to S_I gets rejected. Then next period all four players will be present and given L-type S_I's behavior, the belief of B_2 will be $\pi = 0$ and that of B_1 will be $\pi = d_{t-1}$. In that contingency it is an optimal response of B_2 to offer $\bar{p}(d_{t-1})$ to S_M since he knows that B_1 is playing his equilibrium strategy with the belief d_{t-1}.

Next we will argue that the L-type S_I finds it optimal to accept an offer higher than $\delta^t H$ with probability 1, if she gets two offers. This is because in the event when she gets two offers she knows that rejection will lead the buyer B_1 to play according to the belief d_{t-1} and, two periods from now, the remaining buyer will offer $\delta^{t-2}H$ to S_I. Thus her continuation payoff from rejection is

$$\delta\{\delta^{t-1}Hq(d_{t-1}) + \delta(1-q(d_{t-1}))\delta^{t-2}H\} = \delta\{\delta^{t-1}H\} = \delta^t H$$

Hence she finds it optimal to accept an offer higher than $\delta^t H$ with probability 1.

We need to check that B_2 has no incentive to deviate and make an offer of p^o to S_I such that $p^o > \delta^t H$.

Suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o > \delta^t H$. With probability $q(\pi)$, S_I will get two offers and $B_2's$ will be accepted with probability π. With probability

$(1-q(\pi))$, S_I will get only one offer. B_2 then gets a payoff of

$$(v-p^o)q(\pi)\pi + (1-q(\pi))[(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1})]$$

As shown in the previous appendix, for high values of δ we have $(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. Also for high values of δ, $p^o > p_l^{'}(\pi)$. Thus[12],

$$v_B(\pi) = (v - p_l^{'}(\pi))q(\pi) + (1-q(\pi))\delta v_B(\pi)$$
$$> (v-p^o)q(\pi)\pi + (1-q(\pi))[(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1})]$$

Hence B_2 has no incentive to deviate and make an offer of p^o to S_I.

Lastly, to show that B_2 has no incentive to deviate and make an unacceptable offer to S_M or offer p^0 to S_I such that $p^0 < \delta^t H$ we refer to the analysis in the previous appendix.

E Off-path behavior for the 2-player game where the informed seller's valuation is drawn from a continuous distribution

Suppose the buyer makes an offer of p^0 such that $p^0 > p^t$. We will show that for any $p^0 \in (p^t, p^{t-1})$, the buyer will have no incentive to offer p^0. By definition, we have,

$$p^{t-1} - s^{t-2} = \delta(p^{t-2} - s^{t-2}) \Rightarrow p^{t-1} - s^{t-1} > \delta(p^{t-2} - s^{t-1})$$

since $s^{t-1} < s^{t-2}$. Also,

$$p^t - s^{t-1} = \delta(p^{t-1} - s^{t-1}) \Rightarrow p^t - s^{t-1} < \delta(p^{t-2} - s^{t-1})$$

since $p^{t-2} > p^{t-1}$. This implies that there exists a $\gamma \in (0,1)$ such that

$$\gamma p^{t-1} + (1-\gamma)p^t - s^{t-1} = \delta(p^{t-2} - s^{t-1})$$

Any $p^0 \in (p^t, p^{t-1})$ can be written as $p^0 = \eta p^{t-1} + (1-\eta)p^t$, where $\eta \in (0,1)$.

If $\eta < \gamma$ then rejection takes the posterior to s^{t-1}. The buyer following a rejection randomises between p^{t-1} and p^{t-2} such that the seller with valuation s^{t-1} is indifferent between accepting the offer of p^0 or rejecting it. Since $\eta < \gamma$, such a randomisation is always

[12] This is because B_2 puts a mass point at $p_l^{'}(\pi)$

possible. Also for the buyer, he is offering a higher price and it is getting accepted with the equilibrium probability.

If $\eta > \gamma$ then rejection takes the posterior to $s^{'} \in (s^{t-1}, s^{t-2})$ and the buyer next period offers p^{t-2}. Here $s^{'}$ is such that the seller with such a valuation is indifferent between accepting the offer of p^0 or rejecting it. Since $\eta > \gamma$,

$$p^0 - s^{t-1} > \delta(p^{t-2} - s^{t-1})$$

Also from definition, one can show that

$$p^0 - s^{t-2} < \delta(p^{t-2} - s^{t-1})$$

This shows that such a $s^{'}$exists.

Now, suppose the buyer offers some price p^0 such that $p^0 < p^t$. We will show that for any $p^0 \in (p^{t+1}, p^t)$, the buyer will have no incentive to deviate. For any $p^0 \in (p^{t+1}, p^t)$, there exists a $\alpha \in (0,1)$ such that $p^0 = \alpha p^t + (1-\alpha)p^{t+1}$. By definition, we have

$$p^{t+1} - s^t = \delta[p^t - s^t]$$

$$p^t - s^t > \delta[p^t - s^t]$$

Hence

$$p^0 - s^t > \delta[p^t - s^t]$$

Again by definition,

$$p^{t+1} - s^t = \delta[p^t - s^t] < \delta[p^{t-1} - s^t]$$

$$p^t - s^{t-1} = \delta[p^{t-1} - s^{t-1}] \Rightarrow p^t - s^t > \delta[p^{t-1} - s^t]$$

Hence there exists a $\gamma \in (0,1)$ such that

$$\gamma p^t + (1-\gamma)p^{t+1} - s^t = \delta[p^{t-1} - s^t]$$

Thus if $\alpha < \gamma$, then rejection takes the posterior to s^t. Next period the buyer randomises between offering p^t and p^{t-1}.

If $\alpha > \gamma$, then rejection takes the posterior to some $s^{'} \in (s^t, s^{t-1})$ such that a seller with valuation $s^{'}$ is indifferent between accepting the offer of p^0 or to reject it. As before it can be shown that such a $s^{'}$ exists.

F Out-of-equilibrium behavior for the 4-player game where the informed seller's valuation is drawn from a continuous support (public offers)

We only describe the following two off-path deviations. Others are analogous to the ones with the case where the informed seller's valuation is drawn from a distribution with two-point support.

First, suppose B_2 makes an unacceptable offer to S_M (i.e less than $p_l^{'}(s)$) and B_1 makes an equilibrium offer to S_I. Then rejection of the equilibrium offer by S_I still takes the posterior to s^{t-1}. However, next period, if B_1 offers to S_I, then he randomises between offering p^{t-1} and p^{t-2}. If next period, both the buyers offer to S_M, then two periods from now, the remaining buyer randomises between offering p^{t-1} and p^{t-2} to S_I. Note that when the posterior is s^{t-1}, the buyer is indifferent between offering p^{t-1} and p^{t-2}.

The payoff to the seller with valuation s^{t-1} from accepting an equilibrium offer now is $(p^t - s^{t-1})$. Hence randomisations by the buyers in the subsequent periods should ensure that the continuation payoff to the seller with valuation s^{t-1} from rejecting the equilibrium offer is also $(p^t - s^{t-1})$. We will now show that for high values of δ, such a randomisation is always possible.

Γ_l^c (the minimum continuation payoff to the seller with valuation s^{t-1}; i.e an offer of p^{t-1} in the next period and two periods from now.) is given as,

$$\Gamma_l^c = \delta[q(s^{t-1})(p^{t-1} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-1} - s^{t-1})]$$

$$= \delta[p^{t-1} - s^{t-1}][q(.) + (1 - q(.))\delta] = (p^t - s^{t-1})[q(.) + (1 - q(.))\delta] < (p^t - s^{t-1})$$

(since by definition, $p^t - s^{t-1}) = \delta[p^{t-1} - s^{t-1}]$. This is true for all $\delta < 1$)

Γ_h^c (the maximum continuation payoff to the seller with valuation s^{t-1}; i.e an offer of p^{t-2} in the next period and two periods from now) is given as,

$$\Gamma_h^c = \delta[q(s^{t-1})(p^{t-2} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-2} - s^{t-1})]$$

$$= \delta[(p^{t-2} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

$$> \delta[(p^{t-1} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

(since $p^{t-2} > p^{t-1}$)

$$= (p^t - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)$$

For $\delta = 1$ we have $\Gamma_h^c \geq (p^t - s^{t-1})$ (since $q(.) \to 0$, as $\delta \to 1$). This is because the inequality is strictly maintained when $\delta < 1$, and is not reversed when $\delta \to 1$ (as $p^{t-2} > p^{t-1}$ by definition) . Then by continuity we can say that for high values of δ, we will have $\Gamma_h^c > (p^t - s^{t-1})$. Also, we have $\Gamma_l^c < (p^t - s^{t-1})$. Hence on the equilibrium offer being rejected by the informed seller, offers to S_I can be made by randomising between p^{t-1} and p^{t-2} in a manner, such that the seller with valuation s^{t-1} is indifferent between accepting and rejecting the offer now. In the same way as done in the case of discrete valuations of the informed seller, one can show that the buyer B_2 has no incentive to deviate and make an unacceptable offer to S_M.

Next, suppose B_1 makes an equilibrium offer to S_I and B_2 deviates and makes an offer of p^0 to S_I, such that $p^0 < p^t$. Then the informed seller disregards the lower offer. Rejection takes the posterior to s^{t-1}. Thereafter buyers' behavior in making offers to S_I is exactly the same as described above.

Finally, suppose B_2 deviates and makes an offer of $p^0 > p^t$ to S_I and B_1 makes an equilibrium offer to S_I. Then rejection takes the posterior to s^{t-1}. We will show that for high values of δ, the buyer can always randomise between offering p^{t-1} and p^{t-2} in the next and subsequent periods (if there is no offer to S_I in the next period), such that the seller with valuation s^{t-1} is indifferent between accepting and rejecting the offer.

Any offer $p^0 \in (p^t, p^{t-1})$ is a convex combination of p^t and p^{t-1}. It is already shown above that the minimum continuation payoff to S_I with valuation s^{t-1}, $\Gamma_l^c < p^t - s^{t-1}$. Also,

$$\Gamma_h^c = \delta[q(s^{t-1})(p^{t-2} - s^{t-1}) + (1 - q(s^{t-1}))\delta(p^{t-2} - s^{t-1})]$$

$$= \delta[(p^{t-2} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

$$> \delta[(p^{t-1} - s^{t-1})(q(s^{t-1}) + (1 - q(s^{t-1}))\delta)]$$

Since the inequality is strictly maintained for $\delta < 1$ and not reversed when $\delta \to 1$, we have

$$\Gamma_h^c \geq p^{t-1} - s^{t-1}$$

for $\delta = 1$. Then by continuity we can posit that for high values of δ, we will have $\Gamma_h^c >$

$p^{t-1} - s^{t-1}$.

Hence the suggested randomisation is possible.

G Out-of-equilibrium behavior for the 4-player game where the informed seller's valuation is drawn from a continuous support (private offers)

Specifically we need to describe the behavior of the players in the following three contingencies:

(i) B_2 makes an unacceptable offer to S_M.

(ii) B_2 makes an offer of p^o to S_I such that $p^o < p^t$.

(iii) B_2 makes an offer of p^o to S_I such that $p^o > p^t$.

We denote the above three contingencies by E_1, E_2 and E_3 respectively. We now construct a particular belief system that sustains the equilibrium described in the text.

Suppose B_1 attaches probabilitites λ, λ^2 and λ^3 $(0 < \lambda < 1)$ to E_1, E_2 and E_3 respectively. Thus he thinks that B_2 is going to stick to his equilibrium behavior with probability $(1 - (\lambda + \lambda^2 + \lambda^3))$.

If E_1 or E_2 occurs and B_1 makes an equilibrium offer to S_I, then S_I's probability of accepting the equilibrium offer remains the same. On observing that all four players are present, the common posterior of the buyers will be s^{t-1}. In the subsequent periods when offers are made to S_I, randomisations between p^{t-1} and p^{t-2} are done in a manner to ensure that the continuation payoff to the informed seller with valuation s^{t-1} is $(p^t - s^{t-1})$. If E_3 occurs and all players are observed to be present, then next period B_2 offers $\bar{p}(s^{t-1})$ to S_M.

If the informed seller gets two offers, she accepts an offer $p^0 > p^t$ with probability 1 as long as her valuation is less than $s^{'}$. Here $s^{'}$ is such that

$$p^0 - s^{'} = p^t - s^{t-1}$$

If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

We will now argue that the off-path behavior constitutes a sequentially optimal response by the players to the limiting beliefs as $\lambda \to 0$.

Suppose B_1 makes an equilibrium offer to S_I and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if B_1 finds that all four players are present, he infers that this is due to an out-of-equilibrium play by B_2. Using Bayes' rule he attaches the following probabilities to E_1, E_2 and E_3 respectively.

$$\frac{1}{1+\lambda+\lambda^2} \text{ to } E_1$$

$$\frac{\lambda}{1+\lambda+\lambda^2} \text{ to } E_2$$

$$\frac{\lambda^2}{1+\lambda+\lambda^2} \text{ to } E_3$$

As $\lambda \to 0$, the probability attached to E_1 goes to 1. Thus B_1 believes that his equilibrium offer of p^t to S_I was rejected and the updated belief is s^{t-1}. In the case of E_1 or E_2 the beliefs of B_1 and B_2 coincide. However, in the case of E_3, they differ. Suppose E_3 occurs and B_1's equilibrium offer to S_I gets rejected. Then next period all four players will be present and given S_I's behavior, the belief of B_2 will be $s^{'} > s^{t-1}$ such that

$$p^0 - s^{'} = p^t - s^{t-1}$$

where $p^0 > p^t$ is the out of equilibrium offer made by B_2 to S_I (This in turn implies that the behavior of the informed seller in the contingency E_3 is optimal).

This is because the belief of B_1 is s^{t-1} and B_2, from the subsequent period onwards, plays according to B_1's belief. In the subsequent periods while offers are being made to S_I, randomisations between p^{t-1} and p^{t-2} are done in a manner to ensure that the continuation payoff to S_I is $p^t - s^{t-1}$. As before it is easy to observe that B_2 finds it optimal to play according to B_1's belief, since B_2's belief $(s^{'})$ is greater than that of B_1 (s^{t-1}) .

In the same way as done in the case of discrete valuations of the informed seller, we can show that B_2 will not deviate.

H Proof of lemma (7)

Proof. We only need to show that there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$ and for all $s \in (s^{t+1}, s^t]$,there exists a $p_l(s) \in (p_l^{'}(s), \bar{p}(s))$ with

$$p_l(\pi) = (1-\delta)M + \delta E_s(p)$$

From now on we will write s_t instead of $s_t(\delta)$. For each $\delta \in (\delta^{'}, 1)$ we can construct $d(\delta)$ and the equilibrium strategies as above (assuming existence). Construct the function $G(x)$

as

$$G(x) = x - [\delta E_s^x(p) + (1-\delta)M]$$

We can infer from Appendix (A) that the function $G(.)$ is monotonically increasing in x. Since $E_\pi^x(p) < \bar{p}(\pi)$,

$$\lim_{x \to \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l^{'}(\pi)) = p_l^{'}(s) - [\delta E_s^{p_l^{'}(s)}(p) + (1-\delta)M]$$

By definition $E_\pi^{p_l^{'}(s)}(p) > p_l^{'}(s)$. So for $\delta = 1$, $G(p_l^{'}(\pi))) < 0$. Since $G(.)$ is a continuous function, there exists a $\delta_t^* \geq \bar{\delta}$ such that for all $\delta > \delta_t^*$, $G(p_l^{'}(\pi))) < 0$. By invoking the *Intermediate Value Theorem* we can say that there is a unique $x^* \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $G(x^*) = 0$. This x^* is our required $p_l(\pi)$.

This concludes the proof. ■