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A Note on a Heat Invariant and the Ricci Flow on Surfaces

By

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Abstract

In this short note, we consider the monotonicity of the heat invariant $a_2(g)$ for a Riemannian metric g under the normalized Ricci flow on a closed surface. We show that $a_2(g(t))$ is decreasing under the normalized Ricci flow $g(t)$ in the space of metrics of non-positive curvature.

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Introduction

Let M be an n dimensional compact C^∞ manifold without boundary. Given a C^∞ Riemannian metric g on M. Then, we have the Laplace-Beltrami operator $\Delta = \Delta_g$ acting on C^∞ functions on M, whose spectrum consists of non-negative eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \uparrow \infty.$$

It is well known concerning $\{\lambda_k\}$ that we have the asymptotic expansion

$$\sum_{k=0}^{\infty} e^{-\lambda_k s} \underset{s\downarrow 0}{\sim} (4\pi s)^{-n/2}\{a_0(g) + a_1(g)s + \cdots + a_j(g)s^j + \cdots\},$$

where the coefficients $a_j(g)$ are the heat invariants, first three of which are given by

$$\begin{aligned} a_0(g) &= \mathrm{Vol}(M,g), \quad a_1(g) = \frac{1}{6}\int_M S\, dV_g, \\ a_2(g) &= \frac{1}{360}\int_M [2|R|^2 - 2|\mathrm{Ric}|^2 + 5S^2] dV_g. \end{aligned} \tag{0.1}$$

Here, $S = S(g)$ is the scalar curvature, Ric $=$ Ric(g) is the Ricci tensor and $R = R(g)$ is the Riemannian curvature tensor of (M, g). In the case where M is a closed surface $a_2(g)$ reduces to

$$a_2(g) = \frac{1}{60}\int_M S^2\, dV_g. \tag{0.2}$$

By noticing the Schwarz inequality and the Gauss-Bonnet theorem, we have

$$\int_M S^2 dV_g \geq \Big(\int_M S\, dV_g\Big)^2 \Big/ \mathrm{Vol}(M,g) = \{4\pi\chi(M)\}^2/\mathrm{Vol}(M,g) \tag{0.3}$$

($\chi(M)$ being the Euler characteristic). Thus, the functional $a_2(g)$ attains its minimum at the metric of constant curvature in the space of C^∞ metrics on M with fixed volume.

In the present note we consider the behavior of the invariant $a_2(g)$ given by (0.2) under the normalized Ricci flow on closed surfaces. The normalized Ricci flow (introduced by Hamilton [3]) on a closed surface M is a one-parameter C^∞ family $g(t)$ $(t \geq 0)$ of C^∞ metrics on M which is evolved by the equation

$$\frac{\partial}{\partial t} g_{ij} = (s - S) g_{ij},$$

where s denotes the average scalar curvature given by

$$s := \Big(\int_M S\, dV_g\Big) \Big/ \mathrm{Vol}\,(M,g).$$

Here $\mathrm{Vol}(M,g) = \mathrm{Vol}(M,g(t))$ is constant along $g(t)$, and s is also constant, in fact

$$s = 4\pi\chi(M)/\mathrm{Vol}(M,g(0)).$$

Note that the stationary points of the normalized Ricci flow are the metrics of constant curvature s.

The main result of this note is the following.

Theorem. *Suppose M is a closed surface with $\chi(M) < 0$. Let $g(t)$ $(t \geq 0)$ be a normalized Ricci flow on M such that*

$$S(g(0)) \leq 0. \tag{0.4}$$

Then, $a_2(g(t))$ is monotonously decreasing in t, namely

$$\frac{d}{dt} a_2(g(t)) \leq 0,$$

and the equality holds if and only if $g(t)$ is a metric of (negative) constant curvature s.

Remark. Let $g(t)\,(t \geq 0)$ be any Ricci flow on a surface M with $\chi(M) < 0$. Then, it has been shown by Hamilton [3, Theorems 4.6 and 4.9] that there exists t_0 such that $S(g(t)) \leq 0$ for $\forall t \geq t_0$, and that $g(t)$ converges to a metric of constant negative curvature s as $t \to \infty$.

1. Proof of the theorem

Let $g(t)$ is a one-parameter C^∞ family of C^∞ metrics on the closed surface M. We put

$$h_{ij} = h_{ij}(t) := \frac{\partial}{\partial t} g_{ij}, \quad h^{ij} := g^{ik} h_{kj} \big(= \sum_k g^{ik} h_{kj} \big),$$

which are symmetric 2-tensor fields on M. Put

$$F(t) := \int_M \{S(g(t))\}^2 \, dV_{g(t)} \; (= 60 \, a_2(g(t)) \,).$$

Then, we have the following.

Lemma. *The derivative of $F(t)$ is given by*

$$\frac{dF}{dt} = \int_M \big[2(\nabla_i \nabla_j S) h^{ij} + \big(2\Delta S - \frac{1}{2} S^2 \big) h_j^j \big] \, dV_g \quad (h_j^j := h_{ij} g^{ij}). \tag{1.1}$$

If $g(t)$ is a normalized Ricci flow, then

$$\frac{dF}{dt} = \int_M \big[-2 S \Delta S + S^2 (S - s) \big] \, dV_g. \tag{1.2}$$

Proof. We easily obtain the formulas (1.1) by straightforward calculations following the variation formulas for the volume element, the Levi-Civita connection and its associated curvature tensors (given in [2], for example). The formula (1.2) is derived from (1.1) for $h_{ij} = (s - S) g_{ij}$. □

Put $\widetilde{S} = S - s$. Then, we have

$$\int_M \widetilde{S} \, dV_g = 0, \tag{1.3}$$

and the formula (1.2) is rewritten as

$$\begin{aligned} \frac{dF}{dt} &= \int_M \big[-2 \widetilde{S} \Delta \widetilde{S} + \widetilde{S}^2 S + s S^2 - s^2 S \big] dV_g \\ &= -2 \int_M \widetilde{S} \Delta \widetilde{S} \, dV_g + \int_M \widetilde{S}^2 S \, dV_g + s \int_M S^2 dV_g - s^3 \mathrm{Vol}(M, g). \end{aligned}$$

By virtue of (1.3) we have

$$\int_M \widetilde{S} \Delta \widetilde{S} \, dV_g \geq \lambda_1 \int_M \widetilde{S}^2 dV_g$$

for the non-zero first eigenvalue $\lambda_1 = \lambda_1(g)$ of $\Delta = \Delta_g$. Moreover, by virtue of (0.3) we have

$$\int_M S^2 dV_g \geq s^2 \mathrm{Vol}(M, g).$$

Hence, we get

$$\frac{dF}{dt} \leq -2\lambda_1 \int_M \widetilde{S}^2 dV_g + \int_M \widetilde{S}^2 S\, dV_g = -\int_M (2\lambda_1 - S)\widetilde{S}^2 dV_g \tag{1.4}$$

if $s \leq 0$ (which means $\chi(M) \leq 0$). Thus, we have the following.

Proposition. *Suppose M is a closed surface with $\chi(M) \leq 0$. Let $g(t)$ be a normalized Ricci flow on M. If*

$$\int_M \{2\lambda_1(g(t)) - S(g(t))\}\left(\widetilde{S}(g(t))\right)^2 dV_{g(t)} \geq 0,$$

then, we have

$$\frac{d}{dt} a_2(g(t)) \leq 0\,.$$

Proof of Theorem. By applying the maximum principle to the evolution equation of $S(g(t))$, we get $S(g(t)) \leq 0$ for $\forall t \geq 0$ if $S(g(0)) \leq 0$ ([3, Theorem 3.2]). Moreover, we notice that in (1.4) the equality holds if and only if $\widetilde{S} = 0$. Thus, we obtain Theorem as a corollary of Proposition. □

It is reserved for future discussions to clarify what occurs if the condition (0.4) is removed.

Concluding Remarks

1. The functional $F(g) = \int_M S^2 dV_g$ (called "Calabi energy") was first considered by Calabi [1], and it was shown that in the case where M is a closed surface the critical points of F are metrics of constant curvature (which attain the minimum) and that F is decreasing under the "Calabi flow".

2. As a recent result related to this note we refer to [4], in which it was proved that the determinant of the Laplacian on a closed surface monotonously increases under the normalized Ricci flow.

3. The heat invariant $a_2(g)$ given (0.1) for n dimensional compact Riemannian manifolds has been considered in "Spectral Geometry," particularly to characterize the flat metrics by the spectrum (see [5], [6]).

References

[1] E. Calabi, Extremal Kähler metrics, in *Seminar of Differential Geometry*, ed. S.T. Yau, Annals of Math. Stud. **102**, Princeton Univ. Press. 1982, 259-290.

[2] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Amer. Math. Soc. 2004.

[3] R.S. Hamilton, The Ricci flow on surfaces, Contemporary Math. **71**(1988), 237-262.

[4] A. Kokotov and D. Korotkin, Normalized Ricci flow on Riemannian surfaces and determinant of Laplacian, Lett. Math. Phys. **71**(2005), 241-242.

[5] R. Kuwabara, On the characterization of flat metrics by the spectrum, Comment. Math. Helv. **55**(1980), 427-444.

[6] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, Tôhoku Math. Journ. **25**(1973), 391-403.