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NARRATIVE ANALYSIS OF COLLEGE STUDENTS' INCONSISTECIES IN REPRESENTING DUALITY OF INFINITY

Grace Babarinsa-Ochiedike, Mourat Tchoshanov, Benjamin McDermott

University of Texas at El Paso, USA

gobabarinsa@utep.edu, mouratt@utep.edu, brmcdermott@utep.edu

Abstract

Interpreting students' views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among the students' process-object perceptions. The fluctuations between students' views however reveal an undeveloped duality conception. This paper seeks to examine college students' conception of duality in understanding and representing infinity with the intent to elucidate strategy that could guide researchers in categorizing students' views of infinity into different levels. Data for the study were collected from N=69 college pre-calculus students at one of the southwestern universities in the U.S. using self-report questionnaire and interviews. Data was triangulated using multiple measures analyzed by three independent experts using self-designed coding sheet to assess students' externalization of the duality conception of infinity.

Keywords: Advanced Mathematical Thinking, Levels of Duality Conception, Infinity.

Introduction

Sfard (1991) conjectured two ways of developing a mathematical concept: structurally as an object, and operationally as a process. The structural conception refers to actual infinity, example of which is the infinity of the number of points in a segment, and the operational conception, which is the process of performing algorithms and actions, is dynamic and refers to the potential infinity (Fischbein, 2001). Fischbein claimed that seeing a function or number both as a process and as an object is fundamental for a deeper understanding of mathematics. According to Sfard (1991), the dual nature of mathematical construct can be observed verbally and through various symbolic representations. Majority of researches on infinity carried out in the elementary, secondary and college levels indicate that students' perception of infinity is more of a process. They define infinity as going on and on, continuing forever, endlessly etc. Monaghan (2001) draw our attention to students' usage of the word 'infinite' and 'infinity' in Moreno and Waldegg (1991)'s research. Students perceived infinity to mean an object view and infinite to mean a process view.

Attributable to the danger of assumption that comes with determining students' process-object duality and the dynamic nature of the duality conception (Falk 2010, Bingolbali & Monaghan 2008) scholars warns that care needs to be taken in interpreting students' representations of infinity. We believe that the examination of process-object conception of infinity presented by Monaghan (2001, p. 245-246) does not fully address the complex nature of infinity concept. Duality as a fundamental hidden idea is not explicitly presented. Monaghan takes for granted explicit representations in determining students' view of infinity by using obvious cases. We also disagree with Kolar and Cadez's (2012) interpretation of the symbol ∞ as representing the concept of actual infinity. They stated that "We believe that it represents the concept of actual infinity and indicated the awareness of the respondents about the infinite amount of numbers" (p. 404).

Theoretical Framework

According to Dubinsky et al. (2005), formation of mathematical concepts begins as one transforms an object to form another object. This transformation is referred to as action. This is performed explicitly based on specific instructions. A continuous reflection and performance of this action interiorized the action into a mental process. A process is an action that has been interiorized. With regard to the perspective of Dubinsky et al. (2005) when students repeatedly reflect on their action, they are able to interiorize their action into a mental process. Interiorizing infinity to a process relates to an understanding of potential infinity, whereby, infinity is imagined as performing an endless action, though without imagining the execution of each step. For example, when a student makes as many points as wanted on a line segment to represent infinite number. Relating this peculiarity to our examples, in the case of the *Cookie monster problem* the action of eating half of cookie remaining can be imagined to continue indefinitely. This type of thinking by students signifies a process conception.

The moment students perceives the process as a totality and perform an action on the process, the process is then said to have been encapsulated "into a cognitive object" (Dubinsky et al., 2005, p. 339). A process (e.g. counting natural numbers) can be transformed into an object (e.g. set of natural numbers) by means of encapsulation. Encapsulating this endless process to a complete object relates to a conception of actual infinity (quantity that describes the cardinality or the size of a complete infinite set). For example, when a student assumes the infinite number of points on a line segment as a complete entity, such thinking by students is referred to as object conception. *Actual infinity* entails the completed infinite process of eating half of cookie remaining; and that is acknowledging that the last crumb of cookie is been swallowed. When a process has been transformed into an object, it becomes the person's infinity schema. Actions and processes can always be applied to a schema to produce another cognitive object. This schema represents the "*process-object duality*" (Monaghan, 2001) the least studied of all constructs in APOS theory. This dual nature of mathematical constructs can be observed through various kinds of students' representations (Sfard 1991, Gray & Tall 1994). According to Sfard (1991), theories based on process-object duality though differentiates between a process conception and an object conception of mathematical notions, affirm that when learning a mathematical concept, the process conception precedes the object conception and that it is less abstract being on a lower reduced level of abstraction than the object conception. Hazzan & Zazkis (2005) assert that the means by which students reduce abstraction is neither exhaustive nor mutually exclusive. Modified duality concept development framework known as *Action-Process-Object-Duality* (APOD) (Babarinsa, Tchoshanov, & McDermott, 2012), adapted from the APOS theory was designed and used as well as Tall & Vinner's (1981) "concept definition and concept image" throughout this study to interpret students' intuitions, and their attempts to conceive infinity as a process as well as an object.

Table 1: *Levels of duality conception of infinity*

Conception	Levels	Views	Code
None	Level 0	Blank or Not determinable	ND
Singularity conception	Level 1	Isolated-singular view	'P' or 'O'
	Level 2	Semi-isolated dominant view	'Op' or 'Po'
Duality conception	Level 3	Dual-idiosyncratic view	po (p and o)
	Level 4	Duality view	PO (P and O)

Method of Inquiry

After the self-reported questionnaire administered during class time of the Calculus I sections of the instructors who were willing to let their students voluntarily participate in the study were collected and analyzed by three independent experts using self-designed coding sheet to assess students' externalization of their conception of infinity, students' views were classified into four levels to determine their conception of duality. Five (N=5) of the selected from the Calculus I class voluntarily agreed to participate in the interview and each participant represented a category level of duality conception of infinity.

The interviews with students were conducted for two important reasons. One reason is to gain additional insight into the views that students used to represent infinity as they recall their ways of thinking about the written responses to the questionnaire tasks and worked through related tasks given during interview; check for consistency in their language used to describe infinity as students clarify ambiguous responses to their personal concept definition of infinity, and since most of the participants provided relatively short and simple responses to the open-ended questions. A second reason for conducting the interviews with students is to better probe students' response to the multiple-choice task 4 and its disconnection from the first three tasks. In this way students were able to explicitly talk about their conception of infinity as a process, object or process-object, and I was able to gain further insight into their understanding of infinity and categorize their views as either a process, object or process-object.

The semi-structured interview protocol consisted of two questions related to the Cookie monster task but presented in different context, since it has been established that different representations of the same mathematical problem elicit different student responses (e.g., Arcavi, Tirosh, & Nachmias, 1989; Silver, 1986). During the interviews, some of the participants were first asked to complete the two interview tasks while others were interviewed beginning with their questionnaire responses that required clarification, to get further interpretations on their thinking and externalization of their conception of infinity.

To analyze students' responses and determine their duality conception level, especially because of the fluctuations in students' views from process to object and vice versa based on the task and context, the students' responses were coded and organized into two major views – the dominant views and the recessive views which are further categorized into the singularity conception and duality conception, based on the strength of students' responses/views.

Results and Analysis

In an effort to gain further insight into students' conception of infinity and categorize their views as either a process, object or process-object, the responses to students' questionnaire tasks were compared to the responses to the same tasks during interview and other two related tasks in different contexts. Below we present students' four questionnaire task responses and responses to same tasks during interview. The results from the table below show that there exist fluidity in 3 out of the 5 subjects' view of infinity during the survey and interview. Subjects shifting from one conception level to another is an indication of an unformed concept of infinity.

Table 2: Questionnaire tasks responses during survey and interviews

Task	Subjects	Jose	Vanessa	Susseth	Robin	Emma
Q1	Survey	1	1	3	3	3
	Interview	3	2	2	3	1
Q2	Survey	1	3	3	1	3
	Interview	1	3	3	1	3
Q3	Survey	1	0	0	2	3
	Interview	1	0	0	2	1
Q4	Survey	3	1	3	3	3
	Interview	3	3	3	3	3

Task Q1: "When you think of infinity what comes to your mind?"

Jose (J) in responding to this task wrote "Numbers that goes beyond what we can count on a daily basis". The idea that we are counting makes this a predominantly process view, and that caused him to be rated at level 1. Below is an excerpt of the interview which explains the shift from level 1 to 3.

111. I: Do you have anything to elaborate on that or that's still…?

112. Jose: That still holds true. And when I wrote this, I didn't think of even life. I just thought of… This is the first thing that came to my mind. It was numbers…

113. I: Huh-un! Yeah!

114. J: … And numbers, you know you… you can get one number. Let say 1, and add decimals and decimals. You can put 1.1, 1.13, 1.134.

115. I: Huh-un!

116. J: … and so forth. So if… you can have an infinite number, all the way up until, let's say for example 1.99999…

117. I: Huh-un!

118. J: … and take that number to infinity.

Thinking of infinity as "life" brings his attempt to encapsulate the process into an object. Also the idea of adding repeating decimals non-terminating and taking that number to infinity in lines 114 – 118 suggests infinity as a destination, as a place and thus an attempt of encapsulating the process as a Totality, and clearly a conceptional object. But he also says you can have an infinite number, so using the adjective of infinity, referring to a number, infinite number, this is an object language. His ability to draw the object language in addition to the process language during interview put him on level 3.

Vanessa (V) in responded wrote: “A long list of never ending numbers”. We interpreted “a long list” as a sequence, hence a process (level 1). When asked in an interview, this is what transpired:

G: Ok. Alright, let me take you back to the, [interviewee laughing as I opened to her survey response] take your mind back to this. It’s says when you think of infinity, what comes to your mind? You said a long list of numbers…

V: Never ending numbers

G: Do you still wanna stick to it or you still have more…

V: No, no

G: … explanation you wanna give to that, or you want to explain better to me? When you say long list…

V: I guess, I guess I will add just something that never ends, like that’s infinity; never ends.

She was specifically asked to clarify the phrase “long list” and she said “I guess I will add just something that never ends, like that’s infinity”. That sounds different than just saying “a long list”. So “something” brings the language of object. Also saying “like that’s infinity” is trying to impose the idea of cardinality on “something that never ends”. And that’s where we may interpret it as a set for example, as compared to “a long list”. But still the process view is dominating, hence, her response here was categorized at level 2 (Po).

Emma wrote: “Infinity is forever, an amount that cannot be reached or counted”. There is an indication of both process and object. “Infinity is forever” is kind of like a process language. “An amount” captures the object view and “can never be reached or counted” the process-object view. So the statement suggests anything with boundless amount, and clearly encapsulated process-object language (level 3). She was later asked to look at her response during interview:

G: It talks about when you think of infinity, what comes to your mind. And I like your response, saying "infinity is forever, an amount that cannot be reached or counted". I want you to really throw more light into that for me.

E: Well if you count to numbers like an hundred and that sort of thing. And if you keep going, you can actually count to a million, but you can never really get to infinity because it just goes on and on and on. So in that way you can never count to infinity, you can only count to a million. But even then you'll be really tired. So in that sense, infinity goes on forever because you'll never be able to count it, because by the time you get there, you might be dead. [Both laughing]

G: Interesting! [Laughing] Ok, so when you say an amount, so does it mean that when you're given a certain amount, you can't reach it or what?

E: Yes! Like if I have 5 something that have 5, that I can count to 5. But if someone says am going to give you infinite number of apples, then you can't ever have that many apples because it's too many.

She went clearly to process language of counting as a procedure, even with an example when she was asked to elaborate on "an amount", in order to bring her back to that object language that she used in the survey response. In a way she plays the object language of "amount" that encapsulates the cardinality by process perception of not been able to reach it. So that's why we see her interview response as clearly a process view.

Task 2: The cookie monster sneaks into the kitchen and eats half of a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day.

If the cookie monster continues this process seven days, how much of the cookie has he eaten?

How much is left?

If the process continues, will he ever eat the entire cookie?

All of the participants confirmed their reasoning to the various responses written on the survey.

Task 3: “Draw infinity in the space provided. Explain your drawing below.”

All of the participants confirmed their reasoning to the various responses written on the survey except for Emma that shifted from level 3 to 1 again. Emma drew a line with an arrow on both ends and her explanation to the drawing is that “It’s like a line that never reaches a destination much like the actual infinity”. This is clearly an object view. Below is the interview excerpt:

59. G: Oh Ok! So, that’s the explanation there! Ok! Thank you. Alright. Let’s look at… One I like what you wrote about your drawing; because I saw you draw a line for number (3).

60. E: Uh-hum!

61. G: And I saw the arrows going in both directions. Right? Then it says… Your explanation says “It’s a line than never reaches a destination, but, much like the actual infinity”. What do you mean by that?

62. E: Because, when you’re adding arrow to a line like this, it means the arrow just keeps going and it has no self-stop. So I believe it’s just like infinity because there is no real end to infinity, it just keeps going.

63. G: Infinity just keeps going? Then what do you mean by actual infinity? Let me just know your understanding about actual infinity.

64. E: I think what I meant by actual infinity is just the, thought of infinity going on forever, and never reaching an end.

65. G: Ok. Going on forever?

66. E: Uh-hum!

The idea of infinity keep going and having no real end is predominantly a process view of infinity (level 1).

Task 4: I feel that my conception of infinity is as (check one):

a) A process, e.g. something that goes on and on.

b) An object, e.g. Set of natural numbers is infinite.

c) Both a process and an object.

d) Other:

The multiple-choice task 4 is a self-response task that asks respondents to identify their conception of infinity, whether it is a process (e.g. something that goes on and on), or it is an object (e.g. a set of natural numbers is infinite) or both a process and object or identify other.

All of the participants confirmed their reasoning to the various responses written on the survey except for Vanessa.

129. G: Thank you. Now let's look at this number 4. You're talking about your conception of infinity that is it a process or an object, and you choose a process!

130. V: I think I wanna change my answer to that one. 'Cause I think it can be anything a process… or an object. Like a number like Pi, that goes on forever. And that's a number… an object. And a process is like running a race when we're doing halves or even cookie when we're only eating half every day. So I think infinity can be anything as long as it goes on and on and on and on and on forever.

131. G: Oh! So that's a process? And then the object part is, you said… you give an example of …

132. V: Like Pi, the object could be like Pi. Like a number that never ends or a song like never… like that song that sang never ends or anything really. Just something that never ends. It doesn't matter what it is.

133. G: Okay. Now look at this [the definitions on the multiple choice question 4] so you're good, you're cool with this? The object definition…

134. V: I want to say maybe not like object but for sure like numbers, or infinite. Cause even when you count, that's infinite too, but counting is a process.

135. G: Counting is a process. So, the set of natural numbers is a... is infinite, so you see that as a process also or as an object?

136. V: Hu-mm! No I wanna say it's a process 'cause it's the process of counting. [Pause] Yea! Okay! Never mind. I'm sticking with my answer (a). Yea! It's just something that goes on and on and on.

137. G: So, just… It's a process?

138. V: Yes! Just the process [laughing]

139. G: Well okay! Okay!! No o…

140. V: I know. I'm confusing it. It's because that's how I think in my head when I think about the stuff.

She wrote in her survey response that her conception of infinity was a process but when asked during interview, she kept changing her mind, floating from levels 1 to 3 and back to 1. From process view to process-object and then back to process). It is obvious she has a dominating process view even though she used a strong object example of pi (π). This is an indication of a not well formed conception. She thinks it has to be one or the either. Not accepting it can be both views.

The major outcome of the study is that coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is task-dependent and context-dependent. There exists fluctuations in students' views of infinity which posits challenges for researchers in interpreting students' perceptions of infinity as either a process or an object, and especially in determining the students' process-object duality conception. And this result is supportive of the claim of Falk 2010, Bingolbali and Monaghan (2008) warning that care needs to be taken in interpreting students' representations of infinity.

Conclusion

Interpreting students' views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among students' process-object perceptions. The fluctuations between students' views however reveal an undeveloped duality conception. This study examined college students' conception of duality in understanding and representing infinity with the intent to design strategy that could guide researchers in categorizing students' views of infinity into different levels.

It is known that concept of duality as any other fundamental ideas of mathematics are "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 151).

Results of our study reveal that college students' experiences in traditional Pre-calculus course are not supportive of the development of duality conception. Therefore, it is important to provide college students with relevant experiences to build the concept of duality, which will help them to understand mathematical concepts (e.g., infinity) at a more rigorous level. Understanding the dual nature of mathematical concepts could help students become more knowledgeable and flexible in learning abstract and complex mathematical ideas. "In order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate" (Selden, 2002). Gray and Tall (1994) describes concepts that could be viewed both as a process and an object as procept. Hence, we consider a proceptual perspective as a tool to help students at their earlier stages of learning to understand and overcome the contradictory and counterintuitive nature of infinity concept.

Practical significance of the study is that it helps to recognize misconceptions and start addressing them so students will have a more comprehensive view of fundamental mathematical ideas as they progress through Calculus coursework sequence. If pre-or-miss-conceptions are not timely recognized and addressed, then students' traditional experiences could be easily built on strong 'narrow-minded' mental scripts that could be later transferred to "immature" understanding of mathematical concepts.

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