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### A Multiechelon Inventory Problem with Secondary Market Sales

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# A Multiechelon Inventory Problem with Secondary Market Sales

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We consider a finite-horizon, multiechelon inventory system in which the surplus of stock can be sold (i.e., disposed) in the secondary markets at each stage in the system. What are called *nested echelon order-up-to policies* are shown to be optimal for jointly managing inventory replenishments and secondary market sales. Under a general restriction on model parameters, we establish that it is optimal not to both sell off excess stock and replenish inventory. Secondary market sales complicate the structure of the system, so that the classical Clark and Scarf echelon reformulation no longer allows for the decomposition of the objective function under the optimal policy. We introduce a novel class of policies, referred to as the *disposal saturation policies*, and show that there exists a disposal saturation policy, determined recursively by a single base-stock level at each echelon, that achieves the decomposition of this problem. The resulting optimal replenishment policy is shown to be the echelon base-stock policy. We demonstrate the heuristic performance of this disposal saturation policy through a series of numerical studies: except at extreme ranges of model parameters, the policy provides a very good approximation to the optimal policy while avoiding the curse of dimensionality. We also conduct numerical studies to determine the value of the secondary markets for multistage supply chains and assess its sensitivity to model parameters. The results provide potentially useful insights for companies seeking to enter or develop secondary markets for supply chains.

*Key words:* inventory theory; multiechelon; stock disposals; secondary markets; dynamic programming; optimal policy; heuristics; echelon base-stock policy

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## 1. Introduction

One of the recent developments of significance for supply chain management has been the emergence of secondary markets. Secondary markets have not only become available to supply chains in various industries (Petruzzi and Monahan 2003), but they have also been growing rapidly in size. The secondary markets for electronics components, for example, were estimated at \$15 B in 2008 (Judge 2009), whereas those for pharmaceutical supply chains were worth about \$20 B in 2007, or 6% of the total market for pharmaceuticals products that year (Leininger 2007). In some industries there exist online exchanges to facilitate secondary market transactions (e.g., Converge.com and Virtual Chip Exchange for electronics components), whereas in others specialized firms act as intermediaries for companies to dispose of excess inventory in the supply chain (e.g., QK Healthcare for pharmaceuticals).

Secondary markets have increasingly come to represent the means of dealing with the surplus of stock, which can accumulate in the supply chain for a variety of reasons, including volatility and random shifts in demand, the bullwhip effect, and inadequate information and forecasting systems. An option to dispose

of excess inventory in the supply chain by selling it in the secondary market can increase a company's profits, reduce its financial risks, and improve the mismatch between supply and demand. As Lee and Whang (2001, p. 12) point out, "...secondary markets can thus benefit, in most cases, every member of the supply chain."

At the same time, "...there are no real guidelines for the disposal of excess inventory" (Rosenfield 1989, p. 404), so inventory managers at companies have no conceptual models to guide their thinking when it comes to dealing with excess inventory. Further, although secondary markets have been growing in size, little is known about what exactly has driven that growth or what market conditions contribute to the value of those markets.

In this paper, we address secondary markets for supply chains by considering a centralized, finite-horizon, multiechelon system in which excess inventory can be sold off at each stage in the system. Our contribution is to (i) formulate a multiechelon inventory model with secondary market sales, (ii) establish the structure of the optimal policy, (iii) introduce a class of disposal policies that significantly simplifies the problem while retaining the features of inventory

disposals observed in practice, (iv) numerically evaluate the performance of those policies, and (vi) assess the value of secondary markets and the factors that drive it.

The study of multiechelon inventory systems under stochastic demand, launched by the landmark Clark and Scarf (1960) paper, has focused, almost without exception, on settings where companies do not voluntarily remove stock from the supply chain. Although those models allow for the replenishment of stock at each stage, the only option they traditionally provide for dealing with excess inventory is to hold it in the supply chain (despite potentially significant physical and financial holding costs). Consequently, secondary markets in general have remained largely unexplored in the multiechelon literature.

To the best of our knowledge, the only paper in the supply chain management literature to address the impact of the secondary market is Lee and Whang (2002), which considers a two-period, two-stage model. First, multiple retailers are supplied by a single manufacturer; then, they trade inventories among themselves. Lee and Whang derive the optimal decisions for the retailers and the equilibrium market price for this endogenous secondary market. In their model, the presence of the secondary market may or may not raise the value of the supply chain. In our paper, on the other hand, the supply chain is always better off with secondary markets, and we allow for a secondary market at each stage in the supply chain (rather than just at the retailer). Our secondary markets are exogenous, and each stage in the supply chain is a price taker (in its own secondary market). Instead of focusing on the equilibrium price and order quantities in a decentralized two-period, two-stage supply chain, we seek to determine the joint replenishment and disposal policy for a centralized, multiperiod, multistage system.

## 2. Other Related Literature

Along with Lee and Whang (2002), this work is related to two streams of research literature. The first of those deals with serial multiechelon inventory systems, and its origin is in the classical Clark and Scarf (1960) paper. Clark and Scarf introduced the notion of *echelon inventory* and showed how a multistage inventory model can be reformulated in terms of echelon inventories to allow *additive separation*—a decomposition of the multidimensional problem into a nested sequence of single-dimensional problems. Federgruen and Zipkin (1984) extend the Clark and Scarf (1960) results to the stationary, infinite horizon case. Federgruen (1993) provides an excellent review of multiechelon inventory research.

There is also a number of more recent papers in this area. Chen and Song (2001), for example, allow the

demand to be modulated by an exogenous Markov chain and prove the optimality of the echelon base-stock policies with state-dependent order-up-to levels. Gallego and Özer (2003) establish the Clark–Scarf decomposition for a multiechelon inventory system under advanced demand information. For sufficiently short information horizons, the optimal policy is the echelon base-stock policy in each period; for longer information horizons, the base-stock levels become state dependent. Muharremoglu and Tsitsiklis (2008) replicate the Clark and Scarf (1960) result using an innovative customer-unit decomposition approach. Janakiraman and Muckstadt (2009) employ the same approach to show that when the lead time at the upstream echelon for a capacitated two-stage system is two periods, the optimal decision at the downstream stage is determined by one base-stock level when that level is below the inventory in transit and by another base-stock level otherwise; the optimal decision at the upstream stage has a similar two-tiered structure. DeCroix et al. (2005) introduce the idea of negative demand, representing returns from customers. They establish the optimality of the echelon base-stock policy when returns from customers occur at any, or several stages, in the system or when those returns require a recovery lead time (the latter requires that demands and returns be independent). Van Houtum et al. (2007) assume that the replenishment interval at each stage is an integer multiple of the replenishment interval of the next downstream stage and show the optimality of echelon base-stock policies. Chao and Zhou (2009) generalize this model to allow for batch ordering as well as fixed replenishment intervals.

The second stream of literature relevant to this work deals explicitly with excess inventory. In the single-stage inventory research, Simpson (1978) assumes that items can be stored, disposed of, or remanufactured; once recovered, those items rejoin the items used to satisfy demand. With no lead times, Simpson shows the structure of the optimal policy, with three critical numbers corresponding to the three decisions. Inderfurth (1997) extends those results to positive, identical lead times. Among the papers on single-stage systems with disposals, only Morton (1978) and Rosenfield (1989, 1992) allow stochastic demand. The former develops a sequence of upper and lower cost and policy bounds for the infinite horizon non-stationary problem, whereas the latter assumes stationary demand, finds the optimal number of units to keep in stock, and proves the invariance of that critical number relative to future disposal options.

Regarding research on stock disposals in multiechelon systems, Fukuda (1961) considers a setting where each stage in the system is allowed to return stock to

the next stage upstream. The only way to get stock out of the system in his model is “... to move it step by step up the echelons until it reaches the highest echelon and gets out of it” (p. 222). Actual disposal of stock only happens at the last echelon. Returning stock upstream leaves echelon inventory at all upstream stages unchanged, so the additive separation of the objective function and the optimality of echelon base-stock policies carry over from Clark and Scarf (1960) in a straightforward manner. By contrast, we allow inventory to “get out of the system” directly at each stage. Thus, a secondary market sale in our paper results in immediate echelon inventory reduction at all the upstream echelons, which has important implications for the structure of the problem.

To the best of our knowledge, the only other paper to consider stock disposals in a multiechelon setting is DeCroix (2006), which assumes a recovery facility that receives a stochastic amount of used products where they can be stored, disposed of, or remanufactured. The recovered units can rejoin the flow of material. When this reentry happens at the most upstream stage, the Clark–Scarf decomposition continues to hold. Otherwise, the additive separation of the objective function can only be achieved when the disposal option is eliminated (and echelon quantities redefined to include the inventory at the recovery facility). Thus, although DeCroix (2006) makes an important contribution to the theory of remanufacturing in a multiechelon setting, to achieve the Clark–Scarf decomposition, he restricts the disposal of stock to the very last echelon. In our model, we do not consider remanufacturing; instead, our interest is in supply chains where stock disposals are allowed at each stage.

### 3. Model Description

#### 3.1. Flow of Product in the System

We identify stages (also echelons) for the minimum number of periods that a unit is away from reaching the most downstream stage. When a unit is first ordered, it is in stage  $L$  (external supplier). At the beginning of the next period, this unit is in stage  $L - 1$ , the most upstream stage in the system considered. The unit can be moved down one stage each period, until it reaches stage 0 where it will face customer demand. (The case of multiple periods between stages is addressed explicitly in §7.) The progress of a unit can be delayed by keeping the unit at the same stage until the next period. We allow (excess) inventory to be sold (i.e., disposed) in the secondary market at any stage in the supply chain, including the most downstream stage. The secondary market demand is plentiful, and a disposal results in the immediate removal of the sold stock from the supply chain. Buying from

the secondary markets is not an option. The following parameters describe our system. (A glossary of notation can be found in §8.)

- $x_{jt}$  = the number of units in stage  $j$  at the beginning of period  $t$ , prior to making any decisions, referred to as *the on-hand inventory* for  $j > 0$  and *the net inventory* for  $j = 0$ ;
- $\hat{X}_{jt}$  = the decision variable, referred to as *the disposal decision*, representing the amount disposed from stage  $j$  by the beginning of period  $t + 1$ ;
- $X_{jt}$  = the decision variable, referred to as *the order decision*, representing the amount in stage  $j + 1$  passed downstream to stage  $j$  by the start of period  $t + 1$ .

By the beginning of period  $t + 1$ , stage  $j$  will have received  $X_{jt}$ , sold  $\hat{X}_{jt}$  in the secondary market, and sent  $X_{j-1,t}$  downstream (all of which are decisions made in period  $t$ ). Let  $D_t$  represent the stochastic demand in period  $t$ . The state transition equations are

$$x_{j,t+1} = \begin{cases} x_{0t} - \hat{X}_{0t} + X_{0t} - D_t & \text{if } j = 0, \\ x_{jt} - \hat{X}_{jt} + X_{jt} - X_{j-1,t} & \text{if } j = 1, 2, \dots, L - 1. \end{cases} \quad (1)$$

Figure 1 shows the movement of the product in the system. Let  $x_t := (x_{0t}, x_{1t}, \dots, x_{L-1,t})$  be referred to as *the on-hand inventory state*; vectors  $\hat{X}_t := (\hat{X}_{0t}, \hat{X}_{1t}, \dots, \hat{X}_{L-1,t})$  and  $X_t := (X_{0t}, X_{1t}, \dots, X_{L-1,t})$  are the decision variables of the model.

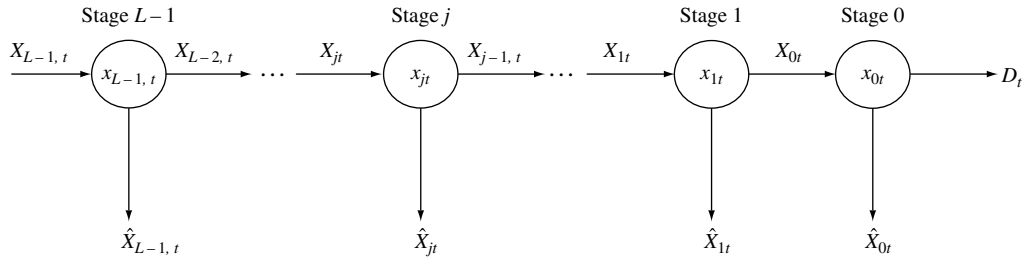
To assess the impact of “demand shocks” on secondary market sales, we allow demands to be modulated by an exogenous Markov chain. Markov modulation has recently been used, for example, in Song and Zipkin (1992, 1996), Chen and Song (2001), and Angelus and Porteus (2008). There exists a countable Markov chain  $\{\omega_t\}$  such that an exogenous state  $\omega_t$ , determined independently of any decisions, can impact the demand distribution in each period.

Let  $\mathcal{X}(x_t)$  be the feasible set given  $x_t$ . The sequence of events is as follows: (1) states  $x_t$  and  $\omega_t$  are observed; (2) the decisions  $(\hat{X}_t, X_t)$  are selected from  $\mathcal{X}(x_t)$ ; (3) ordered amounts are received, and the disposed units are removed from the supply chain; (4) customer demand is observed and satisfied to the extent possible; and (5) costs and revenues are incurred. Thus,

$$\mathcal{X}(x_t) = \{ \hat{X}_{jt}, X_{jt} \geq 0 \mid \hat{X}_{jt} \leq x_{jt}; X_{jt} \leq x_{j+1,t} - \hat{X}_{j+1,t}; 0 \leq j \leq L - 1 \}, \quad (2)$$

so that, at each stage, the sum of the units disposed of and those passed downstream cannot exceed the on-hand inventory at that stage. The external supplier is assumed to have ample stock and is disallowed stock disposals, so that  $x_{L,t} := \infty$  and  $\hat{X}_{L,t} := 0$ .

Figure 1 States, Decisions, and Product Flow in the System



3.2. Costs and Revenues

We assume full backlogging of unsatisfied demand with a unit backlogging cost  $b_t$  and inventory carryover at each stage with a unit on-hand inventory holding cost  $h_{jt}$  at stage  $j$  in period  $t$ . Each unit ordered into stage  $j$  in period  $t$  costs  $\kappa_{jt}$ , and each unit sold in the secondary market at stage  $j$  in period  $t$  generates revenue  $r_{jt}$ . We focus on the case when a sale in the secondary markets generates revenue. At times, disposing of inventory can incur costs rather than revenues. All of the results in this paper apply when  $r_{jt}$  at one or more stages is negative.

Let the states  $x_t$  and  $\omega_t$  and decisions  $\hat{X}_t$  and  $X_t$  in period  $t$  be given. The one-period expected costs (minus revenues) at the most downstream stage ( $j = 0$ ) become

$$b_t E_{D_t|\omega_t} [(D_t - (x_{0t} - \hat{X}_{0t} + X_{0t}))^+] + h_{0t} E_{D_t|\omega_t} [(x_{0t} - \hat{X}_{0t} + X_{0t} - D_t)^+] + \kappa_{0t} X_{0t} - r_{0t} \hat{X}_{0t},$$

and the one-period costs at all the upstream stages are

$$\sum_{j=1}^{L-1} [\kappa_{jt} X_{jt} + h_{jt} (x_{jt} - \hat{X}_{jt} - X_{j-1,t} + X_{jt}) - r_{jt} \hat{X}_{jt}],$$

where, as usual,  $[x]^+ := \max(x, 0)$ , and  $E_{D_t|\omega_t}[\cdot]$  represents the expectation over  $D_t$ , given  $\omega_t$  in period  $t$ . The total expected one-period cost becomes

$$\gamma_t(\omega_t, x_{0t} - \hat{X}_{0t} + X_{0t}) + \sum_{j=0}^{L-1} [(\kappa_{jt} + h_{jt} - h_{j+1,t}) X_{jt} - r_{jt} \hat{X}_{jt} + h_{jt} (x_{jt} - \hat{X}_{jt})], \quad (3)$$

where  $\gamma_t(\omega, x) := (b_t + h_{0t}) E_{D_t|\omega} [(D_t - x)^+] - h_{0t} E_{D_t|\omega} [D_t]$  and  $h_{L,t} := 0$ .

We make the following assumption about the cost and revenue parameters in our model.

ASSUMPTION 1. For each  $j$  in each period  $t$ ,  $\kappa_{jt} \geq r_{jt} - r_{j+1,t}$ .

Assumption 1 acts to prevent “speculative ordering,” the sole purpose of which is to receive revenue from selling the product in the secondary market

rather than satisfying the primary demand at the most downstream stage. Because  $r_{jt} - \kappa_{jt} \leq r_{j+1,t}$ , then ordering a unit into stage  $j$ , while simultaneously disposing of another unit through a secondary market sale, generates less revenue than selling a unit at stage  $j + 1$  directly. Thus, even though selling a unit in the secondary market generates revenue, the total profit on that particular unit is always negative. Consequently, under Assumption 1, a supply chain does not serve its secondary markets directly but rather makes use of them for disposing of excess stock.

Assumption 1 may not always hold: If the secondary market at stage  $j + 1$  is not well developed, a stock disposal may incur costs rather than accrue revenues ( $r_{j+1,t} < 0$ ). In that case, even for larger values of unit order costs at stage  $j$ , the required condition may not be met when the secondary market at stage  $j$  can generate revenue for the supply chain (and  $r_{jt} > 0$ ). Although such settings tend to be uncommon in practice, they may exist, for example, in the chemical industry, where a particular work-in-process chemical can be both unstable and environmentally toxic, so that disposing of it would incur a significant cost. Once that chemical is compounded downstream with other chemicals, it can become a stable product of value and generate revenue in the secondary market.

3.3. Optimality Equations

Let  $T$  be the time horizon for the problem. Let  $F_t(\omega, x_t | \hat{X}_t, X_t)$  denote the minimum expected net present value of the costs (less returns) over periods  $t$  through  $T$ , as of the beginning of period  $t$ , as a function of the on-hand inventory state  $x_t$  and exogenous state  $\omega$ , given decisions  $\hat{X}_t$  and  $X_t$ . Let  $F_t(\omega, x_t)$  denote the best of these:

$$F_t(\omega, x_t) = \min_{\hat{X}_t, X_t \in \mathcal{X}(x_t)} F_t(\omega, x_t | \hat{X}_t, X_t). \quad (4)$$

Let  $\alpha$  be the single-period discount rate. Then, by (3),

$$F_t(\omega, x_t | \hat{X}_t, X_t) = \sum_{j=0}^{L-1} \{(\kappa_{jt} + h_{jt} - h_{j+1,t}) X_{jt} - (r_{jt} + h_{jt}) \hat{X}_{jt} + h_{jt} x_{jt}\} + \gamma_t(\omega, x_{0t} - \hat{X}_{0t} + X_{0t}) + \alpha E_{D_t, \omega_{t+1}|\omega_t} [F_{t+1}(\omega_{t+1}, x_{t+1})], \quad (5)$$

where  $E_{D_t, \omega_{t+1} | \omega_t}[\cdot]$  represents the expectation over both  $D_t$  and  $\omega_{t+1}$ , given  $\omega_t$  in period  $t$ . We will refer to (4) and (5) as *the stage formulation* of the model.

### 3.4. Echelon Formulation

Following the Clark and Scarf (1960) approach, we introduce the following echelon variables:

$y_{jt} := x_{0t} + x_{1t} + \dots + x_{jt}$  referred to as *the initial (echelon) inventory*;

$\hat{Y}_{jt} := y_{jt} - \sum_{i=0}^j \hat{X}_{it}$  referred to as *the postdisposal echelon level*;

$Y_{jt} := \hat{Y}_{jt} + X_{jt}$  referred to as *the replenishment (echelon) level*.

Given that we allow backlogging at the most downstream stage,  $y_{jt}$  represents the echelon *net* inventory at the beginning of the period  $t$ ;  $\hat{Y}_{jt}$  stands for the echelon *net* inventory after disposals at stages 0 through  $j$  have taken place; and  $Y_{jt}$  is the echelon *net* inventory after disposal and ordering because it also includes those units currently in stage  $j + 1$  that are ordered into stage  $j$  in period  $t$ . Going forward, we will, for convenience, continue to treat those variables as being net of the backlogged demand without explicitly stating so.

At the beginning of period  $t + 1$ , the updated echelon quantities are  $y_{j,t+1} = Y_{jt} - D_t$ .

In the stage formulation of the model, the order of decisions  $\hat{X}_{jt}$  and  $X_{jt}$  at any stage  $j$  is inconsequential. Once echelon variables are introduced into the model, this is no longer the case; the manner in which echelon variables are defined necessarily implies a corresponding sequence of decisions. Thus, we assume that the disposal decisions at all stages are made (and executed) first, followed by all the order decisions.

Let  $y_t := (y_{0t}, y_{1t}, \dots, y_{L-1,t})$  be referred as *the echelon state* of the system. Vectors  $\hat{Y}_t := (\hat{Y}_{0t}, \dots, \hat{Y}_{L-1,t})$  and  $Y_t := (Y_{0t}, \dots, Y_{L-1,t})$  are the new decision variables of the model.

LEMMA 1. Let  $\mathcal{Y}(y_t)$  denote the set of feasible decision schedules for state  $y_t$  in period  $t$ :

$$\begin{aligned} \mathcal{Y}(y_t) = \{ & \hat{Y}_t, Y_t \geq 0 \hat{Y}_{0t} \leq y_{0t}; \hat{Y}_{jt} \leq Y_{jt} \leq \hat{Y}_{j+1,t} \\ & \leq \hat{Y}_{jt} + y_{j+1,t} - y_{jt}; 0 \leq j \leq L - 1 \}, \end{aligned}$$

where, for convenience,  $\hat{Y}_{Lt} = y_{Lt} := \infty$  for each  $t$ .

The stock available for disposal at stage  $j$  is  $y_{jt} - y_{j-1,t}$ ; once disposals have taken place at echelons  $1, \dots, j$ , the postdisposal level  $\hat{Y}_{j+1,t}$  is bounded from below by  $\hat{Y}_{jt}$ , the postdisposal level at echelon  $j$ , and from above by the sum of  $\hat{Y}_{jt}$  and the on-hand inventory at stage  $j$ . The replenishment level  $Y_{jt}$  is bounded from below by  $\hat{Y}_{jt}$ , and from above by  $\hat{Y}_{j+1,t}$ .

Let  $f_t(\omega, y_t | \hat{Y}_t, Y_t)$  denote the expected net present value of the costs (less revenues) over periods  $t$  through  $T$ , as of the beginning of period  $t$ , as a function of  $y_t$  and  $\omega$ , given that decisions  $\hat{Y}_t$  and  $Y_t$  are made. Let  $f_t(\omega, y_t)$  denote the best of these. Then,

$$f_t(\omega, y_t) = \min_{\hat{Y}_t, Y_t \in \mathcal{Y}(y_t)} f_t(\omega, y_t | \hat{Y}_t, Y_t), \quad (6)$$

and

$$\begin{aligned} f_t(\omega, y_t | \hat{Y}_t, Y_t) &= \sum_{j=0}^{L-1} r_{jt}(y_{j-1,t} - y_{jt}) + \gamma_t(\omega, Y_{0t}) \\ &+ \sum_{j=0}^{L-1} (b_{jt} Y_{jt} + c_{jt} \hat{Y}_{jt}) + \alpha E[f_{t+1}(\omega_{t+1}, Y_t - D_t)], \end{aligned}$$

where  $b_{jt} := \kappa_{jt} + h_{jt} - h_{j+1,t}$ ,  $c_{jt} := r_{jt} - r_{j+1,t} - \kappa_{jt}$ , and  $r_{Lt} := 0$ . By  $Y_t - D_t$  we mean the vector of  $Y_{jt} - D_t$  for each  $j$  in period  $t$ , and, for notational convenience,  $E[\cdot]$  is used to represent  $E_{D_t, \omega_{t+1} | \omega_t}[\cdot]$ .

ASSUMPTION 2. The terminal value function  $f_{T+1}(\omega, \cdot)$  is convex for each  $\omega$ .

We will refer to the above dynamic program as *the echelon formulation* of the model. Note that one obvious terminal value function (other than zero) is  $f_{T+1}(\omega, y_{T+1}) = \sum_{j=0}^{L-1} \{\kappa_{j,T+1}[y_{j,T+1}]^- - (r_{j,T+1} - r_{j+1,T+1})[y_{j,T+1}]^+\}$ , which, by Assumption 1, is convex.

In general, the Clark–Scarf decomposition cannot be expected to work for  $f_t(\omega, y_t)$ . The upper boundary of the feasible set  $\mathcal{Y}(y_t)$  for each echelon  $j$  depends on two echelon inventories,  $y_{j+1,t}$  and  $y_{jt}$ , rather than just one as is common in Clark–Scarf type models. Thus, each postdisposal level will depend, in general, on those two echelon variables as well as the postdisposal level at the downstream stage. This dependence on multiple echelon inventories cascades from the upper stages downward and renders the objective function dependent on the entire echelon state  $y_t$  instead of just a single echelon inventory. Alternatively, because each stock disposal impacts the feasible region for all upstream decisions, each postdisposal level depends on all downstream states and decisions. Therefore, without special policy restrictions, the system cannot be expected to achieve the Clark–Scarf decomposition.

## 4. Optimal Policy

### 4.1. Nested Echelon Order-up-to Policies

In describing the optimal policy, it is convenient to imagine an organization with two departments charged with implementing the optimal policy: “the stock disposal” department and “the inventory replenishment” department, each with a manager at every echelon.

A policy is a *nested echelon order-up-to* policy if it specifies order-up-to levels  $(\hat{S}_{jt}, S_{jt})$  for  $j = 0, \dots, L - 1$  in each period  $t$  as functions of  $y_t$  and  $\omega$ , such that  $(\hat{Y}_{jt}, Y_{jt})$  are the closest feasible decisions to their respective order-up-to levels, subject to the following: in considering a disposal decision, downstream disposal decisions are held fixed while all other decisions are ignored; in considering a replenishment decision, adjacent disposal decisions are held fixed, and all others are ignored. A disposal manager at echelon  $j$  only recognizes the constraint  $\hat{Y}_{j-1,t} \leq \hat{Y}_{jt} \leq y_{jt} - y_{j-1,t} + \hat{Y}_{j-1,t}$  and selects the point within that interval closest to  $\hat{S}_{jt}$ :  $\hat{Y}_{jt} = \hat{Y}_{j-1,t} \vee [\hat{S}_{jt} \wedge (y_{jt} - y_{j-1,t} + \hat{Y}_{j-1,t})]$ , where  $a \vee b := \max(a, b)$ , and  $a \wedge b := \min(a, b)$ . The amount thus disposed is  $y_{jt} - y_{j-1,t} + \hat{Y}_{j-1,t} - \hat{Y}_{jt}$ . A replenishment manager ignores all (other) inventory decisions and only recognizes the (hard) constraint  $\hat{Y}_{jt} \leq Y_{jt} \leq \hat{Y}_{j+1,t}$ . He selects a point in that interval closest to  $S_{jt}$ :  $Y_{jt} = \hat{Y}_{jt} \vee [S_{jt} \wedge \hat{Y}_{j+1,t}]$ . We formalize this below.

**DEFINITION 1.** Let  $m$  be an integer. A policy is a *nested echelon order-up-to* policy if there exist order-up-to levels  $(\hat{S}_{jt}, S_{jt})$  such that the actual decisions are given recursively as follows:

$$\hat{Y}_{jt} = \hat{Y}_{j-1,t} \vee [\hat{S}_{jt} \wedge (y_{jt} - y_{j-1,t} + \hat{Y}_{j-1,t})], \quad (7)$$

$$Y_{jt} = \hat{Y}_{jt} \vee [S_{jt} \wedge \hat{Y}_{j+1,t}], \quad (8)$$

for  $j = 0, 1, 2, \dots, m$ , where  $\hat{Y}_{-1,t} = y_{-1,t} := 0$  and  $\hat{Y}_{m+1,t} := \infty$ .

Implicit in Definition 1 is the assumption, already stated, that at each stage the disposal decision is made before the order decision, so that the postdisposal level at each echelon is known before the replenishment level is assessed. A reverse order of decisions would necessitate a corresponding change in the above definition, but the fundamental results about the structure of the optimal policy (and the heuristic proposed later in the paper) would remain unaltered.

Because  $\hat{S}_{jt}$  and  $S_{jt}$  are, in general, functions of  $y_t$  and  $\omega$ , then so are  $\hat{Y}_{jt}$  and  $Y_{jt}$ . Once the managers have determined their respective order-up-to levels for a given  $y_t$  and  $\omega$ , the optimal decisions are found directly using the computation given in Definition 1.

Table 1 illustrates this computation of optimal decisions, for a given set of order-up-to levels, for two consecutive periods in a four-echelon system. Columns (1)–(3), which are the given inputs, represent the period, echelon, and initial echelon inventory for some given Markov states. Columns (4) and (7) show the order-up-to levels, also referred to as the postdisposal and replenishment *targets*, for each of the eight managers (four stock disposal managers and four inventory managers). Columns (6) and (9) show actual decisions made.

**Table 1** Illustration of a Nested Echelon Order-up-to Policy

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$t$	$j$	$y_{jt}$	$\hat{S}_{jt}$	$[\hat{Y}_{j-1,t}, y_{jt} - y_{j-1,t} + \hat{Y}_{j-1,t}]$	$\hat{Y}_{jt}$	$S_{jt}$	$[\hat{Y}_{jt}, \hat{Y}_{j+1,t}]$	$Y_{jt}$
1	3	120	100	[70, 110]	100	150	[100, ∞]	150
1	2	80	80	[40, 70]	70	110	[70, 100]	100
1	1	50	40	[25, 45]	40	60	[45, 70]	60
1	0	30	25	[0, 30]	25	50	[25, 45]	45
2	3	115	100	[65, 115]	100	135	[115, ∞]	135
2	2	65	80	[25, 65]	65	90	[65, 115]	90
2	1	25	40	[10, 25]	25	70	[25, 65]	65
2	0	10	20	[0, 10]	10	50	[10, 25]	25

The postdisposal target for echelon 0 in period 1,  $\hat{S}_{01}$ , is seen to be 25. Because the feasible interval (column (5)) is  $[0, y_{01}] = [0, 30]$ , this target is achievable. Once achieved, it becomes the lower end of the feasible region for the postdisposal level at echelon 1. The postdisposal target for echelon 1 in period 1 is  $\hat{S}_{11} = 40$ . The echelon 1 postdisposal level in period 1,  $\hat{Y}_{11}$ , must lie within the interval  $[\hat{Y}_{01}, y_{11} - y_{01} + \hat{Y}_{01}] = [25, 45]$ . Thus, the echelon 1 postdisposal target is achievable. Once realized, the echelon 1 postdisposal level becomes the lower limit of the feasible interval for the replenishment level at the same echelon and the upper limit for the replenishment level at the downstream echelon. The feasible region for echelon 0 replenishment level becomes  $[\hat{Y}_{01}, \hat{Y}_{11}] = [25, 45]$ . The echelon 0 replenishment target,  $S_{01}$ , shown to be 50, is thus not achievable, and the echelon 0 inventory manager chooses the point in the feasible interval closest to his target. Thus,  $Y_{01} = 45$ . The rest of period 1 decisions are made similarly.

The demand in period 1 is for 35 units, so the initial echelon states in period 2 are 35 units less than the replenishment levels decided on in period 1. The optimal decisions in period 2 are then determined in the same nested, bottom-up way. The implementation of the optimal policy proceeds in this manner until the end of the time horizon.

**THEOREM 1.** For every state  $\omega$  and period  $t$ , the objective value function defined in (6) is convex, and a nested echelon order-up-to policy is optimal.

All proofs are in the appendix. The proof of Theorem 1 makes use of an induction within an induction, so that for any set of postdisposal echelon levels, the result can be shown to hold for any set of replenishment levels. This nested structure of the proof is necessitated by the nested nature of the decisions in the problem and their interaction on the boundary of the feasible space. The order-up-to levels of the optimal policy, being functions of both the Markov state and the echelon (inventory) state, are determined by finding the minimum, in each period, for each state  $\omega$



and for each point  $y_t$  in the  $L$ -dimensional state space, of a convex function of  $2L$  state variables ( $y_t$  and  $Y_t$ ). Such a calculation runs very quickly into the curse of dimensionality, rendering the numerical solution of the problem practically impossible.

#### 4.2. Parsimonious Policies

To reduce (though not eliminate) the curse of dimensionality in the state-dependent multiechelon control policy of Theorem 1, we identify special features of the optimal policy.

In the stage formulation of the model, a policy  $(\hat{X}_t, X_t)$  is said to be *parsimonious* if either  $\hat{X}_{jt} = 0$  or  $X_{jt} = 0$  (or both) for every  $j$  in period  $t$ . Under a parsimonious policy, each stage either receives additional stock or disposes of excess inventory, but not both.

**DEFINITION 2.** Let  $(\hat{X}_t, X_t) \in \mathcal{X}(x_t)$  be optimal decisions in period  $t$  given  $x_t$ . A decision set  $(\hat{X}_t^j, X_t^j)$  is the  $j$ th *parsimonious reduction* of  $(\hat{X}_t, X_t)$  if its decisions are recursively given by

$$X_{it}^j := \begin{cases} [X_{0t} - \hat{X}_{0t}]^+ & \text{if } i = 0, \\ [X_{it} - \hat{X}_{it} - (X_{i-1,t} - X_{i-1,t}^j)]^+ & \text{if } 0 < i \leq j, \\ X_{it} & \text{if } j < i \leq L-1; \end{cases}$$

$$\hat{X}_{it}^j := \begin{cases} [\hat{X}_{0t} - X_{0t}]^+ & \text{if } i = 0, \\ [\hat{X}_{it} - X_{it} + (X_{i-1,t} - X_{i-1,t}^j)]^+ & \text{if } 0 < i \leq j, \\ \hat{X}_{j+1,t} + X_{jt} - X_{jt}^j & \text{if } i = j+1, \\ \hat{X}_{it} & \text{if } j+1 < i \leq L-1. \end{cases}$$

**LEMMA 2.** Let  $x_{t+1}$  and  $x_{t+1}^j$  denote on-hand states in period  $t+1$ , obtained by starting with  $x_t$  in period  $t$  and applying policies  $(\hat{X}_t, X_t)$  and  $(\hat{X}_t^j, X_t^j)$ , respectively. For each  $j \leq L-1$ ,

- $X_{it}^j \geq 0, \hat{X}_{it}^j \geq 0$ , for all  $i$ ;
- $X_{it}^j = 0$  or  $\hat{X}_{it}^j = 0$  (or both) for all  $i \leq j$ ;
- $X_{it}^j \leq X_{it}$  for all  $i \leq j$ ;
- $(\hat{X}_t^j, X_t^j) \in \mathcal{X}(x_t)$ ;
- $x_{t+1} = x_{t+1}^j$ .

The  $j$ th parsimonious reduction is parsimonious through echelon  $j$  (part (b)), it reduces the order decision (part (c)), it is feasible (part (d)), and it results in the same on-hand inventory state next period as the original policy (part (e)).

**LEMMA 3.** (a)  $F_t(\omega, x_t | \hat{X}_t^0, X_t^0) \leq F_t(\omega, x_t | \hat{X}_t, X_t)$ ;  
 (b)  $F_t(\omega, x_t | \hat{X}_t^j, X_t^j) \leq F_t(\omega, x_t | \hat{X}_t^{j-1}, X_t^{j-1})$  for all  $j < L-1$ ;  
 (c)  $F_t(\omega, x_t | \hat{X}_t^{L-1}, X_t^{L-1}) \leq F_t(\omega, x_t | \hat{X}_t^{L-2}, X_t^{L-2})$ .

The  $(L-1)$ th parsimonious reduction of  $(\hat{X}_t, X_t)$ , denoted by  $(\hat{X}_t^*, X_t^*)$ , will be referred to as the *parsimonious reduction* of  $(\hat{X}_t, X_t)$ .

**COROLLARY 1.**  $F_t(\omega, x_t | \hat{X}_t^*, X_t^*) \leq F_t(\omega, x_t | \hat{X}_t, X_t)$  for each  $\omega$ .

**THEOREM 2.** There exists a parsimonious optimal policy for each state  $\omega$  in each period  $t$ .

Therefore, under Assumption 1, it is optimal at each stage to either order more stock or dispose of excess inventory, but not both in the same period. (In the absence of Assumption 1, the theorem may not hold because speculative ordering to get extra sales in secondary markets may call for both ordering of more stock and disposing of it at the same time at the same stage.)

Under a parsimonious policy, the size of the optimization space is significantly reduced. The original optimal policy lies within  $\mathcal{X}(x_t)$ , which is a  $2L$ -dimensional subset of  $\mathbb{R}^{+2L}$ . By contrast, the optimal solution under a parsimonious policy is found in a subset of  $\mathcal{X}(x_t)$  obtained by setting zero to one of each pair of optimization variables at each stage. Thus, the feasible set under a parsimonious restriction consists of  $2^L$   $L$ -dimensional subsets of  $\mathcal{X}(x_t)$ . This renders the search for the optimal solution (considerably) less onerous.

#### 5. Disposal Saturation Policies

The curse of dimensionality renders this problem very difficult to solve. The optimality of parsimonious policies helps, but not as much as the Clark–Scarf decomposition in the traditional multiechelon models (i.e., without secondary market sales). To address this, we now identify special means of achieving that decomposition. Fukuda (1961) and DeCroix (2006) accomplish this by constraining stock disposals to only the most upstream stage. Although there are several practical examples where secondary markets can occur at only the last stage, in this paper we are interested in those supply chains for which secondary market sales can occur throughout the supply chain. Therefore, to provide a framework faithful to such supply chains in industry and render the model tractable, we look for a class of policies for which (i) stock disposals are allowed at each stage, (ii) echelon targets are independent of the (echelon) inventory state, and (iii) the resulting objective function is additively separable.

**DEFINITION 3.** A disposal vector  $\hat{X}_t$  in period  $t$  represents a *disposal saturation policy* if there exists a stage  $k_t$  such that  $\hat{X}_{jt} = 0$  for all  $j < k_t$ , if  $k_t > 0$ , and  $\hat{X}_{jt} = x_{jt}$  for all  $j > k_t$ , if  $k_t < L-1$ . We will refer to such  $k_t$  as the *threshold stage* in period  $t$ .

Under a disposal saturation (DS) policy, the firm sells off all stock upstream of some stage  $k_t$ ; there are no disposals downstream of  $k_t$ . Thus, a DS policy allows disposals at each stage, and if any stock is disposed at stage  $k_t$ , then all available stock upstream of that stage is also sold off. This policy starts at the top stage and “saturates” disposals at each stage before initiating disposals at the next stage downstream. Finding the DS policy reduces to finding the threshold stage  $k_t$  (for each  $x_t$ ) and the disposal decision  $\hat{X}_{k_t(x_t),t}$  (or, equivalently,  $\hat{Y}_{k_t(y_t),t}$ ). Once  $k_t(y_t)$  and  $\hat{Y}_{k_t(y_t),t}$  have been determined, the remaining post-disposal levels become  $\hat{Y}_{j_t}(y_t) = y_{j_t}$  for  $j < k_t(y_t)$ , and  $\hat{Y}_{j_t}(y_t) = \hat{Y}_{k_t(y_t),t}(y_t)$  for  $j > k_t$ . Given  $y_t$  and a DS policy, the replenishment levels for  $j = k_t(y_t) + 1, \dots, L - 2$ , are given simply by  $Y_{j_t}(y_t) = \hat{Y}_{k_t(y_t),t}$ .

**5.1. Decomposition Results**

We now show there exists a disposal saturation policy, determined by a set of ordered base-stock levels, that achieves the Clark–Scarf decomposition for the dynamic program in (6). In the process, we make use of two auxiliary results. The first one is adapted from Karush (1959).

LEMMA 4. *If  $f$  is an arbitrary smooth convex function on  $\mathbb{R}$ , then, given  $x \leq y$ ,  $\min_{x \leq \theta \leq y} f(\theta)$  can be expressed as  $g(x) + h(y)$ , where  $g$  is smooth, convex increasing, and  $h$  is smooth, convex decreasing. In particular, if  $f$  has a finite unconstrained minimizer  $S$ , then*

$$g(x) := \begin{cases} f(S) & \text{if } x \leq S, \\ f(x) & \text{otherwise;} \end{cases}$$

$$h(y) := \begin{cases} f(y) - f(S) & \text{if } y \leq S, \\ 0 & \text{otherwise.} \end{cases}$$

If  $f$  is increasing,  $h(y) = 0$ , and  $g(x) = f(x)$ ; if  $f$  is decreasing,  $g(x) = 0$ , and  $h(y) = f(y)$ .

LEMMA 5. *Let  $n$  be an integer. Let  $\{s_j\}$  and  $\{y_j\}$  be two sequences of real numbers such that  $s_0 \geq s_1 \geq \dots \geq s_n$  and  $y_0 \leq y_1 \leq \dots \leq y_n$ , respectively. Exactly one of the following holds:*

- (i)  $s_0 < y_0$ ;
- (ii)  $s_n \geq y_n$ ;
- (iii) there exists a unique  $j, j \in \{1, \dots, n\}$  such that  $y_j > s_j \geq y_{j-1}$ ;
- (iv) there exists a unique  $k, k \in \{0, \dots, n - 1\}$  such that  $s_k \geq y_k > s_{k+1}$ .

LEMMA 6. *Let  $f_{t+1}(\omega, \cdot)$  be smooth and additively convex for each  $\omega$ . For each  $\omega$ , there exists in period  $t$  a vector of base-stock levels  $\hat{S}_t^*(\omega) := \{\hat{S}_{0t}^*(\omega), \dots, \hat{S}_{L-1,t}^*(\omega)\}$  such that*

(a)  $\hat{S}_{0t}^*(\omega) \geq \hat{S}_{1t}^*(\omega) \geq \dots \geq \hat{S}_{L-1,t}^*(\omega)$  for each  $\omega$ ;

(b) given  $y_t, \omega$ , and  $\hat{S}_t^*(\omega)$ , the vector  $\hat{Y}_t^*(\omega, y_t) := \{\hat{Y}_{0t}^*(\omega, y_t), \dots, \hat{Y}_{L-1,t}^*(\omega, y_t)\}$ , where

$$\hat{Y}_{j_t}^*(\omega, y_t) := \begin{cases} \hat{Y}_{j-1,t}^*(\omega, y_t) + y_{j_t} - y_{j-1,t}, & \text{if } \hat{Y}_{j-1,t}^*(\omega, y_t) + y_{j_t} - y_{j-1,t} < \hat{S}_{j_t}^*(\omega), \\ \hat{S}_{j_t}^*(\omega) & \text{if } \hat{Y}_{j-1,t}^*(\omega, y_t) < \hat{S}_{j_t}^*(\omega) \leq \hat{Y}_{j-1,t}^*(\omega, y_t), \\ \hat{Y}_{j-1,t}^*(\omega, y_t) & \text{if } \hat{S}_{j_t}^*(\omega) \leq \hat{Y}_{j-1,t}^*(\omega, y_t), \end{cases} \quad (9)$$

is a disposal saturation policy in period  $t$ ;

(c) given  $y_t, \omega$ , and  $\hat{Y}_t^*(\omega, y_t)$  in period  $t$ , the resulting optimal replenishment policy  $Y_t^*(\omega, y_t)$  is given by the echelon base-stock policy;

(d) given  $y_t, \omega$ , and  $\hat{Y}_t^*(\omega, y_t)$  in period  $t$ , the expected cost function,  $f_t(\omega, y_t | \hat{Y}_t^*)$ , given by

$$f_t(\omega, y_t | \hat{Y}_t^*) = \min_{Y_t \in \mathcal{Y}(y_t | \hat{Y}_t^*)} f_t(\omega, y_t | \hat{Y}_t^*, Y_t), \quad (10)$$

is smooth and additively convex for each  $\omega$  and each  $y_t$  in period  $t$ .

Thus, given additive convexity of the objective function in period  $t + 1$ , there exists a disposal saturation policy, completely specified by a single base-stock level at each echelon, that achieves the Clark–Scarf decomposition of the objective function in period  $t$ .

Let  $\hat{Y}_t^*(\omega, y_t)$  be the disposal saturation policy described in Lemma 6, given  $\omega$  and  $y_t$  in period  $t$ . Let  $f_t^*(\omega, y_t)$  denote the minimum expected net present value of the costs (less revenues) over periods  $t$  through  $T$ , as of the beginning of period  $t$ , as a function of  $y_t$  and  $\omega$ , given the disposal saturation policy  $\hat{Y}_t^*(\omega, y_t)$  in each period  $t$ . Then,

$$f_t^*(\omega, y_t) = \min_{Y_t \in \mathcal{Y}(y_t | \hat{Y}_t^*)} f_t^*(\omega, y_t | \hat{Y}_t^*, Y_t), \quad \text{and}$$

$$f_t^*(\omega, y_t | \hat{Y}_t^*(y_t), Y_t) = \sum_{j=0}^{L-1} r_{j_t}(y_{j-1,t} - y_{j_t}) + \gamma_t(\omega, Y_{0t}) + \sum_{j=0}^{L-1} (b_{j_t} Y_{j_t} + c_{j_t} \hat{Y}_{j_t}^*) + \alpha E[f_{t+1}^*(\omega_{t+1}, Y_t - D_t)]. \quad (11)$$

ASSUMPTION 3. *The terminal function  $f_{T+1}(\omega, \cdot)$  is smooth and additively convex for each  $\omega$ .*

THEOREM 3. *For every state  $\omega$  and period  $t$ , there exists a disposal saturation policy specified by a set of ordered base-stock levels, such that the objective value function in (11) is additively convex and the resulting optimal replenishment policy follows an echelon base-stock policy.*

The disposal saturation policy  $\hat{Y}_t^*$  meets the required criteria: disposals allowed at each stage, state-independence, and the additive convexity of the objective function. This policy also calculates very fast, being determined by single base-stock level for each exogenous state.

**5.2. Heuristic Performance**

Although the disposal saturation policy defined in Lemma 6 achieves the Clark–Scarf decomposition and has the features we desire, it is not clear how closely it approximates the optimal policy. For that purpose, we evaluate its heuristic performance through a series of numerical studies. We start with a set of basic model parameters and vary different pairs of those parameters at a time. The results are presented in the tables below (Tables 3–8) as the percentage by which the total expected cost generated by the DS policy exceeds the total expected cost under the optimal policy. This percentage difference is referred to as *the heuristic performance error*.

Our basic model has two echelons, 20 periods, and three Markov states that correspond to the low-, normal-, and high-demand scenarios and that impact the demand distributions through the multiplier of the mean demand. Given a Markov state  $\omega_t$ , the demand in period  $t$  has the distribution of  $\beta(\omega_t)D_t$ , where  $D_t$  is Poisson distributed with mean of  $4(1.03)^{t-1}$  for the first 12 periods and  $\mu_{12}0.97^{t-12}$  for the last 8 periods. (Period 12 is *the peak demand period*; the shape of the demand curve is not found to impact our numerical results in any significant way.) Transition probabilities and multipliers for the basic model are shown in Table 2. Calculating the expected cost for the basic model with the optimal policy took four hours on a 3 GHz CPU, whereas the same calculation with the DS policy took only 10 seconds.

Unit costs and revenues for the basic model are stationary and as follows:  $(\kappa_0, \kappa_1) = (8, 6)$ ,  $(h_0, h_1) = (2, 1)$ ,  $b_t = 10$ , and  $(r_0, r_1) = (10, 4)$ . The one-period discount factor is 0.95.

We evaluate the expected discounted value of costs (minus revenues from the secondary market sales) over the time horizon, compute the weighted average based on the steady-state probabilities for each of the three Markov states, and compare this value to the

**Table 2 Markov States, Transition Probabilities, and Demand Multipliers**

State $\omega_t$	Transition probabilities			Markov multipliers $\beta(\omega_t)$
	1	2	3	
1	0.70	0.20	0.10	0.333
2	0.25	0.50	0.25	1.0
3	0.10	0.20	0.70	3.0

**Table 3 Heuristic Performance Error—Study 1**

Backlog. cost $b$	Unit disposal revenues $(r_0, r_1)$ (%)					
	(4, 1)	(6, 2)	(8, 3)	(10, 4)	(12, 5)	(14, 6)
8	0.00	0.00	0.03	0.12	0.26	0.43
10	0.00	0.01	0.03	0.14	0.35	0.54
12	0.00	0.01	0.03	0.15	0.41	0.65
14	0.00	0.01	0.03	0.16	0.43	0.75
16	0.00	0.01	0.03	0.17	0.47	0.85
18	0.00	0.01	0.03	0.18	0.48	0.96

corresponding one without secondary market sales. We assume initial on-hand inventory  $(x_{01}, x_{11}) = (4, 4)$  and zero salvage value function, both with and without secondary market sales.

We first explore the effect of secondary market (i.e., disposal) revenues, which we vary from the largest values allowed by Assumption 1,  $(r_0, r_1) = (14, 6)$ , downward, in steps of two at the downstream stage and steps of one at the upstream stage, until the latter is brought down to one. We thus cover a large range of unit disposal revenues, while abiding by Assumption 1.

In Study 1 (Table 3), we vary the disposal revenues against the backlogging cost. The proposed DS policies approximate the optimal very well (within 1%) throughout the shown range of model parameters. The heuristic performance declines when both the backlogging cost and the disposal revenues are high. In that case, although stock disposals are occurring under the optimal policy, they are mostly confined to the downstream stage. With the high backlogging cost, more stock is held at the downstream stage. When the Markov demand shifts to a lower regime, the downstream stage ends up holding, and therefore selling, most of the excess inventory. The performance of the DS policy, under which disposals first deplete the upstream inventory, thus declines. In addition, the disposal revenue differential between the two stages makes secondary market sales at the downstream stage increasingly attractive.

In Study 2 (Table 4), we vary the disposal revenues against the holding costs. As holding costs are increased in Table 4, the performance of the DS

**Table 4 Heuristic Performance Error—Study 2**

Hold. costs $(h_0, h_1)$	Unit disposal revenues $(r_0, r_1)$ (%)					
	(4, 1)	(6, 2)	(8, 3)	(10, 4)	(12, 5)	(14, 6)
(0.5, 0.25)	0.00	0.00	0.00	0.01	0.06	0.64
(1.0, 0.50)	0.00	0.00	0.00	0.02	0.18	0.61
(1.5, 0.75)	0.00	0.00	0.01	0.08	0.31	0.56
(2.0, 1.00)	0.00	0.01	0.03	0.14	0.35	0.54
(2.5, 1.25)	0.00	0.02	0.08	0.24	0.32	0.53
(3.0, 1.50)	0.00	0.04	0.14	0.25	0.30	0.49

**Table 5** Heuristic Performance Error—Study 3

Order costs ( $\kappa_0, \kappa_1$ )	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
(4, 2)	0.84	1.01	1.20	1.39	1.57	1.72
(6, 4)	0.59	0.72	0.83	0.97	1.11	1.24
(8, 6)	0.43	0.54	0.65	0.75	0.85	0.96
(10, 8)	0.33	0.44	0.52	0.60	0.69	0.77
(12, 10)	0.27	0.36	0.43	0.50	0.57	0.65
(14, 12)	0.21	0.29	0.37	0.43	0.48	0.56

heuristic declines: As the difference in the holding costs between the stages grows, it becomes increasingly valuable (for the optimal policy) to sell off excess units at the downstream stage rather than first disposing of those upstream. Thus, at lower disposal revenues, the optimal policy starts to diverge from the heuristic one. On the other hand, when both the unit holding costs and disposal revenues are high, a unit of inventory has a higher chance of being considered in excess as soon as it arrives into the system, so the heuristic performance of the DS policy starts to improve.

In the next four studies, we explore the impact of the unit backlogging cost while assuming the largest unit disposal revenues allowed by Assumption 1. As the heuristic performance is observed to decline in the unit disposal revenues, the numbers displayed below tend to represent *the worst case* scenario for the performance of the DS policy across all unit disposal revenues. In Study 3 (Table 5), for each set of order costs ( $\kappa_0, \kappa_1$ ), we assume the maximum allowed unit revenues of  $(r_0, r_1) = (\kappa_0 + \kappa_1, \kappa_1)$ . Table 5 shows that the DS policy does very well for high order costs and low backlogging costs. The (already) observed deterioration of the performance of the DS heuristic with increasing backlogging costs is (somewhat) offset by higher unit order costs, for which the total amount of inventory held in the system is lower; in particular, the potential excess inventory at the downstream stage declines relative to the upstream stage.

In Study 4 (Table 6), we vary the backlogging cost against the time horizon for the problem. In this study, to minimize the impact of the nonstationary

**Table 6** Heuristic Performance Error—Study 4

Time horizon	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
15	0.36	0.49	0.57	0.65	0.77	0.85
20	0.39	0.53	0.61	0.70	0.84	0.93
25	0.40	0.55	0.63	0.73	0.88	0.97
30	0.41	0.56	0.65	0.75	0.90	1.00
35	0.42	0.57	0.66	0.77	0.92	1.02
40	0.42	0.58	0.67	0.77	0.93	1.03

**Table 7** Heuristic Performance Error—Study 5

Markov multiplier	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
1.5	0.03	0.04	0.05	0.05	0.05	0.06
2.0	0.17	0.21	0.25	0.28	0.31	0.34
2.5	0.31	0.39	0.46	0.53	0.60	0.67
3.0	0.43	0.54	0.65	0.75	0.85	0.96
3.5	0.54	0.67	0.79	0.92	1.07	1.20
4.0	0.61	0.78	0.92	1.10	1.30	1.48

demand on the results, we hold the mean demand throughout the time horizon equal to 5.7, the mean at the peak demand period for the basic model. Table 6 displays the findings and shows how the performance of the DS policy is relatively stable across the time horizon for every backlogging cost considered.

In Studies 5 and 6 (Tables 7 and 8, respectively), we explore the impact of the demand characteristics. First, in Study 5, we vary the backlogging cost against the Markov demand multiplier. The Markov multiplier for the high-demand state ranges from 1.5 to 4 (and the corresponding low-demand Markov multiplier from 0.667 to 0.25). Because the high unit backlogging cost and high Markov multiplier have the identical effect of creating excess inventory at the downstream stage, the performance of the DS heuristic thus declines toward the bottom-right corner.

In Study 6, we vary the Markov chain probability of remaining in the extreme-demand state (i.e., the low-demand or the high-demand state). We fix transition probabilities for the normal-demand state and vary this *in-state probability* from 0.1 to 0.85 in increments of 0.15. We keep the transition probability from an extreme-demand to the normal-demand state as twice that of switching to the other extreme state. With the increasing probability of changing demand regimes, an increasing amount of inventory is held at the downstream stage. Thus, this is where most of the excess inventory is then located when the demand does switch to the lower regime and the heuristic performance of the DS policy starts to decline.

**Table 8** Heuristic Performance Error—Study 6

In-state prob.	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
0.10	0.83	1.04	1.18	1.29	1.38	1.47
0.25	0.84	1.07	1.21	1.31	1.39	1.48
0.40	0.83	1.07	1.24	1.34	1.40	1.48
0.55	0.66	0.87	1.06	1.21	1.31	1.41
0.70	0.43	0.54	0.65	0.75	0.85	0.96
0.85	0.21	0.24	0.27	0.30	0.33	0.36

## 6. Secondary Markets

Secondary markets create value by providing an option for supply chains to sell off excess stock rather than keep it in the system and incur holding costs. We now explore exactly how much value is created by the secondary markets and what factors influence that value. Those questions are worth answering for (i) firms looking to participate as intermediaries in the secondary markets (e.g., Best Buy’s decision in 2009 to enter the electronics secondary market); (ii) supply chains considering setting up secondary markets themselves; and (iii) all secondary market transactions, where information about what drives the value of the option to sell off excess inventory can help with making better pricing and other decisions.

We shed light on those questions with a series of numerical studies. The results are reported in the tables below as (percentage) differences in the expected costs (over the time horizon) between the system with secondary market sales and the one without. We refer to this difference as *the value of the secondary markets* and explore its sensitivity to various system parameters.

### 6.1. Two-Echelon Systems

We first analyze the same two-echelon systems used in the six studies from the previous section on the heuristic performance. We continue to assume zero salvage value at the end of the time horizon.

In Study 1 (Table 9), the value of secondary markets increases with disposal revenues. The largest disposal revenue of  $(r_0, r_1) = (14, 6)$  yields the range of secondary market values from 3.78% to 4.38%, depending on the backlogging cost. The value of secondary markets is not always monotonic in the backlogging cost; instead, at large unit disposal revenues, it is first increasing, then decreasing. As the backlogging cost increases, both the optimal order and disposal quantities seem to increase. Then, as the backlogging cost continues to increase, disposal of excess inventory becomes less attractive because of the high opportunity cost of running out of stock, and the value of the option to sell off excess stock starts to drop. This dynamic is less pronounced at lower disposal revenues, for which the disposed amounts are smaller.

**Table 9** Value of Secondary Markets—Study 1

Backlog. cost $b$	Unit disposal revenues $(r_0, r_1)$ (%)					
	(4, 1)	(6, 2)	(8, 3)	(10, 4)	(12, 5)	(14, 6)
8	0.44	0.65	0.91	1.35	2.19	3.87
10	0.50	0.76	1.07	1.54	2.35	3.98
12	0.53	0.81	1.14	1.62	2.39	3.89
14	0.54	0.83	1.17	1.63	2.40	3.81
16	0.56	0.84	1.19	1.65	2.41	3.76
18	0.58	0.85	1.23	1.69	2.45	3.74

**Table 10** Value of Secondary Markets—Study 2

Hold. costs $(h_0, h_1)$	Unit disposal revenues $(r_0, r_1)$ (%)					
	(4, 1)	(6, 2)	(8, 3)	(10, 4)	(12, 5)	(14, 6)
(0.5, 0.25)	0.40	0.75	1.13	1.58	2.16	3.52
(1.0, 0.50)	0.44	0.75	1.09	1.50	2.13	3.50
(1.5, 0.75)	0.47	0.75	1.07	1.49	2.21	3.66
(2.0, 1.00)	0.50	0.76	1.07	1.54	2.35	3.98
(2.5, 1.25)	0.52	0.77	1.10	1.63	2.58	4.26
(3.0, 1.50)	0.54	0.78	1.15	1.75	2.82	4.45

In Study 2 (Table 10), at low disposal revenues, the value of the secondary markets is increasing in the holding costs: because excess stock can be either held or sold off, as those costs increase, the latter becomes more attractive relative to the former. As disposal revenues grow, increasing the holding costs has a diminishing impact on the optimal policy and a growing impact on the cost function—the costs increases while the secondary sales revenue does not. The value of the option to dispose starts to drop. Eventually, the holding costs become high enough that the optimal policy does change and the value of the secondary markets goes up.

In Study 3 (Table 11) we vary the unit backlogging cost against the unit order costs in the system, assuming the maximum allowed unit revenues of  $(r_0, r_1) = (\kappa_0 + \kappa_1, \kappa_1)$ . Table 11 shows that the value of the secondary markets decreases with the unit order costs. This decline is more pronounced for lower values of the backlogging cost, where a small change in order costs can represent a large change in the amount of the product ordered (and held) in the system and consequently in the quantity of excess inventory sold off.

Study 4 (Table 12), shows how the value of secondary markets is declining with the length of the time horizon across all the backlogging costs analyzed; the supply chains with shorter time horizons thus tend to benefit more from secondary markets. The value of the secondary markets seems to be convex in the time horizon, and the decreasing differences point to a noninsignificant asymptotic limit (2–2.5%) of the option value as the time horizon increases.

**Table 11** Value of Secondary Markets—Study 3

Order costs $(\kappa_0, \kappa_1)$	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
(4, 2)	4.77	4.38	4.15	4.14	4.17	4.20
(6, 4)	4.38	4.18	3.92	3.91	3.94	3.98
(8, 6)	3.87	3.98	3.89	3.81	3.76	3.74
(10, 8)	3.50	3.65	3.72	3.69	3.68	3.67
(12, 10)	3.12	3.43	3.49	3.54	3.57	3.63
(14, 12)	2.80	3.15	3.36	3.37	3.42	3.51

**Table 12** Value of Secondary Markets—Study 4

Time horizon	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
15	4.44	4.74	4.68	4.53	4.56	4.64
20	3.71	3.83	3.68	3.51	3.53	3.59
25	3.29	3.31	3.10	2.92	2.93	2.98
30	3.04	2.98	2.74	2.54	2.56	2.59
35	2.86	2.77	2.50	2.29	2.30	2.33
40	2.75	2.61	2.33	2.12	2.12	2.15

**Table 13** Value of Secondary Markets—Study 5

Markov multiplier	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
1.5	0.43	0.58	0.66	0.68	0.70	0.71
2.0	1.36	1.55	1.65	1.67	1.69	1.73
2.5	2.62	2.78	2.79	2.82	2.77	2.76
3.0	3.87	3.98	3.89	3.81	3.76	3.74
3.5	4.95	5.03	4.83	4.66	4.62	4.59
4.0	5.75	5.78	5.52	5.42	5.34	5.31

In Study 5 (Table 13), the secondary markets are (considerably) more valuable for larger Markov multipliers. The greater the demand shocks, the more likely the supply chain is to accumulate excess inventory and thus the more valuable the option to sell it off. Thus, although at the demand multiplier of 1.5 secondary markets matter little to the total costs, at the demand multiplier of 4 they can represent a significant source of cost savings for the supply chain.

In Study 6 (Table 14), the in-state probability impacts positively the value of secondary markets. A higher in-state probability results in bigger inventory levels because the expected mean demand (at the high-demand state) is higher and the chance of switching to the low-demand state is lower. When the system does switch to the low-demand state, there is an excess inventory to be sold off and the option to do so becomes more valuable.

## 6.2. Higher-Echelon Systems

An important research question concerns the value of secondary markets for higher-echelon systems.

**Table 14** Value of Secondary Markets—Study 6

In-state prob.	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
0.10	1.59	2.10	2.49	2.72	2.87	3.00
0.25	1.77	2.30	2.75	3.01	3.16	3.29
0.40	2.23	2.73	3.12	3.34	3.47	3.55
0.55	2.92	3.16	3.45	3.59	3.67	3.74
0.70	3.87	3.98	3.89	3.81	3.76	3.74
0.85	4.08	4.31	4.31	4.23	4.13	4.03

**Table 15** Unit Costs and Revenues for Higher-Echelon Systems Studied

Supply chain length	Unit order costs	Unit holding costs	Unit disposal revenues
3 echelons	(5, 5, 4)	(1.2, 1, 0.8)	(14, 9, 4)
4 echelons	(4, 4, 3, 3)	(1.0, 0.8, 0.7, 0.5)	(14, 10, 6, 3)
5 echelons	(4, 3, 3, 2, 2)	(1.0, 0.7, 0.6, 0.4, 3)	(14, 10, 7, 4, 2)

Finding the optimal policy for such systems runs quickly into the curse of dimensionality: even for a three-echelon system, finding the total cost under the optimal (parsimonious) policy is estimated to take about five months, rendering any such calculation completely impractical, if not impossible. Therefore, instead of the optimal policy, we make use of the disposal saturation heuristic to calculate the value of secondary markets for higher-echelon supply chains.

Next, we numerically analyze a series of three-, four-, and five-echelon systems. We evaluate the combinations of model parameters from Studies 5 and 6 in the previous section. The demand characteristics remain unchanged from those studies. To provide a meaningful comparison across systems with a different number of stages, we take the sum of the unit order costs and the sum of the unit holding costs across all the stages in the system to be the same as those used in the original studies 5 and 6 above. We again assume the largest unit disposal revenues allowed by Assumption 1. The relevant model parameters are presented in Table 15, from left to right, from the most downstream stage to the most upstream stage.

For the 20-period time horizon with three Markov states, finding the total expected cost with the disposal saturation policy took 17 seconds for a three-echelon system, 30 seconds for a four-echelon system, and 45 seconds for a five-echelon system. Results are in the Tables 16–21.

In Tables 16–21, the value of the secondary markets is increasing uniformly in the number of stages in the supply chain. The first differences of this value are decreasing in the number of stages, so most of the value of the secondary markets for supply chains may be captured with moderate-length supply chains.

**Table 16** Value of Secondary Markets—Study 5 (Three Echelons)

Markov multiplier	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
1.5	0.63	0.83	0.94	0.98	1.04	1.10
2.0	1.56	1.78	1.93	2.01	2.07	2.14
2.5	2.53	2.70	2.85	2.91	2.95	2.97
3.0	3.32	3.46	3.56	3.60	3.59	3.59
3.5	3.94	4.04	4.11	4.11	4.08	4.04
4.0	4.42	4.49	4.53	4.51	4.44	4.38

**Table 17 Value of Secondary Markets—Study 6 (Three Echelons)**

In-state prob.	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
0.10	0.73	0.99	1.25	1.41	1.52	1.61
0.25	0.91	1.19	1.49	1.67	1.81	1.91
0.40	1.29	1.65	1.96	2.19	2.35	2.47
0.55	2.20	2.52	2.79	2.95	3.07	3.13
0.70	3.32	3.46	3.56	3.60	3.59	3.59
0.85	4.56	4.42	4.19	3.97	3.87	3.80

**Table 18 Value of Secondary Markets—Study 5 (Four Echelons)**

Markov multiplier	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
1.5	0.91	1.13	1.30	1.36	1.45	1.54
2.0	1.97	2.24	2.44	2.55	2.65	2.72
2.5	2.99	3.22	3.41	3.50	3.57	3.61
3.0	3.78	3.96	4.13	4.18	4.22	4.22
3.5	4.38	4.52	4.66	4.67	4.68	4.66
4.0	4.84	4.95	5.06	5.05	5.03	4.98

**Table 19 Value of Secondary Markets—Study 6 (Four Echelons)**

In-state prob.	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
0.10	0.90	1.16	1.43	1.59	1.69	1.78
0.25	1.15	1.45	1.76	1.94	2.07	2.16
0.40	1.57	1.94	2.26	2.48	2.63	2.74
0.55	2.45	2.83	3.16	3.35	3.48	3.56
0.70	3.78	3.97	4.13	4.18	4.22	4.22
0.85	4.70	4.66	4.65	4.56	4.50	4.42

**Table 20 Value of Secondary Markets—Study 5 (Five Echelons)**

Markov multiplier	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
1.5	1.17	1.43	1.61	1.68	1.78	1.88
2.0	2.32	2.61	2.82	2.93	3.03	3.10
2.5	3.32	3.56	3.77	3.85	3.92	3.94
3.0	4.07	4.29	4.44	4.48	4.51	4.50
3.5	4.62	4.80	4.92	4.93	4.93	4.89
4.0	5.03	5.19	5.27	5.26	5.24	5.17

**Table 21 Value of Secondary Markets—Study 6 (Five Echelons)**

In-state prob.	Unit backlogging cost $b$ (%)					
	8	10	12	14	16	18
0.10	1.11	1.42	1.70	1.84	1.92	1.98
0.25	1.39	1.74	2.03	2.19	2.30	2.38
0.40	1.88	2.23	2.57	2.75	2.87	2.96
0.55	2.73	3.14	3.42	3.59	3.71	3.78
0.70	4.07	4.29	4.44	4.49	4.51	4.50
0.85	4.96	4.97	4.95	4.86	4.79	4.70

Further, the Markov multiplier of demand and the in-state probability remain important drivers of this value for higher-echelon systems.

## 7. Discussion

### 7.1. Lead-Time Considerations

In our model we assume a single-period lead time between successive stages in the system. Although this assumption covers a number of settings encountered in practice, there also exist supply chains in which it may not apply. Therefore, we now show how this restriction can be relaxed without affecting the structure of our results.

Assuming a multiperiod lead time, say  $\lambda$ , between some two adjacent stages, say  $j$  and  $j + 1$ , is equivalent to adding stages  $j + 1, j + 2, \dots, j + \lambda - 1$ , at which inventory can be neither deferred nor physically disposed of (the original stage  $j + 1$  is now relabeled as  $j + \lambda$ ). The key to handling such stages is to make use of a *forward sale* into the closest available secondary market (looking downstream). Forward sales are generally common in secondary markets and are therefore readily available to inventory managers in most supply chains.

Suppose, for example, that a product is transported on a ship from Asia to the United States, that the trip takes six weeks, and that the lead time between stages is one week. If, after two weeks in transit, (a portion of the) inventory on that ship is no longer needed, it can be sold (four weeks) forward into the secondary market at the port of entry. The analysis of this system then proceeds exactly *as if* those units were *physically* removed from the ship, even though the physical removal happens only four weeks later. In other words, the inventory sold forward in a secondary market is simply taken out of the record of stock on the ship. Thus, a forward sale allows us to treat stages physically constrained to hold inventory in the same manner as those that are not. The limitation that stages  $j + 1, j + 2, \dots, j + \lambda - 1$  also do not allow inventory to be held at those locations is handled directly through constraints in the feasible set. In particular, if inventory cannot be deferred at stage  $j$ , it suffices to include a constraint  $Y_{jt} = \hat{Y}_{j+1,t}$ , referred to as a *no-deferral constraint*, in the definition of the feasible region.

It is straightforward to verify that with forward sales and no-deferral constraints, having multiperiod lead times between stages does not impact the form of the optimal policy. Further, although the definition of the parsimonious policy has to be adjusted somewhat (so that the decisions under the parsimonious policy for stages  $j + 1, \dots, j + \lambda - 1$  are exactly the same as those for the original optimal policy), its structure remains the same: at each stage there is still only

one decision to make instead of two. The structure of the disposal saturation policies, and the Clark–Scarf decomposition they achieve, goes through unaltered with forward sales agreements and no-deferral constraints. Thus, with those two modifications to the problem formulation, our results continue to hold when multiple periods are allowed between stages.

## 7.2. Secondary Markets

We have shown that there exists a number of reasonable situations where secondary markets provide significant value to the supply chain. This finding is supported by observations from industry, cited in the introduction, where secondary markets can represent multi-billion dollar industries. At the same time, we saw that there also exist settings in which secondary markets provide little value, such as those when unit disposal revenues are small or when there is little volatility in demand. Aside from the unit disposal revenues, the two factors with the biggest impact on the value of secondary markets are the intensity of the Markov multiplier and the in-state probability. We can, therefore, expect that supply chains with most to gain from secondary markets are those that exhibit demand shocks—shifts from normal demand levels to other, very different demand regimes or other forms of demand volatility. Our numerical studies also show that secondary markets tend to be especially valuable to supply chains for products such as those in the high-tech industry, where due to “... very short product life cycles, excess inventory of components and parts could result in huge obsolescence costs” (Lee and Whang 2001, p. 13). Even for long-life cycle products, the secondary markets were found to generate 2%–3% cost savings across the supply chain. Under those circumstances, secondary markets can provide considerable help with the mismatch between supply and demand.

Interestingly, certain cost parameters seem to have a very complex impact on secondary markets. Although the unit order costs affect this value directly in a negative way, increasing the unit holding or backloging costs can result in a very interdependent behavior, so the resulting change in the value of secondary markets also depends on the level of other model parameters, such as the unit disposal revenues. When assessing the sensitivity of secondary markets, a comprehensive view of all the cost and revenues parameters is required.

Although disposal saturation policies generate a relatively significant value of the secondary markets for systems with more than two stages, it is presently not known how close those policies come to the optimal ones for such higher-order systems. Further research on developing effective heuristics (and related bounds) for larger scale systems with secondary market sales would be worthwhile, in the

spirit of Gallego and Özer (2003), for example, who show that myopic policies are optimal for stationary multiechelon inventory problems for both finite and infinite horizon problems (without secondary market sales).

We have assumed zero salvage value throughout the paper. In practice, secondary markets can also be used to sell excess inventory at the end of the time horizon, so the values assessed for those markets in this paper tend to underestimate their actual value. For higher-echelon systems, we have used the DS policy rather than the exact optimal policy to assess the value of secondary markets. Such an approximate calculation arguably represents the *actual* value of secondary markets, given that, in practice, the optimal policy would not be possible to determine and the best a firm could do is to make use of the disposal saturation policy. Our research thus paves the way for companies to make use of the type of results presented in this paper to (1) simultaneously manage their inventory replenishments and secondary market sales, (2) estimate the value of existing or potential secondary markets for their own supply chains, and (3) identify drivers of secondary market value and manage those for greater profit.

## 8. Glossary of Notation

### Model Parameters

- $b_t$ : unit backloging cost in period  $t$ ;
- $\kappa_{jt}$ : unit order cost at stage  $j$  in period  $t$ ;
- $h_{jt}$ : unit holding cost at stage  $j$  in period  $t$ ;
- $r_{jt}$ : unit disposal revenue at stage  $j$  in period  $t$ ;
- $\alpha$ : one-period discount factor;
- $L$ : number of stages in the supply chain;
- $T$ : number of periods in the time horizon.

### State Variables

- $x_{jt}$ : on-hand inventory at stage  $j$  at the beginning of period  $t$ ;
- $y_{jt}$ : echelon inventory at stage  $j$  at the beginning of period  $t$ ;
- $\omega_t$ : Markov chain exogenous state in period  $t$ .

### Decision Variables

- $X_{jt}$ : order decision at stage  $j$  at the beginning of period  $t$ ;
- $\hat{X}_{jt}$ : disposal decision at stage  $j$  at the beginning of period  $t$ ;
- $Y_{jt}$ : replenishment echelon level at stage  $j$  in period  $t$ ;
- $\hat{Y}_{jt}$ : postdisposal echelon level at stage  $j$  in period  $t$ .



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## Appendix

**PROOF OF THEOREM 1.** We prove the convexity result of the theorem by induction. By Assumption 2,  $f_{T+1}(\omega, \cdot)$  is convex for each  $\omega$ . Assume inductively that  $f_{i+1}(\omega, \cdot)$  is convex for each  $\omega$ . The optimality equations given in (6) can be rewritten as follows:

$$f_t(\omega, y_t) = \sum_{j=0}^{L-1} r_{jt}(y_{j-1,t} - y_{jt}) + \min_{\hat{Y}_{jt} \leq \hat{Y}_{j+1,t} \leq y_{j+1,t} - y_{jt} + \hat{Y}_{jt}} \left\{ \sum_{j=0}^{L-1} c_{jt} \hat{Y}_{jt} + G_{t+1}(\omega, \hat{Y}_t) \right\}, \quad (12)$$

where

$$G_{t+1}(\omega, \hat{Y}_t) := \min_{\hat{Y}_{jt} \leq \hat{Y}_{j+1,t} \leq y_{j+1,t} - y_{jt} + \hat{Y}_{jt}} \left\{ \gamma_t(\omega, Y_{0t}) + \sum_{j=0}^{L-1} b_{jt} Y_{jt} + \alpha E[f_{t+1}(\omega_{t+1}, Y_t - D_t)] \right\}. \quad (13)$$

By assumption, the right-hand side (RHS) of (13) is convex in  $Y_t$ . Furthermore, the RHS is being minimized over a convex set  $\mathbb{A}_{\hat{Y}} := [\hat{Y}_{0t}, \hat{Y}_{1t}] \times [\hat{Y}_{1t}, \hat{Y}_{2t}] \times \dots \times [\hat{Y}_{L-1,t}, \infty)$ . Then, by convexity preservation under minimization (CPUM),  $G_{t+1}(\omega, \cdot)$  is convex (see for example, Heyman and Sobel 1984, p. 525). Let  $\hat{Y}_t^k := \{\hat{Y}_{0t}, \hat{Y}_{1t}, \dots, \hat{Y}_{kt}\}$ , where  $\hat{Y}_t^{L-1} = \hat{Y}_t$  and  $\hat{Y}_t^0 = \hat{Y}_{0t}$ . For each  $\omega$  and  $y_t$ , define  $H_{t+1}^k(\omega, y_t, \hat{Y}_t^k)$ , for  $k = 0, 1, \dots, L-2$ , recursively as

$$H_{t+1}^k(\omega, y_t, \hat{Y}_t^k) := \min_{\hat{Y}_{kt} \leq \hat{Y}_{k+1,t} \leq y_{k+1,t} - y_{kt} + \hat{Y}_{kt}} \left\{ c_{k+1,t} \hat{Y}_{k+1,t} + H_{t+1}^{k+1}(\omega, y_t, \hat{Y}_t^{k+1}) \right\}, \quad (14)$$

where  $H_{t+1}^{L-1}(\omega, y_t, \hat{Y}_t^{L-1}) := G_{t+1}(\omega, \hat{Y}_t)$ . Because  $G_{t+1}(\omega, \cdot)$  is convex, then  $H_{t+1}^{L-1}(\omega, \cdot)$  is jointly convex. Assume inductively that  $H_{t+1}^k(\omega, y_t, \hat{Y}_t^k)$  is jointly convex. (Note that we are conducting a proof by induction within another proof by induction, which is permissible because we are relying only on the convexity of  $G_{t+1}(\omega, \cdot)$ .) Then, the RHS of (14) is jointly convex. Further, the RHS is minimized over the convex set  $\mathbb{A}_{\hat{Y}}^k := [\hat{Y}_{kt}, y_{k+1,t} - y_{kt} + \hat{Y}_{kt}]$ . We again apply convexity preservation under minimization to get that  $H_{t+1}^k(\omega, y_t, \hat{Y}_t^k)$  is jointly convex. Therefore,  $H_{t+1}^0(\omega, y_t, \hat{Y}_t^0)$  is convex, because  $\hat{Y}_t^0 = \hat{Y}_{0t}$ . We can now rewrite (12) as

$$f_t(\omega, y_t) = \sum_{j=0}^{L-1} r_{jt}(y_{j-1,t} - y_{jt}) + \min_{\hat{Y}_{0t} \leq y_{0t}} \{c_{0t} \hat{Y}_{0t} + H_{t+1}^0(\omega, y_t, \hat{Y}_t^0)\}. \quad (15)$$

The RHS of (15) is convex. By CPUM,  $f_t(\omega, y_t)$  is convex in  $y_t$ , which completes the proof.

Next, let  $\hat{S}_{0t}(\omega, y_t)$  be the unconstrained minimizer of the RHS of (15). Then,  $\hat{Y}_{0t}(\omega, y_t) = \hat{S}_{0t}(\omega, y_t) \wedge y_{0t}$ . Assume inductively that  $\hat{Y}_{0t}(\omega, y_t), \dots, \hat{Y}_{jt}(\omega, y_t)$  are given by (7). Then,

$$H_{t+1}^j(\omega, y_t, \hat{Y}_t^j) = \min_{\hat{Y}_{jt}(\omega, y_t) \leq \hat{Y}_{j+1,t} \leq y_{j+1,t} - y_{jt} + \hat{Y}_{jt}(\omega, y_t)} \{c_{j+1,t} \hat{Y}_{j+1,t} + H_{t+1}^{j+1}(\omega, y_t, \hat{Y}_t^{j+1})\}, \quad (16)$$

where decisions labeled with  $(\omega, y_t)$  have already been determined. As already established above, the RHS of (16) is convex. Let  $\hat{S}_{j+1,t}(\omega, y_t)$  be its unconstrained minimizer. Then,  $\hat{Y}_{j+1,t}(\omega, y_t) = \hat{Y}_{jt}(\omega, y_t) \vee [\hat{S}_{j+1,t}(\omega, y_t) \wedge (y_{j+1,t} - y_{jt} + \hat{Y}_{jt}(\omega, y_t))]$ . Finally, because the RHS of (13) is convex in  $Y_t$ , let  $S_{jt}(\omega, y_t)$  be its unconstrained minimizer. The optimal postdisposal level becomes  $Y_{jt}(\omega, y_t) = \hat{Y}_{jt}(\omega, y_t) \vee [S_{jt}(\omega, y_t) \wedge \hat{Y}_{j+1,t}(\omega, y_t)]$ .

**PROOF OF LEMMA 2.** Part (c) is true for  $X_{0t}$ . Assume  $X_{i-1,t}^j \leq X_{i-1,t}$  for some  $i-1 \leq j$ . Then,  $X_{it}^j = [X_{it} - \hat{X}_{it} - (X_{i-1,t} - X_{i-1,t}^j)]^+ \leq [X_{it} - \hat{X}_{it}]^+$ . Because  $\hat{X}_{it} \geq 0$ ,  $X_{it}^j \leq X_{it}$  (for all  $i \leq j$ ).

To prove (d), first consider  $i \leq j$ . Because  $\hat{X}_t, X_t \in \mathcal{X}(x_t^*)$ , then  $\hat{X}_{it}^j = (\hat{X}_{it} - X_{0t})^+ \leq \hat{X}_{it} \leq x_{0t}$ . Also,  $X_{i-1,t} \leq x_{it} - \hat{X}_{it}$ , for  $i \geq 1$ . Because  $\hat{X}_{it}^j = [\hat{X}_{it} - X_{it} + (X_{i-1,t} - X_{i-1,t}^j)]^+$  for  $i \geq 1$ , then  $\hat{X}_{it}^j \leq (x_{it} - X_{it} - X_{i-1,t}^j)^+ \leq x_{it}$  for all  $i \leq j$ . Next, if  $\hat{X}_{it}^j = 0$ , then, by part (b) and the feasibility of  $(\hat{X}_t, X_t)$ ,  $X_{i-1,t}^j \leq X_{i-1,t} \leq x_{it} - \hat{X}_{it} \leq x_{it}$ . If  $\hat{X}_{it}^j > 0$ , then  $\hat{X}_{it}^j + X_{i-1,t}^j = \hat{X}_{it} + X_{i-1,t} - X_{it}$ . By the feasibility of  $\hat{X}_t$  and  $X_t$ ,  $\hat{X}_{it}^j + X_{i-1,t}^j \leq x_{it} - X_{it} \leq x_{it}$ . If  $i = j$ , then by Definition 2,  $\hat{X}_{j+1,t}^j + X_{jt}^j = \hat{X}_{j+1,t} + X_{jt}$ . For  $i > j$ , the feasibility of  $(\hat{X}_t^j, X_t^j)$  carries over directly from the feasibility of  $(\hat{X}_t, X_t)$ .

To prove (e), first consider  $0 < i \leq j$ . Use (1) to get  $x_{i,t+1}^j = x_{it}^j + X_{it}^j - \hat{X}_{it}^j - X_{i-1,t}^j$ . Assume  $X_{it}^j = 0$ . By Definition 2,  $x_{i,t+1}^j = x_{it} - \hat{X}_{it} + X_{it} - X_{i-1,t} = x_{i,t+1}$ . If  $\hat{X}_{it}^j = 0$ , then by definition of  $X_{it}^j$ , we get the same. If  $i = 0$ , then either  $X_{0t}^j = 0$  or  $\hat{X}_{0t}^j = 0$  leads directly to  $x_{0,t+1}^j = x_{0,t+1}$ . Next, if  $i = j+1$ , then

$$\begin{aligned} & x_{j+1,t+1}^j \\ &= x_{j+1,t} + X_{j+1,t}^j - \hat{X}_{j+1,t}^j - X_{jt}^j \quad [\text{by (1)}] \\ &= x_{j+1,t} + X_{j+1,t}^j - (\hat{X}_{j+1,t} + X_{jt} - X_{jt}^j) - X_{jt}^j \quad [\text{by Definition 2}] \\ &= x_{j+1,t} + X_{j+1,t} - \hat{X}_{j+1,t} - X_{jt} \quad [\text{by Definition 2}] \\ &= x_{j+1,t+1} \quad [\text{by (1)}]. \end{aligned}$$

For  $i > j+1$ , the result follows directly from Definition 2.

**PROOF OF LEMMA 3.** Fix  $\omega$ . By (5) and Definition 2,  $F_t(\omega, x_t | \hat{X}_t, X_t) - F_t(\omega, x_t | \hat{X}_t^0, X_t^0)$  becomes

$$\begin{aligned} & (\kappa_{0t} + h_{0t} - h_{1t})[X_{0t} - X_{0t}^0] - (r_{0t} + h_{0t})[\hat{X}_{0t} - \hat{X}_{0t}^0] \\ & - (r_{1t} + h_{1t})[\hat{X}_{1t} - \hat{X}_{1t}^0] \\ & = (\kappa_{0t} + h_{0t} - h_{1t})[X_{0t} - X_{0t}^0] - (r_{0t} + h_{0t})[\hat{X}_{0t} - \hat{X}_{0t}^0] \\ & - (r_{1t} + h_{1t})[\hat{X}_{1t} - (\hat{X}_{1t} + X_{0t} - X_{0t}^0)] \\ & = (\kappa_{0t} + h_{0t} + r_{1t})[X_{0t} - X_{0t}^0] - (r_{0t} + h_{0t})[\hat{X}_{0t} - \hat{X}_{0t}^0]. \quad (17) \end{aligned}$$

By Lemma 2(b), either  $X_{0t}^0 = 0$  or  $\hat{X}_{0t}^0 = 0$ ; thus, by Definition 2, (17) reduces to either  $(\kappa_{0t} + r_{1t} - r_{0t})X_{0t}$  or  $(\kappa_{0t} + r_{1t} - r_{0t})\hat{X}_{0t}$ , both of which are positive by Assumption 1.

To prove (b), we proceed similarly:  $F_t(\omega, x_t | \hat{X}_t^{j-1}, X_t^{j-1}) - F_t(\omega, x_t | \hat{X}_t^j, X_t^j)$  becomes

$$\begin{aligned} & (\kappa_{jt} + h_{jt} - h_{j+1,t})[X_{jt} - X_{jt}^j] - (r_{jt} + h_{jt})[\hat{X}_{jt}^{j-1} - \hat{X}_{jt}^j] \\ & - (r_{j+1,t} + h_{j+1,t})[\hat{X}_{j+1,t} - \hat{X}_{j+1,t}^j] \\ & = (\kappa_{jt} + h_{jt} - h_{j+1,t})[X_{jt} - X_{jt}^j] \\ & - (r_{jt} + h_{jt})[\hat{X}_{jt} + X_{j-1,t} - X_{j-1,t}^{j-1} - \hat{X}_{jt}^j] \\ & - (r_{j+1,t} + h_{j+1,t})[\hat{X}_{j+1,t} - (\hat{X}_{j+1,t} + X_{jt} - X_{jt}^j)] \\ & = (\kappa_{jt} + h_{jt} + r_{j+1,t})[X_{jt} - X_{jt}^j] \\ & - (r_{jt} + h_{jt})[\hat{X}_{jt} + X_{j-1,t} - X_{j-1,t}^{j-1} - \hat{X}_{jt}^j], \quad (18) \end{aligned}$$

because  $X_{j-1,t}^{j-1} = X_{j-1,t}^j$ . By Lemma 2(b), first assume  $\hat{X}_{jt}^j = 0$ . Then, by Definition 2,  $\hat{X}_{jt} + X_{j-1,t} - X_{j-1,t}^j = X_{jt} - X_{jt}^j$ , and (18) reduces to  $(\kappa_{jt} + r_{j+1,t} - r_{jt})[X_{jt} - X_{jt}^j]$ , which is positive by Assumption 1 and Lemma 2(c). If  $X_{jt}^j = 0$ , then, by Definition 2, (18) becomes  $(\kappa_{jt} + r_{j+1,t} - r_{jt})X_{jt}$ , which is positive. Part (c) is established similarly.

**PROOF OF LEMMA 5.** Suppose first that  $s_0 < y_0$ . Then, given the  $y_{i+1} \geq y_i$ ,  $y_n \geq \dots \geq y_0 > s_0 \geq s_1 \geq \dots \geq s_n$ , so that none of the (ii), (iii), and (iv) can be true. Now suppose that  $s_0 \geq y_0$  (i.e., part (i) is not true). Then, either  $s_n \geq y_n$  or  $s_n < y_n$  holds. Suppose that statement (ii),  $s_n \geq y_n$ , is true. Then,  $s_0 \geq s_1 \geq \dots \geq s_n \geq y_n \geq \dots \geq y_0$ , and neither (iii) nor (iv) can be true.

Now suppose that  $s_n < y_n$  and  $s_0 \geq y_0$  (i.e., (i) and (ii) are not true). Assume there exists a  $j \in \{1, \dots, n\}$  such that  $y_{j-1} \leq s_j < y_j$ . Then,  $s_0 \geq \dots \geq s_{j-1} \geq s_j \geq y_{j-1} \geq y_{j-2}$ ; thus, there is no  $k \in \{0, \dots, j-1\}$  such that  $y_k > s_{k+1}$ . Similarly,  $y_n \geq \dots \geq y_j > s_j \geq s_{j+1} \geq \dots \geq s_n$ , and there is no  $k \in \{j, \dots, n\}$  such that  $s_k \geq x_k$ . Thus, (iv) cannot be true. Let there exist a  $j^* \neq j$  such that  $y_{j^*} > s_{j^*} \geq y_{j^*-1}$ . If  $j^* < j$ , then  $s_{j^*} \geq s_j \geq y_{j-1} \geq y_{j^*}$ , a contradiction. If  $j^* > j$ , then  $s_{j^*} \leq s_j < y_j \leq y_{j^*-1}$ , also a contradiction. Thus,  $j$  is unique.

Now suppose that (i), (ii), and (iii) are not true. To prove uniqueness, let there exist a  $k \in \{0, \dots, n-1\}$ , such that  $s_k \geq y_k > s_{k+1}$ . Assume an  $i \neq k$ , such that  $s_i \geq y_i > s_{i+1}$ . If  $i > k$ , then,  $s_i \leq s_{k+1} < y_k \leq y_i$ , which is a contradiction. If  $i < k$ , then  $s_{i+1} \geq s_k \geq y_k \geq y_i$ , which is a contradiction. Thus,  $k$  is unique. To prove existence, we distinguish three cases.

*Case (1):* There does not exist any  $i$ ,  $i \in \{0, 1, \dots, n\}$ , such that  $y_n > s_i \geq y_0$ . Because  $s_n < y_n$ , there exists a  $j$ ,  $j \in \{0, 1, \dots, n\}$ , such that  $y_n > s_j \geq s_{j+1} \geq \dots \geq s_n$ . Let  $k_n := \min\{j \mid s_j < y_n\}$ . Thus,  $s_{k_n} = \max\{s_j \mid s_j < y_n\}$ . Because  $s_0 \geq y_0$ , there exists a  $j'$ ,  $j' \in \{0, 1, \dots, n\}$ , such that  $s_0 \geq s_1 \geq \dots \geq s_{j'} \geq y_0$ . Let  $k_0 := \max\{j' \mid s_{j'} \geq y_0\}$ . Thus,  $s_{k_0} = \min\{s_{j'} \mid s_{j'} \geq y_0\}$ . Because there is no  $i$  such that  $y_n > s_i \geq y_0$ , then  $s_{k_n} < y_0$  and  $s_{k_0} \geq y_n$ . By definition of  $s_{k_0}$ ,  $s_{k_0+1} < y_0$  ( $k_0 < n$  because, by the original assumptions,  $s_n < y_n$ ). Thus,  $k_n = k_0 + 1$  and  $s_{k_0} \geq y_n \geq y_{k_0} \geq y_0 > s_{k_0+1}$ . The required condition holds with  $k = k_0$ .

*Case (2):* There exists a single  $i \in \{0, 1, \dots, n\}$ , such that  $y_n > s_i \geq y_0$ . If  $s_i \geq y_i$ , then  $s_i \geq y_i \geq y_0 > s_{i+1}$ , and the required

condition holds with  $k = i$ . If  $s_i < y_i$ , then  $s_{i-1} \geq y_n \geq y_{i-1} > s_i$ ; the required condition holds with  $k = i - 1$ .

*Case (3):* There exist multiple  $i \in \{0, 1, \dots, n\}$ , such that  $y_n > s_i \geq y_0$ . Let  $i_1 = \min\{i \mid y_n > s_i \geq y_0\}$ ;  $i_2 = \max\{i \mid y_n > s_i \geq y_0\}$ . Thus,  $y_n > s_{i_1} \geq \dots \geq s_{i_2} \geq y_0$ . Suppose that  $s_{i_1} < y_{i_1-1}$ . Then, by definition,  $s_{i_1-1} \geq y_n \geq y_{i_1-1} > s_{i_1}$ , and the required condition holds with  $k = i_1 - 1$ . Suppose instead  $s_{i_1} \geq y_{i_1-1}$ . Then,  $s_{i_1} \geq y_{i_1}$  (otherwise,  $y_{i_1} > s_{i_1} \geq y_{i_1-1}$ , which contradicts the assumption that (iii) is not true). Let  $i^* := \max\{i \mid i_1 \geq i \geq i_2; s_i \geq y_i\}$ . Thus,  $s_{i^*} \geq y_{i^*+1}$  and  $s_{i^*+1} < y_{i^*+1}$ . Because  $s_{i^*+1} < y_{i^*+1}$ , then  $s_{i^*+1} < y_{i^*}$ ; otherwise the initial assumption is violated. Thus,  $s_{i^*} \geq y_{i^*} > s_{i^*+1}$ . The required condition holds with  $k = i^*$ .

Finally, assume there exists no  $k$ ,  $k \in \{0, \dots, n-1\}$ , such that  $s_k \geq y_k > s_{k+1}$ . Then we must have (i)  $s_0 \geq \dots \geq s_n \geq y_n \geq \dots \geq y_0$ , (ii)  $y_n \geq \dots \geq y_0 \geq s_0 \geq \dots \geq s_n$ , or (iii) there exists a  $j$  such that  $y_n \geq \dots \geq y_j > s_j \geq y_{j-1} \geq \dots \geq y_0$ . This completes the proof.

**PROOF OF LEMMA 6.** There exist convex  $f_{j,t+1}$  such that  $f_{t+1}(\omega, y_t) = \sum_{j=0}^{L-1} f_{j,t+1}(\omega, y_{jt})$ . Let  $\hat{Y}_t$  be a DS policy with the threshold stage  $k$ . Then,  $\hat{Y}_{jt} = y_{jt}$  for  $j < k$ , and  $\hat{Y}_{jt} = Y_{jt} = \hat{Y}_{kt}$  for  $j \geq k$ . Define

$$\begin{aligned} & \tilde{f}_{j,t+1}(\omega, y) \\ & := \begin{cases} \gamma_t(\omega, y) + b_{jt}y + \alpha E[f_{j,t+1}(\omega_{t+1}, y - D_t)] & \text{if } j = 0, \\ b_{jt}y + \alpha E[f_{j,t+1}(\omega_{t+1}, y - D_t)] & \text{if } j > 0 \end{cases} \end{aligned}$$

for each  $j$ . We first construct the base-stock vector  $\hat{S}_t^*(\omega)$ , then prove the results of the lemma. Using  $\tilde{f}_{j,t+1}(\omega, y)$ , Equation (10) becomes

$$\begin{aligned} f_t(\omega, y_t | \hat{Y}_t) & = \sum_{j=0}^{L-1} (r_{j+1,t} - r_{jt})y_{jt} + \sum_{j=0}^{k-1} c_{jt}y_{jt} + \sum_{j=k}^{L-1} c_{jt}\hat{Y}_{kt} \\ & + \sum_{j=k}^{L-2} \tilde{f}_{j,t+1}(\omega, \hat{Y}_{kt}) + G_{t+1}^k(\omega, \hat{Y}_{kt}, y_t), \quad (19) \end{aligned}$$

where

$$\begin{aligned} G_{t+1}^k(\omega, \hat{Y}_{kt}, y_t) & = \min_{\substack{y_{jt} \leq Y_{jt} \leq y_{j+1,t} \\ j=0, \dots, k-2}} \left[ \sum_{j=0}^{k-2} \tilde{f}_{j,t+1}(\omega, Y_{jt}) \right] \\ & + \min_{y_{k-1,t} \leq Y_{k-1,t} \leq \hat{Y}_{kt}} \tilde{f}_{k-1,t+1}(\omega, Y_{k-1,t}) \\ & + \min_{\hat{Y}_{kt} \leq Y_{L-1,t}} \tilde{f}_{L-1,t+1}(\omega, Y_{L-1,t}). \quad (20) \end{aligned}$$

Because  $\tilde{f}_{j,t+1}(\omega, \cdot)$  is convex for each  $\omega$ , then, by CPUM,  $G_{t+1}^k(\omega, \cdot)$  is jointly convex in  $\hat{Y}_{kt}$  and  $y_t$ . Because  $\tilde{f}_{j,t+1}(\omega, \cdot)$  is also smooth, then by Lemma 4, for each  $\omega$  and each  $j$ ,  $0 \leq j \leq k-1$ , there exist smooth convex functions  $g_{j,t+1}(\omega, \cdot)$  and  $h_{j,t+1}(\omega, \cdot)$ , as well as a smooth convex function  $g_{L-1,t+1}(\omega, \cdot)$  (with  $h_{-1,t+1} = g_{-1,t+1} := 0$ ), such that  $G_{t+1}^k(\omega, \hat{Y}_{kt}, y_t)$  becomes

$$\begin{aligned} & \sum_{j=0}^{k-2} (g_{j,t+1}(\omega, y_{jt}) + h_{j,t+1}(\omega, y_{j+1,t})) + g_{k-1,t+1}(\omega, y_{k-1,t}) \\ & + h_{k-1,t+1}(\omega, \hat{Y}_{kt}) + g_{L-1,t+1}(\omega, \hat{Y}_{kt}). \quad (21) \end{aligned}$$

Because  $\tilde{f}_{j,t+1}(\omega, \cdot)$  is smooth and convex, there exist, in particular for  $j \geq k$ , a smooth convex increasing function  $g_{j,t+1}(\omega, \cdot)$  and a smooth convex decreasing function  $h_{j,t+1}(\omega, \cdot)$ , such that  $f_{j,t+1} = g_{j,t+1} + h_{j,t+1}$ . Thus,  $g_{j,t+1}(\omega, \cdot)$  and  $h_{j,t+1}(\omega, \cdot)$  are now defined for all  $\omega$  and  $j$ . For each  $j$ ,  $0 \leq j \leq L-1$ , define  $H_{j,t+1}(\omega, y) := h_{j-1,t+1}(\omega, y_{jt}) + g_{j,t+1}(\omega, y_{jt}) + c_{jt}(\omega)y$ . Then,  $\tilde{H}_{j,t+1}(\omega, \cdot)$  is convex and smooth for each  $\omega$ . We make use of  $H_{j,t+1}(\omega, \cdot)$ , (21) and (20) to rewrite the original equation in (19) as

$$f_t(\omega, y_t | \hat{Y}_t) = \sum_{j=0}^{L-1} (r_{j+1,t} - r_{jt})y_{jt} + \sum_{j=0}^{k-1} H_{j,t+1}(\omega, y_{jt}) + \sum_{j=k}^{L-1} H_{j,t+1}(\omega, \hat{Y}_{kt}). \quad (22)$$

For each  $\omega$ , and  $k$ ,  $0 \leq k \leq L-1$ , define the following.

$$G_{k,t+1}(\omega, \hat{Y}_{kt}) := \sum_{j=k}^{L-1} H_{j,t+1}(\omega, \hat{Y}_{kt}), \quad \text{and}$$

$$\hat{S}_{kt}(\omega) := \arg \min_{\hat{Y}_{kt}} G_{k,t+1}(\omega, \hat{Y}_{kt}).$$

Thus,  $G_{k,t+1}(\omega, \cdot)$  is smooth and convex for each  $\omega$ , and  $\hat{S}_{kt}(\omega)$  is its unconstrained minimizer.

We construct our desired base-stock vector  $\hat{S}_t^*(\omega)$  recursively as follows:

$$\hat{S}_{jt}^*(\omega) = \begin{cases} \hat{S}_{L-1,t}(\omega) & \text{if } j = L-1, \\ \max[\hat{S}_{jt}(\omega), \hat{S}_{j+1,t}^*(\omega)] & \text{if } j < L-1. \end{cases} \quad (23)$$

We now verify the desired properties of  $\hat{S}_t^*(\omega)$ . Part (a) follows directly from (23).

To prove (b), we make use of Lemma 5. Let  $\omega$  and  $y_t$  be given, and  $\hat{Y}_t^*(\omega, y_t)$  be as in (9).

Assume case (i):  $\hat{S}_{0t}^*(\omega) < y_{0t}$ . Then, by (9),  $\hat{Y}_{0t}^*(\omega, y_t) = \hat{S}_{0t}^*(\omega) < y_{0t}$ . Because  $y_{0t} \leq y_{1t}$ , by the construction of  $\hat{S}_t^*(\omega)$ , we get  $\hat{S}_{1t}^*(\omega) \leq \hat{S}_{0t}^*(\omega) = \hat{Y}_{0t}^*(\omega, y_t)$ . Thus,  $\hat{Y}_{1t}^*(\omega, y_t) = \hat{Y}_{0t}^*(\omega, y_t)$ . Assume inductively that  $\hat{Y}_{jt}^*(\omega, y_t) = \hat{Y}_{0t}^*(\omega, y_t)$  for some  $j < L-1$ . We get  $\hat{S}_{j+1,t}^*(\omega) \leq \hat{S}_{0t}^*(\omega) = \hat{Y}_{0t}^*(\omega, y_t) = \hat{Y}_{jt}^*(\omega, y_t)$ . Thus,  $\hat{Y}_{j+1,t}^*(\omega, y_t) = \hat{Y}_{0t}^*(\omega, y_t)$ . Consequently,  $\hat{Y}_{kt}^*(\omega, y_t) = \hat{Y}_{0t}^*(\omega, y_t)$  for all  $k$ , and  $\hat{Y}_t^*(\omega, y_t)$  is a DS policy (all upstream echelons are disposal saturated).

Assume case (ii):  $\hat{S}_{L-1,t}^*(\omega) \geq y_{L-1,t}$ . Then,  $\hat{S}_{0t}^*(\omega) \geq \hat{S}_{L-1,t}^*(\omega) \geq y_{L-1,t} \geq y_{0t}$ , and we get  $\hat{Y}_{0t}^*(\omega, y_t) = y_{0t}$ . Assume inductively that  $\hat{Y}_{jt}^*(\omega, y_t) = y_{jt}$  for some  $j < L-1$ . Then,  $\hat{Y}_{j+1,t}^*(\omega, y_t) + y_{j+1,t} - y_{jt} = y_{j+1,t}$ , and we get  $\hat{S}_{j+1,t}^*(\omega) \geq \hat{S}_{L-1,t}^*(\omega) \geq y_{L-1,t} \geq y_{j+1,t}$ , which gives  $\hat{Y}_{j+1,t}^*(\omega, y_t) = y_{j+1,t}$ . So, in this case, there are no disposals, and  $\hat{Y}_t^*(\omega, y_t)$  is a DS policy.

Assume case (iii): there exists a unique  $j$ ,  $j \in \{1, \dots, L-1\}$  such that  $y_{jt} > \hat{S}_{jt}^*(\omega) \geq y_{j-1,t}$ . Then,  $\hat{Y}_{jt}^*(\omega, y_t) = \hat{S}_{jt}^*(\omega)$ . Use the induction steps from case (i) to show that  $\hat{Y}_{it}^*(\omega, y_t) = \hat{Y}_{jt}^*(\omega, y_t)$  for all  $i > j$ . Use the induction steps from case (ii) to show that  $\hat{Y}_{it}^*(\omega, y_t) = y_{it}$  for all  $i < j$ . Thus, stages upstream of  $j$  are disposal saturated, and downstream stages have no disposal, so  $\hat{Y}_t^*(\omega, y_t)$  is a DS policy.

Assume case (iv): there exists a unique  $k$ ,  $k \in \{0, \dots, L-2\}$  such that  $\hat{S}_{kt}^*(\omega) \geq y_{kt} > \hat{S}_{k+1,t}^*(\omega)$ . Thus,  $\hat{Y}_{kt}^*(\omega, y_t) = y_{kt}$ . Use the induction steps from case (i) to show that  $\hat{Y}_{it}^*(\omega, y_t) = y_{it}$  for all  $i > k$ . Use the induction steps from case (ii) to show that  $\hat{Y}_{it}^*(\omega, y_t) = y_{it}$  for all  $i < k$ , so that  $\hat{Y}_t^*(\omega, y_t)$  is a DS policy. This completes the proof of part (b).

The proof of part (c) follows directly from expression (20).

To prove (d), first define, for convenience,  $\bar{f}_t(\omega, y_t) := f_t(\omega, y_t | \hat{Y}_t^*)$ . Next, define

$$\bar{f}_{0t}(\omega, y_{0t}) := \begin{cases} (r_{1t} - r_{0t})y_{0t} + H_{0,t+1}(\omega, y_{0t}) + G_{1,t+1}(\omega, \hat{S}_{1t}^*(\omega)) & \text{if } y_{0t} \leq \hat{S}_{1t}^*(\omega), \\ (r_{1t} - r_{0t})y_{0t} + G_{0,t+1}(\omega, y_{0t}) & \text{if } \hat{S}_{0t}^*(\omega) \geq y_{0t} > \hat{S}_{1t}^*(\omega), \\ (r_{1t} - r_{0t})y_{0t} + G_{0,t+1}(\omega, \hat{S}_{0t}^*(\omega)) & \text{if } y_{0t} > \hat{S}_{0t}^*(\omega); \end{cases}$$

and, for  $0 < j \leq L-1$ ,

$$\bar{f}_{jt}(\omega, y_{jt}) := \begin{cases} (r_{j+1,t} - r_{jt})y_{jt} + H_{j,t+1}(\omega, y_{jt}) + G_{j+1,t+1}(\omega, \hat{S}_{j+1,t}^*(\omega)) - G_{j,t+1}(\omega, \hat{S}_{jt}^*(\omega)) & \text{if } y_{jt} \leq \hat{S}_{j+1,t}^*(\omega), \\ (r_{j+1,t} - r_{jt})y_{jt} + G_{j,t+1}(\omega, y_{jt}) - G_{j,t+1}(\omega, \hat{S}_{jt}^*(\omega)) & \text{if } \hat{S}_{jt}^*(\omega) \geq y_{jt} > \hat{S}_{j+1,t}^*(\omega), \\ (r_{j+1,t} - r_{jt})y_{jt} & \text{if } y_{jt} > \hat{S}_{jt}^*(\omega), \end{cases}$$

where, for convenience,  $\hat{S}_{L,t}^*(\omega) := 0$  for all  $t$  and  $\omega$ .

Because  $H_{jt}(\omega, \cdot)$  and  $G_{j,t+1}(\omega, \cdot)$  are smooth and convex for each  $j$  and  $\omega$ , then  $\bar{f}_{jt}(\omega, \cdot)$  is also smooth and convex. It remains to verify that  $\bar{f}_{jt}(\omega, y_{jt}) = \sum_{i=0}^{L-1} \bar{f}_{i,t+1}(\omega, y_{it})$  for each  $\omega$ . We apply Lemma 5. Let  $\omega$ ,  $y_t$ , and  $\hat{Y}_t^*(\omega, y_t)$  be given, and  $\bar{f}_{jt}(\omega, y_{jt})$  be as given above.

Assume case (i):  $\hat{S}_{0t}^*(\omega) < y_{0t}$ . By part (b),  $\hat{Y}_{jt}^*(\omega, y_t) = \hat{Y}_{0t}^*(\omega, y_t) = \hat{S}_{0t}^*(\omega) < y_{0t} \leq y_{jt}$  for every  $j$ , and using the definition of  $\bar{f}_{jt}(\omega, y_{jt})$ , we get

$$\sum_{j=0}^{L-1} \bar{f}_{j,t+1}(\omega, y_{jt}) = \sum_{j=0}^{L-1} (r_{j+1,t} - r_{jt})y_{jt} + G_{0,t+1}(\omega, \hat{S}_{0t}^*(\omega)),$$

which, by Equation (22), is exactly  $\bar{f}_t(\omega, y_t)$  with the threshold stage  $k=0$  and  $\hat{Y}_{0t}^*(\omega, y_t) = \hat{S}_{0t}^*(\omega)$ .

Assume case (ii):  $\hat{S}_{L-1,t}^*(\omega) \geq y_{L-1,t}$ . By part (b),  $\hat{Y}_{jt}^*(\omega, y_t) = y_{jt}$  for all  $j$ , and we get

$$\sum_{j=0}^{L-1} \bar{f}_{j,t+1}(\omega, y_{jt}) = \sum_{j=0}^{L-1} (r_{j+1,t} - r_{jt})y_{jt} + \sum_{j=0}^{L-1} H_{j,t+1}(\omega, y_{jt}),$$

which, by Equation (22), is exactly  $\bar{f}_t(\omega, y_t)$  when there are no disposals, and  $\hat{Y}_{jt}^*(\omega, y_t) = y_{jt}$ .

Assume case (iii): Given such a  $j$ , by part (b),  $\hat{Y}_{it}^*(\omega, y_t) = \hat{Y}_{jt}^*(\omega, y_t)$  for all  $i > j$ , and  $\hat{Y}_{it}^*(\omega, y_t) = y_{it}$  for all  $i < j$ . Using the definition of  $\bar{f}_{jt}(\omega, y_{jt})$ , we get

$$\sum_{j=0}^{L-1} \bar{f}_{i,t+1}(\omega, y_{it}) = \sum_{i=0}^{L-1} (r_{i+1,t} - r_{it})y_{it} + \sum_{i=0}^{j-1} H_{i,t+1}(\omega, y_{it}) + G_{j,t+1}(\omega, \hat{S}_{jt}^*(\omega)),$$

which, by Equation (22), is exactly  $\bar{f}_i(\omega, y_i)$  with the threshold stage  $j$ , and  $\hat{Y}_{ji}^*(\omega, y_i) = \hat{S}_{ji}^*(\omega)$ .

Assume case (iv). Given such a  $k$ , by part (b),  $\hat{Y}_{it}^*(\omega, y_i) = y_{kt}$  for all  $i > k$ , and  $\hat{Y}_{it}^*(\omega, y_i) = y_{it}$  for all  $i < k$ . Using the definition of  $\bar{f}_{ji}(\omega, y_{jt})$ , we get

$$\sum_{j=0}^{L-1} \bar{f}_{j,t+1}(\omega, y_{jt}) = \sum_{i=0}^{L-1} (r_{j+1,t} - r_{jt}) y_{jt} + \sum_{j=0}^{k-1} H_{j,t+1}(\omega, y_{jt}) + G_{j,t+1}(\omega, y_{kt}),$$

which, by Equation (22), is  $\bar{f}_i(\omega, y_i)$  with the threshold stage  $k+1$ , and  $\hat{Y}_{k+1,t}^*(\omega, y_i) = y_{kt}$ .

**PROOF OF THEOREM 3.** By Assumption 3,  $f_{T+1}^*(\omega, \cdot) = f_{T+1}(\omega, \cdot)$  is additively convex and smooth for every  $\omega$ . Assume inductively that  $f_{t+1}^*(\omega, \cdot)$  is additively convex and smooth for every  $\omega$ . Apply Lemma 6: There exists in period  $t$  a disposal saturation policy specified by a set of ordered base-stock levels, one at each echelon; the resulting optimal inventory policy is the echelon base-stock policy; and  $f_t^*(\omega, \cdot)$  is additively convex and smooth for every  $\omega$ . This completes the proof.

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