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A Note on Symmetric Differential Operators and Binomial Coefficients

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Abstract

In this note we derive some identities concerning the binomial coefficients by considering a certain n-th order symmetric differential operator on $\mathbb{R}^m$ associated to the function $p(x,\xi)(x \in \mathbb{R}^m)$ which is a homogeneous polynomial in ξ.

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Introduction

Let $\binom{n}{k}$ denote the binomial coefficients, namely

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1}$$

Various formulas for the binomial coefficients are well known (see e.g. [1], [2]). For example we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{2}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \tag{3}$$

$$\sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} \binom{n}{k} = 0 \qquad (n \geq r), \tag{4}$$

which are easily obtained from (1). (The last one is obtained by differentiating (1) r times relative to x, dividing by $r!$, and putting $x = -1$.)

In this note we consider a certain linear symmetric differential operator, and derive some identities concerning the binomial coefficients (Corollaries 5 and 6).

1. Symmetric differential operators

Let $C_0^\infty(\mathbb{R}^m)$ denote the space of complex-valued C^∞ functions on $\mathbb{R}^m$ with compact support. Suppose the space $C_0^\infty(\mathbb{R}^m)$ is endowed with the inner product $(\cdot,\cdot)$ defined by

$$(f,g) := \int_{\mathbb{R}^m} f(x)\overline{g(x)}\,dx_1\cdots dx_m \qquad (f,g\in C_0^\infty(\mathbb{R}^m)).$$

Let D_j $(j=1,\ldots,m)$ denote the differential operator $\frac{1}{i}\frac{\partial}{\partial x_j}$ $(i:=\sqrt{-1})$. Then, D_j is a *symmetric* operator, namely,

$$(D_j f,g) = (f,D_j g) \qquad (f,g\in C_0^\infty(\mathbb{R}^m))$$

holds.

Let us consider a function $p(x,\xi)$ of variables $(x_1,\ldots,x_m,\xi_1,\ldots,\xi_m)$ which is a polynomial in ξ_j's :

$$p(x,\xi) = \sum_{p=0}^{n}\left[\sum_{j_1,j_2,\ldots,j_p} a^{j_1j_2\cdots j_p}(x)\xi_{j_1}\xi_{j_2}\cdots\xi_{j_p}\right],$$

where $a^{j_1j_2\cdots j_p}(x)$'s are symmetric with respect to the indeces $j_1,j_2,\ldots,j_p$.

The function $p(x,\xi)$ is regarded as an "observable" in the phase space $T^*\mathbb{R}^m$ of classical mechanics. In theory of quantum mechanics the classical observable $p(x,\xi)$ corresponds to a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^m)$ according to the corresponding rule of variables:

$$\xi_j \mapsto D_j, \qquad x_j \mapsto x_j\times.$$

We consider the symmetric (formally self-adjoint) operator corresponding to the homogeneous polynomial of degree n given by

$$p_n(x,\xi) = \sum_{j_1,j_2,\ldots,j_n} a^{j_1j_2\cdots j_n}(x)\xi_{j_1}\xi_{j_2}\cdots\xi_{j_n}.$$

By applying the corresponding rule directly to $p_n(x,\xi)$ we get the n-th order differential operator

$$P_n = \sum_{j_1,j_2,\ldots,j_n} a^{j_1j_2\cdots j_n}(x)D_{j_1}D_{j_2}\cdots D_{j_n}.$$

Lemma 1 *The adjoint operator P_n^* of P_n is given by*

$$\begin{aligned} P_n^* &= \sum_{j_1,\ldots,j_n} D_{j_1}\cdots D_{j_n}\left(\bar{a}^{j_1\cdots j_n}(x)\,\cdot\,\right) \\ &= \sum_{p=0}^{n}\binom{n}{p}\sum_{j_{p+1},\ldots,j_n}\Big(\sum_{j_1,\ldots,j_p} D_{j_1}\cdots D_{j_p}\bar{a}^{j_1\cdots j_n}(x)\Big)D_{j_{p+1}}\cdots D_{j_n}, \end{aligned}$$

where $\bar{a}^{j_1\cdots j_n}(x)$ denotes the complex conjugate of $a^{j_1\cdots j_n}(x)$.

Remark The property $(P_n^*)^* = P_n$ (formally) derives the formula (4). In fact, by virtue of Lemma 1 we have

$$(P_n^*)^* = \sum_{p=0}^{n}(-1)^p\binom{n}{p}\Big\{\sum_{q=0}^{n-p}\binom{n-p}{q}\sum_{j_1,\ldots,j_n}\big(D_{j_1}\cdots D_{j_{p+q}}a^{j_1\cdots j_n}(x)\big)D_{j_{p+q+1}}\cdots D_{j_n}\Big\}.$$

The $(n-r)$-th order differential term of $(P_n^*)^*$ is given by

$$\sum_{p+q=r}(-1)^p\binom{n}{p}\binom{n-p}{q}\sum_{j_1,\ldots,j_n}\big(D_{j_1,\ldots,j_r}a^{j_1\cdots j_r}(x)\big)D_{j_{r+1}}\cdots D_{j_n}.$$

Hence, for $1 \le r \le n$ we have

$$\begin{aligned} 0 &= \sum_{p+q=r}(-1)^p\binom{n}{p}\binom{n-p}{q} \\ &= \sum_{p=0}^{r}(-1)^p\binom{n}{n-p}\binom{n-p}{r-p} = \sum_{p=0}^{r}(-1)^p\binom{n}{n-p}\binom{n-p}{n-r}, \end{aligned}$$

that is nothing but the formula (4). □

In order to obtain the symmetric operator P corresponding to $p_n(x,\xi)$ we put

$$\begin{aligned} P &= \sum_{j_1,\ldots,j_n} a^{j_1\cdots j_n}(x)D_{j_1}\cdots D_{j_n} \\ &\quad + \sum_{p=1}^{n} c_{n-p}\left[\sum_{j_1,\ldots,j_n}\big(D_{j_1}\cdots D_{j_p}a^{j_1\cdots j_n}(x)\big)D_{j_{p+1}}\cdots D_{j_n}\right], \end{aligned} \tag{5}$$

where $a^{j_1\cdots j_n}(x)$'s are real-valued functions, and c_{n-p}'s are complex constants.

Proposition 2 *The operator P is symmetric, i.e., $P^* = P$ if and only if the coefficients c_{n-p} $(p = 1, 2, \ldots n)$ satisfy*

$$\begin{aligned} c_{n-p} &= (-1)^p\bar{c}_{n-p} + (-1)^{p-1}\binom{n-p+1}{1}\bar{c}_{n-p+1} \\ &\quad + (-1)^{p-2}\binom{n-p+2}{2}\bar{c}_{n-p+2} + \cdots \\ &\quad \cdots - \binom{n-1}{p-1}\bar{c}_{n-1} + \binom{n}{p}. \end{aligned} \tag{6}$$

Proof. The assertion is directly derived by comparing the coefficients of $(n-p)$-th order differential terms in P and P^*. □

As examples of symmetric operators of the form (5) we have the following:

$$\sum_j a^j(x)D_j + \frac{1}{2}\sum_j D_j a^j(x),$$
$$\sum_{j,k} a^{jk}(x)D_jD_k + \sum_k \big(\sum_j D_j a^{jk}(x)\big)D_k,$$
$$\sum_{j,k,l} a^{jkl}(x)D_jD_kD_l + \frac{3}{2}\sum_{k,l}\big(\sum_j D_j a^{jkl}(x)\big)D_kD_l - \frac{1}{4}\sum_{j,k,l} D_jD_kD_l a^{jkl}(x).$$

Observing these examples we assume the coefficients c_{n-p} $(p = 1, 2, \ldots, n)$ to be

$$c_{n-p} = \begin{cases} \text{a real number} & (p : \text{odd}) \\ 0 & (p : \text{even}) \end{cases} \tag{7}$$

Theorem 3 *For any $n \in \mathbb{N}$, and any real valued functions $a^{j_1\cdots j_n}(x)$ there exists an unique n-th order symmetric differential operator P of the form (5) satisfying the condition (7).*

Proof. First we show the existence of P (cf. [3, Lemma 4.2]). Let $Q_0 := \sum a^{j_1\cdots j_n}(x)D_{j_1}\cdots D_{j_n}(= P_n)$. Put

$$Q_1 := \frac{1}{2}(Q_0 + Q_0^*).$$

Then, by means of Lemma 1 Q_1 is a symmetric operator with the n-th order term being equal to Q_0, and the coefficients

$$\frac{1}{2}\binom{n}{p}\sum_{j_1,\ldots,j_p} D_{j_1}\cdots D_{j_p}a^{j_1\cdots j_p\cdots j_n}(x)$$

of the $(n-p)$-th order term of Q_1 are real if p is even. Let P_{n-2} denote the $(n-2)$-th order term of Q_1, and put

$$Q_2 := Q_1 - \frac{1}{2}(P_{n-2} + P_{n-2}^*).$$

Then, Q_2 is a symmetric operator of the form (5) with c_{n-p} being real and $c_{n-2} = 0$.

Next, let P_{n-4} be the $(n-4)$-th order term of Q_2, and put

$$Q_4 := Q_2 - \frac{1}{2}(P_{n-4} + P_{n-4}^*).$$

Then, Q_4 is a symmetric operator of the form (5) with c_{n-p} being real and $c_{n-2} = c_{n-4} = 0$. Thus by continuing this process we get $Q_2, Q_4, Q_6, \ldots$, and we obtain the required operator P as Q_{n-1} if n is odd, or Q_n if n is even.

Next, we show that the coefficients c_{n-p} is uniquely determined by the condition (6) under the assumption (7).

Suppose <u>n is odd</u>. The condition (6) for $p = 1, 2, \dots$ gives a system of linear equations for $c_{n-1}, c_{n-3}, \dots, c_2, c_0$ as follows:

$$\begin{aligned}
2c_{n-1} &= \binom{n}{1}, \\
\binom{n-1}{1} c_{n-1} &= \binom{n}{2}, \\
2c_{n-3} + \binom{n-1}{2} c_{n-1} &= \binom{n}{3}, \\
\binom{n-3}{1} c_{n-3} + \binom{n-1}{3} c_{n-1} &= \binom{n}{4}, \\
\cdots\cdots\cdots & \\
\binom{2}{1} c_2 + \binom{4}{3} c_4 + \cdots\cdots + \binom{n-1}{n-2} c_{n-1} &= \binom{n}{n-1}, \\
2c_0 + \binom{2}{2} c_2 + \binom{4}{4} c_4 + \cdots\cdots + \binom{n-1}{n-1} c_{n-1} &= \binom{n}{n}.
\end{aligned}$$

It is easy to see that the rank of the $(n \times (n+1)/2)$-matrix of the coefficients of the above linear equations is equal to $(n+1)/2$. Hence, the solution (if exists) is unique.

If <u>n is even</u>, the linear equations for $c_{n-1}, c_{n-3}, \dots, c_1$ is the following:

$$\begin{aligned}
2c_{n-1} &= \binom{n}{1}, \\
\binom{n-1}{1} c_{n-1} &= \binom{n}{2}, \\
2c_{n-3} + \binom{n-1}{2} c_{n-1} &= \binom{n}{3}, \\
\binom{n-3}{1} c_{n-3} + \binom{n-1}{3} c_{n-1} &= \binom{n}{4}, \\
\cdots\cdots\cdots & \\
2c_1 + \binom{3}{2} c_3 + \cdots\cdots + \binom{n-1}{n-2} c_{n-1} &= \binom{n}{n-1}, \\
\binom{1}{1} c_1 + \binom{3}{3} c_3 + \cdots\cdots + \binom{n-1}{n-1} c_{n-1} &= \binom{n}{n}.
\end{aligned}$$

This system similarly derives the uniqueness of the solution. □

2. Properties of binomial coefficients

From the system of linear equations for $c_{n-1}, c_{n-3}, \ldots$ in the preceding section we have the following.

Theorem 4 *Let $1 \le k \le (n+1)/2$. The following two systems of linear equations for $c_{n-1}, c_{n-3}, \ldots, c_{n-2k+1}$ are equivalent each other :*

$$\begin{bmatrix} 2 & & & & & \\ \binom{n-1}{2} & 2 & & & \mathbf{0} & \\ \binom{n-1}{4} & \binom{n-3}{2} & 2 & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-4} & \binom{n-3}{2k-6} & \cdots & \binom{n-2k+5}{2} & 2 & \\ \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+5}{4} & \binom{n-2k+3}{2} & 2 \end{bmatrix} \begin{bmatrix} c_{n-1} \\ c_{n-3} \\ \vdots \\ \vdots \\ c_{n-2k+3} \\ c_{n-2k+1} \end{bmatrix} = \begin{bmatrix} \binom{n}{1} \\ \binom{n}{3} \\ \vdots \\ \vdots \\ \binom{n}{2k-3} \\ \binom{n}{2k-1} \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} \binom{n-1}{1} & & & & & \\ \binom{n-1}{3} & \binom{n-3}{1} & & & \mathbf{0} & \\ \binom{n-1}{5} & \binom{n-3}{3} & \binom{n-5}{1} & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-3} & \binom{n-3}{2k-5} & \cdots & \cdots & \binom{n-2k+3}{1} & \\ \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \cdots & \binom{n-2k+3}{3} & \binom{n-2k+1}{1} \end{bmatrix} \begin{bmatrix} c_{n-1} \\ c_{n-3} \\ \vdots \\ \vdots \\ c_{n-2k+3} \\ c_{n-2k+1} \end{bmatrix} = \begin{bmatrix} \binom{n}{2} \\ \binom{n}{4} \\ \vdots \\ \vdots \\ \binom{n}{2k-2} \\ \binom{n}{2k} \end{bmatrix}. \tag{9}$$

Proof. The system (8) of linear equations is obtained from (6) in Proposition 2 for odd $p = 1, 3, \ldots, 2k-1$. On the other hand, the system (9) is obtained from (6) for even $p = 2, 4, \ldots, 2k$. These two systems of linear equation have the same solution associated to the unique symmetric differential operator P (Theorem 3). □

Note Cramer's formulas for the solution c_{n-2k+1} of (8) and (9), and we have the following.

Corollary 5 *For $n, k \in \mathbb{N}$ with $1 \le k \le (n+1)/2$ we have the following* *identity, which is equal to* $(-1)^{k-1}c_{n-2k+1}$ *:*

$$\frac{1}{2^k}\begin{vmatrix} \binom{n}{1} & 2 & & & \\ \binom{n}{3} & \binom{n-1}{2} & 2 & & \text{\Large 0} \\ \binom{n}{5} & \binom{n-1}{4} & \binom{n-3}{2} & \ddots & \\ \vdots & \vdots & \vdots & & 2 \\ \binom{n}{2k-1} & \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+3}{2} \end{vmatrix}$$

$$= \frac{(n-2k-1)!!}{(n-1)!!}\begin{vmatrix} \binom{n}{2} & \binom{n-1}{1} & & & \\ \binom{n}{4} & \binom{n-1}{3} & \binom{n-3}{1} & & \text{\Large 0} \\ \binom{n}{6} & \binom{n-1}{5} & \binom{n-3}{3} & \ddots & \\ \vdots & \vdots & \vdots & & \binom{n-2k+3}{1} \\ \binom{n}{2k} & \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \binom{n-2k+3}{3} \end{vmatrix}. \quad (10)$$

Remark If $k = 1$, (10) means

$$\frac{1}{2}\binom{n}{1} = \frac{1}{n-1}\binom{n}{2} \ (= c_{n-1}).$$

Table for c_{n-p}

n	c_{n-1}	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	c_{n-6}	c_{n-7}	c_{n-8}	c_{n-9}	c_{n-10}
1	$\frac{1}{2}$									
2	1	0								
3	$\frac{3}{2}$	0	$-\frac{1}{4}$							
4	2	0	-1	0						
5	$\frac{5}{2}$	0	$-\frac{5}{2}$	0	$\frac{1}{2}$					
6	3	0	-5	0	3	0				
7	$\frac{7}{2}$	0	$-\frac{35}{4}$	0	$\frac{21}{2}$	0	$-\frac{17}{8}$			
8	4	0	-14	0	28	0	-17	0		
9	$\frac{9}{2}$	0	-21	0	63	0	$-\frac{153}{2}$	0	$\frac{31}{2}$	
10	5	0	-30	0	126	0	-255	0	155	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2}\binom{n}{1}$	0	$-\frac{1}{4}\binom{n}{3}$	0	$\frac{1}{2}\binom{n}{5}$	0	$-\frac{17}{8}\binom{n}{7}$	0	$\frac{31}{2}\binom{n}{9}$	0

Finally, by considering the case $n = 2k$ we have the following from the last equation in (9).

Corollary 6 *For even* $n\,(\in \mathbb{N})$ *we have*

$$
\begin{aligned}
&\sum_{k=1}^{n/2} c_{n-2k+1} \\
&= \sum_{k=1}^{n/2} \frac{(-1)^{k-1}}{2^k}
\left|\begin{array}{ccccc}
\binom{n}{1} & 2 & & & \\
\binom{n}{3} & \binom{n-1}{2} & 2 & & \mbox{\Large 0} \\
\binom{n}{5} & \binom{n-1}{4} & \binom{n-3}{2} & \ddots & \\
\vdots & \vdots & \vdots & & 2 \\
\binom{n}{2k-1} & \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+3}{2}
\end{array}\right| \\
&= \sum_{k=1}^{n/2} (-1)^{k-1} \frac{(n-2k-1)!!}{(n-1)!!}
\left|\begin{array}{ccccc}
\binom{n}{2} & \binom{n-1}{1} & & & \\
\binom{n}{4} & \binom{n-1}{3} & \binom{n-3}{1} & & \mbox{\Large 0} \\
\binom{n}{6} & \binom{n-1}{5} & \binom{n-3}{3} & \ddots & \\
\vdots & \vdots & \vdots & & \binom{n-2k+3}{1} \\
\binom{n}{2k} & \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \binom{n-2k+3}{3}
\end{array}\right| \\
&= 1\,.
\end{aligned}
$$

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