Logic for Web Scientists

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1 Sets

Definition: Set

A *set* is an unordered collection of objects, without duplicates. A set A containing the objects a, b and c is written as $A = \{a, b, c\}$.

Definition: Empty Set

The *empty set* (written $\emptyset$ or $\{\}$) is the set containing nothing.

Definition: Set Membership

An object a is a *member* of a set A (written $a \in A$) if it is contained within that collection.

Note: $a \in A$ can be read as "a is a member of A" or "a belongs to A".

Definition: Set Equality

Two sets which contain the same objects are considered to be equal (the order of the objects is unimportant).

Example: If $A = \{a, b, c\}$, $B = \{b, a, c\}$ and $C = \{a, b, d\}$, then $A = B$ but $A \neq C$

Definition: Cardinality

The *cardinality* of a set A (written $|A|$ or $\#A$) is the number of members of A.

Example: If $A = \{a, b, c\}$, then $|A| = 3$

Definition: Subset

A set A is a *subset* of a set B (written $A \subseteq B$) if every member of A is also a member of B.

Example: If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, c, d\}$, then $A \subseteq B$, but $A \nsubseteq C$

Definition: Strict Subset

A set A is a *strict subset* of a set B (written $A \subset B$) if every member of A is also a member of B, and $A \neq B$.

Example: If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, b\}$, then $A \subset B$, but $A \not\subset C$

Definition: Set Intersection

The *intersection* of two sets A and B (written as $A \cap B$) is the set containing every object which is **both** a member of A **and** a member of B.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cap B = \{a, c\}$

Definition: Set Union

The *union* of two sets A and B (written as $A \cup B$) is the set containing every object that is a member of A **or** a member of B, **or** a member of both A and B.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cup B = \{a, b, c, d\}$

Mnemonic: $\cup$ stands for U(nion)

Definition: Set Difference

The *difference* of two sets A and B (written as $A - B$) is the set of every object that is a member of A but not a member of B.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A - B = \{b\}$

Note: $(A - B) \neq (B - A)$

Definition: Powerset

The *powerset* of A (written $\mathbb{P}(A)$ or 2^A) is the set containing all possible subsets of A, including A and the empty set.

Example: If $A = \{a, b, c\}$, then

$$\mathbb{P}(A) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \{\}\}$$

Note: $|\mathbb{P}(A)| = 2^{|A|}$.

Definition: Set Comprehension

Rather than explicitly list the members of a set as $A = \{a_1, \ldots, a_n\}$,we can define a set by specifying the properties that its members must have. This is known as *set comprehension.*

Set comprehension is expressed using *set-builder notation*, for which the general form is $\{x : \phi(x)\}$, where x is a variable and $\phi(x)$ is a predicate containing x which holds true for all members of the set. $\{x : \phi(x)\}$ can be read as "the set of x for which $\phi(x)$ is true".

Example: $\{x : x \in \mathbb{Z} \wedge x > 0\}$
The set of positive integers - $\mathbb{Z}$ is the set of integers.
Read as: "the set of x's where x is an integer and x is greater than zero".

Example: $\{x : x \in \mathbb{Z} \wedge x = x^2\}$
The set of integers which are equal to their square: $\{0, 1\}$

Example: $\{\langle x, y\rangle : x \in A \wedge y \in B\}$
The set of pairs $\langle x, y\rangle$ where x is a member of set A and y is a member of set B. This is the definition of the Cartesian product $A \times B$ using set-builder notation.

Definition: Tuple

A *tuple* is an ordered collection of objects, which may include duplicates. The tuple containing a, b, c and a, in that order, is written $\langle a, b, c, a\rangle$

Definition: Arity

The *degree* or *arity* of a tuple is the number of objects in the tuple.

Definition: Pair

A tuple containing two objects (a tuple of arity 2) is known as a *pair*.

Definition: Cartesian Product

The *Cartesian product* of two sets A and B (written $A \times B$) is a set of pairs, where each pair contains one member from A and one member from B, and which contains all possible combinations of members from A and B.

Example: If $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then

$$A \times B = \{\langle a, c\rangle, \langle a, d\rangle, \langle a, e\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle b, e\rangle, \langle c, c\rangle, \langle c, d\rangle, \langle c, e\rangle\}$$

Note: $|A \times B| = |A| * |B|$

Definition: Binary Relation

A *binary relation* R from set A to set B is a set of pairs, where each pair contains one member from A and one member from B.

Example: If $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then a possible relation R from A to B might be:

$$R = \{\langle a, c\rangle, \langle a, e\rangle, \langle b, d\rangle, \langle c, c\rangle, \langle c, d\rangle, \}$$

Note: $R \subseteq A \times B$

Definition: Domain

The *domain* of a binary relation R is the set that the relation goes *from*.

Example: The domain of R in the above example is A.

Definition: Range

The *range* of a binary relation R is the set that the relation goes *to*.

Example: The range of R in the above example is B.

Mnemonic: the range of a cannon is the distance **to** which it can fire a cannonball.

2 Logic

Definition: Predicate

A *predicate* is a truth-valued expression. That is, a predicate can either be *true* or *false*.
Example: "$a \in A$" is a predicate (either a is a member of A, in which case "$a \in A$" is *true*, or a is not a member of A, in which case "$a \in A$" is *false*). "$A \times B$" is not a predicate, because its value is a set.

Definition: Logical Operators

Predicates may be combined to form *compound predicates* by using the *logical operators*: conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) and implication ($\Rightarrow$) .

Definition: Conjunction (logical and)

The *conjunction* of two predicates ϕ and ψ (written as $\phi \wedge \psi$, and read as "ϕ and ψ") is *true* if both ϕ **and** ψ are *true*.

Mnemonic: $\wedge$ stands for A(nd)

ϕ	ψ	$\phi \wedge \psi$
false	false	false
false	true	false
true	false	false
true	true	true

Definition: Disjunction (logical or)

The *disjunction* of two predicates ϕ and ψ (written as $\phi \vee \psi$, and read as "ϕ or ψ") is *true* if either ϕ is *true* **or** ψ is *true* (**or** if both ϕ and ψ are *true* – $\vee$ is the inclusive-or).

ϕ	ψ	$\phi \vee \psi$
false	false	false
false	true	true
true	false	true
true	true	true

Definition: Negation

The *negation* of a predicate ϕ (written as $\neg\phi$, and read as "not ϕ") is *true* if ϕ is *false*.

ϕ	$\neg\phi$
false	true
true	false

Definition: Implication

ϕ	ψ	$\phi \Rightarrow \psi$
false	false	true
false	true	true
true	false	false
true	true	true