ON THE COHOMOLOGY OF FROBENIUSIAN MODEL LIE ALGEBRAS

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ABSTRACT. We compute the first and second cohomology groups with coefficients in the adjoint module of frobeniusian model algebras whose parameters move in a dense open subset of $\mathbb{C}^{p-1}$, and obtain upper bounds for the dimension of cohomology groups of frobeniusian Lie algebras. Moreover, it is shown that for a dense open subset of $\mathbb{C}^{p-1}$ the deformations of model algebras also belong to the family. Therefore any frobeniusian non-model algebra contracts on some element of the model whose parameters move on a finite union of hyperplanes. Further applications as the nullity of Rim's quadratic map sq_1 are obtained.

1. INTRODUCTION

Lie algebra model theory arises from the attempts to describe certain neighbourhoods of Lie algebras in the variety of Lie algebra laws $\mathcal{L}^n$ in terms of particular properties. While such properties as solvability are nonstable for deformations, other as semisimplicity are preserved, to such an extent that it implies rigidity. There are intermediate properties, usually extracted from Differential Geometry, and which play an important role in the study of forms and differential systems, as for example the existence of linear contact or symplectic forms or r-contact systems [7]. Here we will consider the property of having a linear form whose differential is symplectic. These algebras, which are solvable and are usually called frobeniusian, were classified up to contraction by Goze in [6]. Now the model is not unique for the described algebras: there exists a family on $(p-1)$ parameters, which shows that models need not to be isolated. Most of these objects have been studied in the frame of perturbation theory [5]. This approach allows some simplifications in cohomological questions, such as the deformation equation of Nijenhuis and Richardson [12], which reduces to analyze a finite system.

From the physical point of view, solvable Lie algebras often occur as Lie algebras of symmetry groups of differential equations, which, added to the symplectic structure, of extraordinary importance, justifies the interest of analyzing frobeniusian Lie algebras.

In this paper we calculate the first and second cohomology groups (with coefficients in the adjoint module) of frobeniusian model algebras whose parameters move in a dense open subset of $\mathbb{C}^{p-1}$. This allows to obtain upper bounds for the dimension of cohomology groups of frobeniusian Lie algebras. Moreover, it is shown that for a dense open subset of $\mathbb{C}^{p-1}$ the deformations of model algebras whose parameters belong to the subset also belong to the family. Therefore any frobeniusian non-model algebra contracts on some element of the model whose parameters move on a finite union of hyperplanes of $\mathbb{C}^{p-1}$. Further applications as the nullity of Rim's quadratic map sq_1 are obtained.

Any Lie algebra considered in this work is finite dimensional over the field $\mathbb{C}$. Moreover, any n-dimensional Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$ is identified with its law μ in $\mathcal{L}^n$. Moreover, we convene that nonwritten brackets are either zero or obtained by antisymmetry.

2. Lie algebra models

Let $\mathfrak{L}^n$ be the variety of Lie algebra laws over $\mathbb{C}^n$. As known, the linear group $GL(n, \mathbb{C})$ acts on $\mathbb{C}^n$ by changes of basis. Recall that a Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \lambda)$ is said to contract to the law μ if $\mu \in \overline{O(\lambda)}$, where $\overline{O(\lambda)}$ is the closure of the orbit of λ by the group action (in either the metric or Zariski topology [1]). It is easy to verify that this definition is quite the same as the more classical one: let $(\mathbb{C}^n, \mu_0)$ be a point in $\mathcal{L}^n$ [10] and $\{f_t\}$ a sequence of isomorphisms in $\mathbb{C}^n$. Clearly the element $(\mathbb{C}^n, \mu_t)$ with

$$\mu_t = f^{-1} \circ \mu_0 (f_t \times f_t)$$

belongs to the orbit $\mathcal{O}(\mu_0)$. If the limit $\lim_{t \to \infty} \mu_t$ exists, it defines a law of $\mathcal{L}^n$ called a contraction of μ_0.
As told before, for analyzing Lie algebra models the concept of perturbation is more convenient. This notion is better adapted for the study of certain topological properties concerning the orbits of laws in the variety $\mathcal{L}^n$, and is placed within the internal set theory (I.S.T. [11]).

Definition 1. *A perturbation μ of a Lie algebra law μ_0 in $\mathcal{L}^n$ is a law satisfying*

$$\mu (X, Y) \simeq \mu_0 (X, Y), \quad X, Y \in \mathbb{C}^n \text{ standard}$$

i.e., the structure constants of μ are infinitely close to those of μ_0 (the basis being fixed).

Perturbations of a law can be decomposed according to the following rule [8]:

$$\mu = \mu_0 + \varepsilon_1\varphi_1 + ... + \varepsilon_1..\varepsilon_k\varphi_k$$

where $\varphi_1, .., \varphi_k$ are (standard) 2-cochains independent of $\varepsilon_1, .., \varepsilon_k \simeq 0$. The integer k is called the length of the perturbation. If $k = 1$, they correspond to the infinitesimal deformations.
Let now $\mathfrak{g}_0 = (\mathbb{C}^n.\mu_0)$ be a Lie algebra and (P) be an internal property concerning the Lie algebra laws.

Definition 2. *The algebra $\mathfrak{g}_0$ is called a semimodel relative to property (P) if any Lie algebra law μ satisfying (P) contracts on μ_0.*
The semimodel is called a model if, in addition, any perturbation of μ_0 satisfies (P).

Remark 1. *The neighbourhoods of model algebras are entirely characterized by property (P) ([2], [3]). As examples of properties whose analysis lead to interesting questions we could enumerate those related to differential invariants of forms and systems on a Lie algebra.*

Example 1. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional Lie algebra and ω a linear form. Then ω is called a contact form if $\omega \wedge (d\omega)^n \neq 0$, where $d\omega(X,Y) = -\omega([X,Y])$ corresponds to the contragradient representation of $\mathfrak{g}$. Let (P) be the property "there exists a linear contact form on $\mathfrak{g}$". It is not difficult to see that the Heisenberg Lie algebra $\mathfrak{h}_n$ given by*

$$\left\{ d\omega_{2n+1} = \sum_{k=0}^{n-1} \omega_{2k+1} \wedge \omega_{2k+2} \right.$$

is the unique model relative to this property [6].

Another example illustrating the interest of models for physical applications is the following:

Example 2. *Let $\mathfrak{g} = \mathfrak{so}(3)$. It can be easily seen, without using the fact that this algebra is rigid for being simple, that this algebra is the unique model for the property "any linear form on $\mathfrak{g}$ is a contact form". This implies in particular that this property is valid only in dimension three* [5].

It could also happen that a model is not unique, even that a model for a property (P) does not exist (as happens for the r-contact systems [5]). Therefore the definition of (semi-)model has to be generalized.

Definition 3. *A family F of Lie algebras satisfying an internal property (P) is called a multiple semimodel relative to (P) if any Lie algebra satisfying it contracts to an algebra of the family.*
If in addition any perturbation of elements in F also satisfies (P), the family is called a multiple model.

In order to get a minimal family of models, we also define irreducible multiple models:

Definition 4. *Let $F = \{\mathfrak{g}_i = (\mathbb{C}^n, \mu_i)\}_{i\in I}$ be a multiple model relative to property (P). Then F is called irreducible if for any $i, j \in I$ we have $\mu_i \notin \overline{O(\mu_j)}$.*

The property we consider here is "there exists a linear form ω whose differential $d\omega$ is symplectic". Recall that a linear form ω on a $2n$-dimensional Lie algebra $\mathfrak{g}$ is called symplectic if it is closed and $\omega^n \neq 0$. Considering the differentials $d\omega$ instead of ω we obtain the frobeniusian Lie algebras:

Definition 5. *A $2n$-dimensional Lie algebra $\mathfrak{g}$ is called frobeniusian if there exists $\omega \in \mathfrak{g}^*$ such that $(d\omega)^n \neq 0$.*

Semimodels of frobeniusian Lie algebras were studied in [6] and [7]. Here models exist, but uniqueness is lost. Let $(\varphi) := (\varphi_1, .., \varphi_{p-1}) \in \mathbb{C}^{p-1}$.

Theorem 1. [5] *Let $\{F_\varphi \mid (\varphi) \in \mathbb{C}^{p-1}\}$ be the family on $(p-1)$-parameters of $2p$-dimensional Lie algebras given by*

$$\begin{cases} d\omega_1 = \omega_1 \wedge \omega_2 + \sum_{k=1}^{p-1} \omega_{2k+1} \wedge \omega_{2k+2} \\ d\omega_2 = 0 \\ d\omega_{2k+1} = \varphi_k \omega_2 \wedge \omega_{2k+1},\ 1 \leq k \leq p-1 \\ d\omega_{2k+2} = -(1+\varphi_k)\,\omega_2 \wedge \omega_{2k+2},\ 1 \leq k \leq p-1 \end{cases}$$

where $\{\omega_1, .., \omega_{2p}\}$ is a basis of $(\mathbb{C}^{2p})^*$. The family $\{F_\varphi\}$ is an irreducible multiple model for the property "there exists a linear form whose differential is symplectic".

It can be easily seen that the algebras F_φ admit the following graduation: if $\{X_1, .., X_{2p}\}$ is a dual basis to $\{\omega_1, .., \omega_{2p}\}$, then $F_\varphi = (F_\varphi)_0 \oplus (F_\varphi)_1 \oplus (F_\varphi)_2$, where $(F_\varphi)_0 = \mathbb{C}X_2, (F_\varphi)_1 = \sum_{k=3}^{2p} \mathbb{C}X_k$ and $(F_\varphi)_2 = \mathbb{C}X_1$. This decompostion will be of importance for cohomological computations.

Let Ω_1 denote the union of the following subsets of $\mathbb{C}^{p-1}$

$$\{1+\varphi_i+\varphi_j=0\}_{1\leq i,j\leq p-1}$$
$$\{2+\varphi_i+\varphi_j=0\}_{1\leq i,j\leq p-1}$$
$$\{\varphi_i-\varphi_j=0\}_{1\leq i\neq j\leq p-1}$$
$$\{\varphi_i=0\}_{1\leq i\leq p-1}$$
$$\{\varphi_i+1=0\}_{1\leq i\leq p-1}$$
$$\{2\varphi_i+1=0\}_{1\leq i\leq p-1}$$

Lemma 1. *If* $(\varphi)\notin\Omega_1$, *then* $\dim Der\,(F_\varphi)=3p-1$.

Proof. Let $f\in Der\,(F_\varphi)$ and let $\{X_1,..,X_{2p}\}$ be a dual basis to $\{\omega_1,..,\omega_{2p}\}$. Let $f\,(X_i)=f_i^jX_j\;\;(1\leq i\leq 2p)$. For $1\leq k\leq p-1$ the conditions $[f\,(X_2)\,,X_{2k+1}]+[X_2,f\,(X_{2k+1})]=\varphi_kf\,(X_{2k+1})$ imply $f_{2k+1}^j=0$ for $j\neq 1,2k+1$, as $(\varphi)\notin\Omega$, and $f_2^{2k+2}=-\left(1+\varphi_k\right)f_{2k+1}^1$. Similarly we obtain $f_{2k+2}^j=0$ for $j\neq 1,2k+2$ and $f_2^{2k+1}=-\varphi_kf_{2k+1}^1$. As X_2 does not belong to the derived subalgebra D^1F_φ, we have $f_2^2=0$.
Therefore, any nontrivial derivation f modulus the image $ad\,(F_\varphi)$ is given by

$$f\,(X_i)\;\;(mod\;\;ad\,(F_\varphi))=\begin{cases}0 & i=1,2\\ \lambda_iX_i & i\text{ odd and }\geq 3\\ -\lambda_iX_i & i\text{ even and }\geq 4\end{cases}$$

where $\lambda_i\in\mathbb{C}$. This is a linear combination of the derivations $f_i\;\;(1\leq i\leq p-1)$ defined by

$$\begin{cases} f_i\,(X_{2i+1})=X_{2i+1}\\ f_i\,(X_{2i+2})=-X_{2i+2}\end{cases}$$

This shows that the $\{f_i\}_{1\leq i\leq p-1}$ form a basis of $H^1\,(F_\varphi,F_\varphi)$, thus dim $H^1\,(F_\varphi,F_\varphi)=p-1$. The algebra $Der\,(F_\varphi)$ is easily seen to be isomorphic to the semidirect product $adF_\varphi\oplus\sum_{i=1}^{p-1}\mathbb{C}\,\{f_i\}$ given by

$$\left\{[ad\,(X_{2i+1+t})\,,f_i]=-\left(-1\right)^t ad\,(X_{2i+1+t})\,,\;1\leq i\leq p-1,\;t\in\{0,1\}\right.$$

Clearly dim $Der\,(F_\varphi)=3p-1$. □

Corollary 1. *If* $p\geq 2$, *then* $dimH^1\,(F_\varphi,F_\varphi)=p-1$.

Now let $\mathfrak{g}$ be a frobeniusian Lie algebra not belonging to the family $\{F_\varphi\mid\varphi\in\mathbb{C}^{p-1}\}$. Then $\mathfrak{g}$ contracts to some element F_{φ_0} in view of theorem 1. We denote it by $\mathfrak{g}\rightarrowtail F_{\varphi_0}$. By the general properties of contractions [2], $\mathfrak{g}$ satisfies the following conditions

1. $\dim Der\,(\mathfrak{g})<\dim Der\,(F_{\varphi_0})$

2. $\dim [\mathfrak{g}, \mathfrak{g}] \geq \dim [F_{\varphi_0}, F_{\varphi_0}]$
3. $\dim Z(\mathfrak{g}) \leq \dim Z(F_{\varphi_0})$

where $Der(\circ)$ denotes the algebra of derivations, $[\circ, \circ]$ the commutator algebra and $Z(\circ)$ the center. As the algebras F_φ are centerless, we have $Z(\mathfrak{g}) = \{0\}$ for a frobeniusian Lie algebra.

Proposition 1. *Let $p \geq 2$ and $\mathfrak{g}$ be a frobeniusian Lie algebra $\mathfrak{g} \rightarrowtail F(\varphi)$ such that $(\varphi) \notin \Omega_1$. Then $\dim H^1(\mathfrak{g}, \mathfrak{g}) \leq p - 2$.*

Proof. As $Z(\mathfrak{g}) = \{0\}$, the adjoint representation is faithful and we obtain

$$\dim H^1(\mathfrak{g}, \mathfrak{g}) = \dim Der(\mathfrak{g}) - 2p < \dim Der(F_\varphi) - 2p$$

The result follows from the previous lemma. □

One of the reasons for isolating the parameters of Ω is the similar structure of its derivations, which leads to isomorphisms of the corresponding derivation algebras. In particular, we will obtain complete Lie algebras. Recall that a Lie algebra is called complete if $Z(\mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = \{0\}$.

Theorem 2. *Let $p \geq 2$. If $(\varphi) \notin \Omega_1$, the Lie algebra $Der(F_\varphi)$ is a 3-step solvable complete Lie algebra. Moreover, for any $(\varphi_0) \neq (\varphi_0')$ we have*

$$Der(F_{\varphi_0}) \simeq Der\left(F_{\varphi_0'}\right)$$

Proof. The structural equations of $Der(F_\varphi)$ for any $(\varphi) \notin \Omega_1$ are given by

$$\begin{cases} d\omega_1 = \omega_1 \wedge \omega_2 + \sum_{k=1}^{p-1} \omega_{2k+1} \wedge \omega_{2k+2} \\ d\omega_2 = 0 \\ d\omega_{2k+1} = \varphi_k \omega_2 \wedge \omega_{2k+1} + \omega_{2p+k} \wedge \omega_{2k+1}, \ 1 \leq k \leq p-1 \\ d\omega_{2k+2} = -(1+\varphi_k)\,\omega_2 \wedge \omega_{2k+2} - \omega_{2p+k} \wedge \omega_{2k+2}, \ 1 \leq k \leq p-1 \\ d\omega_{2p+k} = 0, \ 1 \leq k \leq p-1 \end{cases}$$

over the basis $\{\omega_1, .., \omega_{2p}, \omega_{2p+1}, .., \omega_{3p-1}\}$. The proof of completeness is routine, as well as the fact that the algebra is 3-step solvable. For the last assertion, let $(\varphi_0) \neq (\varphi_0')$ and consider the change of basis given by

$$\begin{cases} X_2' = X_2 + \sum_{i=1}^{p-1} (\varphi_i' - \varphi_i)\, X_{2p+i} \\ X_i' = X_i, \ i \neq 2 \end{cases}$$

□

Remark 2. *Observe in particular that Der* (F_φ) *cannot arise as a contraction of a Lie algebra, since it is complete* [3]. *Moreover, it is easily seen that this Lie algebra is rigid* [1], *as it is isomorphic to the semidirect product* $\mathfrak{h}_{p-1} \oplus \mathfrak{t}$ *of the* $(2p-1)$*-dimensional Heisenberg Lie algebra* $\mathfrak{h}_{p-1}$ *with a maximal torus of derivations.*

3. Chevalley cohomology

In this section we determine the cohomology groups $H^2(F_\varphi, F_\varphi)$ whenever φ lies in some dense open subset of $\mathbb{C}^{p-1}$. Specifically, let Ω_2 be the union of following hyperplanes in $\mathbb{C}^{p-1}$:

$$\{1+\varphi_i-\varphi_j=0\}$$
$$\{\varphi_i+\varphi_j=0\}$$
$$\{2+\varphi_i=0\}$$
$$\{1-\varphi_i=0\}$$
$$\{1+2\varphi_i-\varphi_j=0\}$$
$$\{1+2\varphi_i+\varphi_j=0\}$$
$$\{2\varphi_j-\varphi_j=0\}$$
$$\{2+2\varphi_i+\varphi_j=0\}$$

and $\Omega = \Omega_1 \cup \Omega_2$. Clearly this is a finite union of hyperplanes, and therefore its complementary in $\mathbb{C}^{p-1}$ is dense. In what follows we always suppose that $(\varphi) \notin \Omega$. In the previous section we have seen that the algebras F_φ are graded. Therefore, the cohomology spaces inherit a graduation (see [9], as well as for the notation used), and we have $H^2(F_\varphi, F_\varphi) = F_{-k}H^2(F_\varphi, F_\varphi)$ for some positive integer k, where $F_{-k}H^2(F_\varphi, F_\varphi) = \sum_{j\geq -k}^{1} H_k^2(F_\varphi, F_\varphi)$. For brevity in the notation, we will denote the coboundary operator $\delta_{\mu_\varphi,p} : C^p(F_\varphi, F_\varphi) \rightarrow C^{p+1}(F_\varphi, F_\varphi)$ simply by δ_{μ_φ}

Lemma 2. *For* $p \geq 2$ *we have* $H^2(F_\varphi, F_\varphi) = F_{-2}H^2(F_\varphi, F_\varphi)$.

Proof. Let $\psi \in Z^2(F_\varphi, F_\varphi)$ with $\psi(X_i, X_j) = \sum_k a_{ij}^k X_k$ $(1 \leq i < j \leq 2p)$. Considering the coboundary operator δ_{μ_φ} for the triples $\{X_1, X_{2k+1}, X_{2k+2}\}_{1\leq k\leq p-1}$ we obtain

$$\delta_{\mu_\varphi}(\psi)(X_1, X_{2k+1}, X_{2k+2}) = -a_{1,2k+1}^2(1+\varphi_k)X_{2k+2} - a_{1,2k+2}^2\varphi_k X_{2k+1} + \\ + \sum_{i\neq 2k+1,2k+2} \alpha_i X_i, \qquad (\alpha_i \in \mathbb{C})$$

Since $(\varphi) \notin \Omega$ we have $\varphi_k(1+\varphi_k) \neq 0$ for any k, and thus $a_{1,2k+1}^2 = a_{1,2k+2}^2 = 0$ $(1 \leq k \leq p-1)$. This shows that $Z_{-3}^2(F_\varphi, F_\varphi) = 0$. Now

$Z_k^2(F_\varphi, F_\varphi) = 0$ for $k \geq 2$, as follows immediately from the graduation. □

Therefore, the determination of the cohomology spaces reduces to the computation of the distinct subspaces $H_k^2(F_\varphi, F_\varphi)$. We begin determining bases for the cocycle spaces $Z_k^2(F_\varphi, F_\varphi)$.

Lemma 3. *dim* $Z_{-2}^2(F_\varphi, F_\varphi) = 1$.

Proof. Let $\psi \in Z_{-2}^2(F_\varphi, F_\varphi)$. Then its generic form is

$$\begin{array}{ll} \psi(X_1, X_2) & = a_{12}^2 X_2 \\ \psi(X_i, X_j) & = a_{ij}^2 X_2,\ 3 \leq i,j \leq 2p \\ \psi(X_1, X_j) & = \sum_{t=3}^{2p} a_{1,j}^t X_t,\ j \geq 3 \end{array}$$

For $p \geq 4$ we can always find $3 \leq l \leq 2p$ such that $[X_i, X_j] = [X_i, X_k] = [X_j, X_k] = 0$. Then

$$\delta_{\mu_\varphi}(\psi)(X_i, X_j, X_l) = a_{ij}^2 [X_2, X_l] + \sum_{t \neq l} \alpha_t X_t,\ \alpha_t \in \mathbb{C}$$

where

$$[X_2, X_l] = \begin{cases} \varphi_k X_{2k+1} \text{ if } l = 2k+1 \\ -(1+\varphi_k) X_{2k+2} \text{ if } l = 2k+2 \end{cases}$$

This implies $a_{ij}^2 = 0$ whenever $[X_i, X_j] = 0$. For $p = 2, 3$ the nullity follows from considering the triples $\{X_2, X_i, X_j\}$.
Applying δ_{μ_φ} to the triples $\{X_2, X_{2k+1}, X_{2k+2}\}$ we obtain

$$a_{12}^2 = a_{2k+1,2k+2}^2,\ 1 \leq k \leq p-1$$

Finally, taking $\{X_1, X_2, X_j\}_{j \geq 3}$ we get

$$\begin{array}{rl} a_{1,j}^k & = 0,\ k \neq j \\ a_{1,j}^j + \beta_j a_{12}^2 & = 0 \end{array}$$

Thus $Z_{-2}^2(F_\varphi, F_\varphi)$ is generated by the cocycle ψ_{12}^2 defined by

$$\begin{cases} (X_1, X_2) \mapsto X_2 \\ (X_1, X_{2k+1}) \mapsto -\varphi_k X_{2k+1},\ 1 \leq k \leq p-1 \\ (X_1, X_{2k+2}) \mapsto (1+\varphi_k) X_{2k+2},\ 1 \leq k \leq p-1 \\ (X_{2k+1}, X_{2k+2}) \mapsto X_2,\ 1 \leq k \leq p-1 \end{cases}$$

□

Lemma 4. *dim* $Z_{-1}^2(F_\varphi, F_\varphi) = 4(p-1)$.

Proof. A cocycle $\psi \in Z^2_{-1}(F_\varphi, F_\varphi)$ has the generic form

$$\begin{array}{ll} \psi(X_2, X_j) & = a^2_{2j} X_2,\ j \geq 3 \\ \psi(X_1, X_2) & = \sum_{j=3}^{2p} a^j_{12} X_j \\ \psi(X_i, X_j) & = \sum_{t=3}^{2p} a^t_{ij} X_t,\ 3 \leq i, j \leq 2p \\ \psi(X_1, X_j) & = a^1_{1j} X_1,\ 3 \leq j \leq 2p \end{array}$$

and satisfies the following equations, for any $1 \leq k \leq p-1$:

$$\begin{aligned} a^{2k+2-r}_{12} + (-1)^r a^2_{2,2k+1+r} + (r + \varphi_k) a^1_{1,2k+1+r} &= 0 \\ a^{2k+2-r}_{12} - (1 + \varphi_k - r) a^2_{2,2k+1+r} + (-1)^r (r + \varphi_k) a^{2k+2-r}_{2k+1,2k+2} &= 0 \\ a^{2k+2-r}_{12} + (-1)^r (r + \varphi_j) a^{2k+2-r}_{2j+1,2j+2} &= 0 \\ \varphi_j a^2_{2,2k+1+r} + (-1)^r (r + \varphi_k) a^{2j+1}_{2k+1+r,2j+1} &= 0 \\ (1 + \varphi_j) a^2_{2,2k+1+r} - (-1)^r (r + \varphi_k) a^{2j+2}_{2k+1+r,2j+2} &= 0 \end{aligned}$$

where $1 \leq j \neq k \leq p-1$ and $r \in \{0, 1\}$. From the structure of this system it is not difficult to extract a basis. One is given by the cocycles $\psi^2_{2,2k+1}, \psi^2_{2,2k+2}, \psi^{2k+1}_{12}$, ψ^{2k+2}_{12} $(1 \leq k \leq p-1)$ defined by :

1. $\psi^2_{2,2k+1}$:
$$\left\{ \begin{array}{ll} (X_1, X_{2k+1}) \mapsto X_1; \quad (X_2, X_{2k+1}) \mapsto -\varphi_k X_2 & \\ (X_{2k+1}, X_{2k+2}) \mapsto -(1 + \varphi_k) X_{2k+2} & \\ (X_{2k+1}, X_{2k'+1}) \mapsto \varphi_{k'} X_{2k'+1} & (k \neq k') \\ (X_{2k+1}, X_{2k'+2}) \mapsto -(1 + \varphi_{k'}) X_{2k'+2} & (k \neq k') \end{array} \right.$$
2. $\psi^2_{2,2k+2}$:
$$\left\{ \begin{array}{ll} (X_1, X_{2k+2}) \mapsto X_1; \quad (X_2, X_{2k+2}) \mapsto (1 + \varphi_k) X_2 & \\ (X_{2k+1}, X_{2k+2}) \mapsto -\varphi_k X_{2k+1} & \\ (X_{2k+2}, X_{2k'+1}) \mapsto \varphi_{k'} X_{2k'+1} & (k \neq k') \\ (X_{2k+2}, X_{2k'+2}) \mapsto -(1 + \varphi_{k'}) X_{2k'+2} & (k \neq k') \end{array} \right.$$
3. ψ^{2k+1}_{12}:
$$\left\{ \begin{array}{ll} (X_1, X_2) \mapsto (1 + \varphi_k) X_{2k+1} & \\ (X_1, X_{2k+2}) \mapsto -X_1 & \\ (X_{2j+1}, X_{2j+2}) \mapsto X_{2k+1} & ; 1 \leq j \leq p-1 \end{array} \right.$$
4. ψ^{2k+2}_{12}:
$$\left\{ \begin{array}{ll} (X_1, X_2) \mapsto \varphi_k X_{2k+2} & \\ (X_1, X_{2k+1}) \mapsto -X_1 & \\ (X_{2j+1}, X_{2j+2}) \mapsto -X_{2k+2} & ; 1 \leq j \leq p-1 \end{array} \right.$$

Therefore dim $Z^2_{-1}(F_\varphi, F_\varphi) = 4(p-1)$. □

Lemma 5. dim $Z^2_0(F_\varphi, F_\varphi) = 4p^2 - 8p + 5$.

Proof. Any cocycle ψ belonging to this space has the generic form

$$\begin{array}{ll} \psi(X_2, X_{2k+1+r}) & = \sum_{j=3}^{2p} a^j_{2,2k+1+r} X_j,\ r = 0,1 \\ \psi(X_1, X_2) & = a^1_{12} X_1 \\ \psi(X_i, X_j) & = a^1_{ij} X_1,\ 3 \leq i,j \leq 2p \end{array}$$

Now, by the coboundary operator δ_{μ_φ} the following equations must be satisfied:

$$\begin{array}{rl} a^1_{12} + a^{2k+1}_{2,2k+1} + a^{2k+2}_{2,2k+2} & = 0,\ 1 \leq k \leq p-1 \\ a^{2t+1}_{2,2k+1} + (\varphi_k - \varphi_t) a^1_{2k+1,2t+2} + a^{2k+2}_{2,2t+2} & = 0,\ 1 \leq k \neq t \leq p-1 \\ a^{2k+2}_{2,2t+1} + (1 + \varphi_t + \varphi_k) a^1_{2k+1,2t+1} - a^{2t+2}_{2,2k+1} & = 0,\ 1 \leq k \neq t \leq p-1 \\ a^{2t+1}_{2,2k+2} - (1 + \varphi_t + \varphi_k) a^1_{2k+2,2t+2} - a^{2k+1}_{2,2t+2} & = 0,\ 1 \leq k \neq t \leq p-1 \end{array}$$

From the equations we deduce the following linearly independent cocycles:

1. ψ^1_{12}:

$$\left\{ \begin{array}{l} (X_1, X_2) \mapsto X_1 \\ (X_2, X_{2k+2}) \mapsto -X_{2k+2} \quad ,1 \leq k \leq p-1 \end{array} \right.$$

2. $\psi^{2k+1}_{2,2k+1}$ $(1 \leq k \leq p-1)$:

$$\left\{ \begin{array}{ll} (X_2, X_{2k+1}) \mapsto X_{2k+1}, & 1 \leq k \leq p-1 \\ (X_2, X_{2k+2}) \mapsto -X_{2k+2}, & 1 \leq k \leq p-1 \end{array} \right.$$

3. $\psi^{2t+2}_{2,2k+1}$ $(1 \leq k \leq p-1)$:

$$\left\{ \begin{array}{ll} (X_2, X_{2k+1}) \mapsto -(1 + \varphi_t + \varphi_k) X_{2t+2}, & 1 \leq k \neq t \leq p-1 \\ (X_{2k+1}, X_{2t+1}) \mapsto -X_1 & 1 \leq k \neq t \leq p-1 \end{array} \right.$$

4. $\psi^{2t+1}_{2,2k+2}$ $(1 \leq k \leq p-1)$:

$$\left\{ \begin{array}{ll} (X_2, X_{2k+2}) \mapsto (1 + \varphi_t + \varphi_k) X_{2t+1}, & 1 \leq k \neq t \leq p-1 \\ (X_{2k+2}, X_{2t+2}) \mapsto X_1 & 1 \leq k \neq t \leq p-1 \end{array} \right.$$

5. $\psi^{2t+2}_{2,2k+2}$ $(1 \leq k \leq p-1)$:

$$\left\{ \begin{array}{ll} (X_2, X_{2k+2}) \mapsto (\varphi_k - \varphi_t) X_{2t+2}, & 1 \leq k \neq t \leq p-1 \\ (X_{2k+2}, X_{2t+1}) \mapsto -X_1 & 1 \leq k \neq t \leq p-1 \end{array} \right.$$

To these we have to add those cocycles for which the operator δ_{μ_φ} gives no conditions:

6. $\psi^{1}_{2k+1,2k+2}$ $(1 \leq k \leq p-1)$:

$$\{(X_{2k+1}, X_{2k+2}) \mapsto X_1$$

7. $\psi^{2k+2}_{2,2k+1}$ $(1 \leq k \leq p-1)$:

$$\{(X_2, X_{2k+1}) \mapsto X_{2k+2}$$

8. $\psi^{2k+1}_{2,2k+2}$ $(1 \leq k \leq p-1)$:

$$\{(X_2, X_{2k+2}) \mapsto X_{2k+1}$$

Adding up, we obtain

$\dim\ Z^2_0 (F_\varphi, F_\varphi) = 1 + 4(p-1) + 4(p-1)(p-2) = 4p^2 - 8p + 5$

□

Lemma 6. $\dim\ Z^2_1 (F_\varphi, F_\varphi) = 2(p-1)$.

Proof. These cocycles have the generic form $\psi(X_2, X_j) = a_{2,j} X_1$ for $3 \leq j \leq 2p$. The coboundary operator gives no conditions on the coefficients, thus we have the basis
$\psi^1_{2,j}$ $(3 \leq j \leq 2p)$:

$$\{(X_2, X_j) \mapsto X_1$$

of dimension $2(p-1)$. □

Proposition 2. *If* $(\varphi) \notin \Omega$ *then* $\dim\ H^2 (F_\varphi, F_\varphi) = p - 1$.

Proof. By the preceding lemmas $\dim\ Z^2 (F_\varphi, F_\varphi) = 4p^2 - 2p$. Now $\dim\ B^2 (F_\varphi, F_\varphi) = 4p^2 - (3p-1)$, since $\dim Der(F_\varphi) = 3p - 1$ by lemma 1. □

Proposition 3. *For* $p \geq 2$ *the following holds:*

1. $B^2_{-2} (F_\varphi, F_\varphi) = Z^2_{-2} (F_\varphi, F_\varphi)$
2. $B^2_{-1} (F_\varphi, F_\varphi) = Z^2_{-1} (F_\varphi, F_\varphi)$
3. $\dim Z^2_0 (F_\varphi, F_\varphi) - \dim B^2_0 (F_\varphi, F_\varphi) = p - 1$
4. $Z^2_1 (F_\varphi, F_\varphi) = B^2_1 (F_\varphi, F_\varphi)$

Proof. The proof of *(i)*,*(ii)* and *(iv)* follows immediately by application of the coboundary operator to an element $f \in GL(2p, \mathbb{C})$. Let $\theta \in B^2_0 (F_\varphi, F_\varphi)$ be the coboundary given by

$$\begin{cases} \theta(X_1, X_2) = X_1 \\ \theta(X_2, X_{2k+1}) = \varphi_k X_{2k+1},\ 1 \leq k \leq p-1 \\ \theta(X_2, X_{2k+2}) = -(1 + \varphi_k) X_{2k+2},\ 1 \leq k \leq p-1 \end{cases}$$

It is a straightforward verification that

$$\psi_{12}^{1}+\sum_{k=1}^{p-1}\psi_{2,2k+1}^{2k+1}=\theta$$

Thus a basis of $H^2(F_\varphi, F_\varphi)$ is given by the classes $\left[\psi_{2,2k+1}^{2k+1}\right]$ for $1 \leq k \leq p-1$. □

Recall that the mapping that associates the product $\psi \circ \psi$ to a 2-cocycle ψ induces a map $sq_1 : H^2(F_\varphi, F_\varphi) \rightarrow H^3(F_\varphi, F_\varphi)$ called the Rim map [13]. The image $sq_1(\psi)$ gives the first obstruction to the integability of the class of ψ.

Corollary 2. *For any $p \geq 2$ the Rim map $sq_1 : H^2(F_\varphi, F_\varphi) \rightarrow H^3(F_\varphi, F_\varphi)$ is identically zero.*

Proof. For the representatives $\psi_{2,2k+1}^{2k+1}$ $(1 \leq k \leq p-1)$ of the nontrivial cohomology classes we have $\psi_{2,2k+1}^{2k+1} \circ \psi_{2,2k+1}^{2k+1} = 0$ since $X_2 \notin \psi_{2,2k+1}^{2k+1}(F_\varphi, F_\varphi)$. Therefore the mapping is everywhere zero. □

It can also be easily seen that the cocycles $\psi_{2,2k+1}^{2k+1}$ are linearly expandable, i.e., they define a linear deformation $F_\varphi + t\psi_{2,2k+1}^{2k+1}$. The importance of this fact follows from the next

Theorem 3. *If $(\varphi) \notin \Omega$, almost any infinitesimal deformation of F_φ lies in the set $\{F_\varphi \mid (\varphi) \notin \Omega\}$.*

Proof. We have seen that any nontrivial cohomology class admits a linearly expandable representative. For any $1 \leq k \leq p-1$ we have

$$F_\varphi + t\psi_{2,2k+1}^{2k+1} \simeq F(\varphi_1, .., \varphi_k + t, .., \varphi_{p-1})$$

and for small t we have $(\varphi_1, .., \varphi_k + t, .., \varphi_{p-1}) \notin \Omega$. □

Corollary 3. *If $(\varphi) \notin \Omega$, any linear deformation $F_\varphi + t\psi_{2,2k+1}^{2k+1}$ belongs to the family $\{F_\varphi \mid (\varphi) \in \mathbb{C}^{p-1}\}$.*

This corollary shows that frobeniusian model Lie algebras admitting deformations not belonging to the model are relatively scarce.

Corollary 4. *Let $\mathfrak{g}$ be a non-model frobeniusian Lie algebra. Then $\mathfrak{g}$ contracts on some model Lie algebra F_φ for $(\varphi) \in \Omega$.*

Remark 3. *In particular, from this corollary we deduce that there exists only a finite number of hyperplanes in $\mathbb{C}^{p-1}$ such that the algebras F_φ whose parameters (φ) belong to these spaces allow deformations which lie outside the multiple model.*

Corollary 5. *If* $\mathfrak{g} \notin \{F_\varphi \mid (\varphi) \notin \Omega\}$ *is frobeniusian, then* $\dim\ H^2(F_{\varphi_0}, F_{\varphi_0}) > p-1$ *for any* F_{φ_0} *such that* $F_{\varphi_0} \in \overline{\mathcal{O}(\mathfrak{g})}$.

Conclusion. *It has been shown that most of the Lie algebras admitting a linear form whose differential is symplectic already belong to the model, which constitutes a surprising result when compared with other properties such as the* r*-contact systems. The cohomology of the family* $\{F_\varphi\}$ *shows that the search for non-model frobeniusian Lie algebras reduces to study the parameters lying in* Ω*. This implies that these algebras depend on less than* $p-1$ *parameters, due to the relations they satisfy. This translates in either more generators of the corresponding Lie algebra or eigenvalues of multiplicity* $r \geq 2$ *for the semisimple derivations of the algebra. Due to these particularities, their cohomology has to be computed for each case, since the structure depends heavily on the values the parameters take within* Ω.

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