J. Math. Tokushima Univ.
Vol. 32 (1998), 1-13

Decompositions of Boolean functions and hypergraphs

By

Toru ISHIHARA

Department of Mathematics and Comuputer Sciences,
Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minamijosanjima, Tokushima 770, JAPAN
e-mail adress, ishihara@ias.tokushima-u.ac.jp
(Received September 12,1998)

Abstract

Boolean functions are closely reltated to hypergraphs. In fact, Ibaraki and Kameta(1993) sutudied relations between coteries (intersecting simple hypergraphs) and positive Boolean functions. In this paper, we shall show that the set of all simple hypergraphs is lattice-isomorphic to the set of all positive Boolean functions. A decompositions of a given function into a conjunction of self-dual functions were studied by Ibaraki, Kameta(1993) and Bioch, Ibaraki(1995). For a given dual-minor function, using a certain corresponding hypergraph, we shall give the general condition for the decomposition.

1991 Mathematics Subject Classification. Primary 06E30; Secondary 05C65

Introduction

A coterie is an intersecting simple hypergraph on a finite set $U = \{1, 2, \cdots, n\}$. It is used as a mechanism to realize mutual exclusion in a distributed system [7],[6]. Coteries was sutudied by using Boolean functions in [8]. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$, the set of all true vectors is denoted by $T(f)$. The set of all minimal true vectors is written by $minT(f)$. By identifing a vector in $\{0,1\}^n$ with a subset of U, we can consider $minT(f)$ as a simple hypergraph. There is the one to one corresponence between the set of all positve Boolean functions and a set of simple hypergraphs on U [8]. A positive function f is dual-minor(resp. self-dual) if and only if $minT(f)$ is a coterie(resp. *ND*-coterie) [8]. In this paper, firstly, we shall describe some relations between hypergraphs and Boolean functions. Main results were alredy given in [3],[6], [8], but we treate them in a slighly different manner. A transversal hypergraph is very important in our treatment. For a given hypergraph H, the family of minimal edges which meet all the edges of

H constitute a simple hypergraph, called the transversal hypergraph of H, and denoted by TrH. A simple hypergraph H is a coterie if and only if $H \subseteq TrH$. For any function f, we have $TrT(f) \subseteq T(f^d)$, where f^d is the dual function of f. From these facts, we get many relations between coteries and Boolean functions.

In Section 2, for a Boolean function f, we divide $\{0,1\}^n$ into four parts $TT(f), TF(f), FT(f)$ and $FF(f)$, where $TT(f) = \{X \in \{0,1\}^n; f(X) = 1, f^d(X) = 1\}, TF(f) = \{X \in \{0,1\};\ f(X) = 1,\ f^d(X) = 0\}$ and so on. A function is dual-minor if and only if $TF(f) = \emptyset$. The part $FT(f)$ plays an important roll to investigate a dual-minor functin f. We introduce a natural order and natural operations into the set of all simple hypergraphs and show that it is lattice-isomorphic to the set of all positive Boolean functions, the free distributive latice, with the least element and the greatest element adjointed, in Section 3. We describe, in Section 4, relations between coteries and positive Boolean functions as explained above. We investigate the set of all simple hypergraphs as a distributive lattice in Section 5.

Decompositions of coteries are reduced to decompositions of Boolean functins [8],[3]. Bioch and Ibaraki [3] obtained a condition for the decomposition of a given dual-minor functin f into a conjugation of self-dual functions; $f = (f + f^d g_1)(f + f^d g_2) \cdots (f + f^d g_k)$, where each g_i is self-dual. Using the part $FT(f)$, we investigate conditon for the general deconposition of a dual-minor function f in Secton 6. For a positive dual-minor function f, as will be discussed in the last Secttion, the problem of finding a decomposition of f into a conjugation of positive selu-dual functions is reduced to the problem of finding a family of maximal intersecting subsets $m_1, m_2, \cdots, m_k$ of $minFT(f)$ such that $\cup_{i=1}^k m_i = minFT(f)$. From our results, we can show that any decomposition of a dual-minor function is one given by Bioch and Ibaraki.

2. Definitions and basic facts

Let $U = \{u_1, u_2, \cdots, u_n\}$ be a finite set. Then, U will always stand for a set of size $n \geq 1$, and we identify U with $[n] = \{1, 2, \cdots, n\}$. The power set of U is the set of all subsets of U and it is denoted by $P_n = P(U)$. With a natural order, namely $X \leq Y$ if $X \subseteq Y$, P_n is a partially odered set or briefly a poset. We shall identify P_n with $\{0,1\}^n$ such that $X = (x_1, x_2, \cdots, x_n) \in \{0,1\}^n$ represents the subset which contains the i-th element if and only if $x_i = 1$. A subset of P_n represents a hypergraph which has no multiedge but may have loops. Let $H(P_n)$ denotes the set of all such hypergraphs. A Boolean function is a mapping $f : \{0,1\}^n \to \{0,1\}$. The set of all true vectors $\{X \in \{0,1\}^n; f(X) = 1\}$ is denoted by $T(f)$. Then, it represents a hypergraph in $H(P_n)$. The mapping of the set of all Boolean functions $\{f : \{0,1\}^n \to \{0,1\}\}$ to $H(P_n)$ which takes f to $T(f)$ gives a one to one correspondence. A Boolean function is called a function in short. The *dual* of a functin f, denoted f^d, is defined by $f^d(X) = \bar{f}(\bar{X})$, where $\bar{f}$ and $\bar{X}$ denote the complements of f and x respectively. The *contra-dual* f^* of f is defined by $f^*(X) = f(\bar{X})$. Boolean funtions are ordered naturally, i.e., $f \leq g$ if and only if $f(X) \leq g(X)$ for all $X \in \{0.1\}^n$. It is evident that $f \leq g$ if and only if $T(f) \subseteq T(g)$. A function f is called *dual-minor* if $f \leq f^d$, *dual-major* if $f \geq f^d$ and *self-dual* if $f = f^d$. For a function f, we put $F(f) = \{X \in \{0,1\}^n; f(X) = 0\}$, then $T(\bar{f}) = F(f)$. Set

$$TT(f) = T(ff^d) = T(f) \cap T(f^d) = \{X \in \{0,1\}^n; X \in T(f), \bar{X} \in F(f)\},$$

$$TF(f) = T(f\bar{f}^d) = T(f) \cap F(f^d) = T(ff^*) = \{X \in \{0,1\}^n; X \in T(f), \bar{X} \in T(f)\},$$

$$FT(f) = T(\bar{f}f^d) = F(f) \cap T(f^d) = \{X \in \{0,1\}^n; X \in F(f), \bar{X} \in F(f)\},$$

$$FF(f) = T(\bar{f}\bar{f}^d) = F(f) \cap F(f^d) = \{X \in \{0,1\}^n; X \in F(f), \bar{X} \in T(f)\}.$$

Then $T(f) \subseteq T(f^d)$ (resp. $T(f^d) \subseteq T(f)$) if and only if $TF(f) = \emptyset$ (resp. $FT(f) = \emptyset$), and we have(see Lemma 1 in [3] and Property 1.5 in [8])

Lemma 1. Let f be a Boolean function.

(1) $FF(f) = \{X \in \{0,1\}^n; \bar{X} \in TT(f)\}$.
(2) f is dual-minor if and only if $TF(f) = \emptyset$.
(3) f is dual-major if and only if $FT(f) = \emptyset$.
(4) f is self-dual if and only if $TF(f) = FT(f) = \emptyset$.

For any set A, we denote its cardinal number by $|A|$. From (1) of the above lemma, we get $|T(f)| = |TT(f)| + |TF(f)| = |FF(f)| + |TF(f)| = |F(f^d)| = |P_n| - |T(f^d)| = 2^n - |T(f^d)|$.

Proposition 2. Let f be a Boolean function.
(1) $|T(f^d)| = 2^n - |T(f)|$.
(2) If it is dual-minor, $|T(f)| \leq 2^{n-1} \leq |T(f^d)|$.
(3) If it is dual-mayjor, $|T(f^d)| \leq 2^{n-1} \leq |T(f)|$.
(4) If it is self-dual, $|T(f)| = 2^{n-1}$.
(5) If it satisfies $|T(f)| = 2^{n-1}$, and it is dual-minor or dual-major, it is self-dual.

Let $H \in H(P_n)$ be a hypergraph. An edge $X \in H$ is called *minimal* (resp. *maximal*) if there is no edge $Y \in H$ with $Y < X$(resp.$Y > X$). If edges $X, Y \in H$ satysfy that $X \subseteq Y$, then $X = Y$, H is called a *simple* hypergraph. Let denote the family of all simple hypergraphs on $U = [n]$ by SH_n. Put $minH = \{X \in H; X \text{ is minimal}\}$ and $maxH = \{X \in H; X \text{ is maximal}\}$. Then, $minH$ and $maxH$ are simple for every $H \in H(P_n)$. If a set $T \in P_n$ satisfies $T \cap X \neq \emptyset$ for every $X \in H$, it is called a *transversal* of H. The family of minimal tansversals of H constitutes a simple hypergraph on P_n called the *transversal hypergrah* of H, and denoted by TrH. If any two eddges X, Y of a hyergrah H satisfy $X \cap Y \neq \emptyset$, it is said to be *intersecting*. It is evident that H is intesecting if and only if it satisfies that $H \subseteq TrH$.

Proposition 3. For a Boolean function f, it holds $TrT(f) \subseteq T(f^d)$.

Proof. For a set X contained in $TrT(f)$, it holds that $X \cap Y \neq \emptyset$ for every $Y \in T(f)$. Hence, for $X \in TrT(f)$, $\bar{X}$ does not contain any $Y \in T(f)$ and $\bar{X}$ is not contained in $T(f)$, i.e., $X \in T(f^d)$.

If $T(f)$ is intersecting, $T(f) \subseteq TrT(f) \subseteq T(f^d)$, it hold $TF(f) = \emptyset$. Hence, from Lemma1 we have

Proposition 4. For a Boolean function f, if $T(f)$ is intersecting, f is dual-minor.

3. Positive function, simple hypergraphs, free distributive lattices

A Boolean function f is said to be *positive* if $X \leq Y$ implies that $f(X) \leq f(Y)$. A set(hypergraph) $H \in H(P_n)$ is called a *monotone decreasing set system* (also called an *ideal*) if $X \in H$ and $Y < X$ implies that $Y \in H$. A set H is a *monotone increasing set system*(also called a *filter*) if $X \in H$ and $Y > X$ implies that $Y \in H$. It is well known that there is the one to one correspondence between the set of all ideals (resp. filters) in $H(P_n)$ and SH_n, which takes an ideal(resp. a filter) H to $maxH$ (resp. $minH$). If a function f is positive, it is evident that $T(f)$ (resp. $F(f)$) is a filter (resp. an ideal). Thus, we have the one to one corresopndence between the set of all positive functions, which we denote by PF_n, and SH_n which sends a positive function f to the simple hypergraph $minT(f)$. We put

$$H_f = minT(f).$$

The corespondece which takes $f \in PF_n$ to $maxF(f)$ is also a bijection of PF_n onto SH_n.

A set of all Boolean functions, denoted by F_n, is a lattice with natural order and natural operations. We give a partial order in SH_n. Let $H_1, H_2 \in SH_n$. We difine $H_1 \leq H_2$ if for any edge $X_1 \in H_1$, there exists an edge $X_2 \in H_2$ such that $X_2 \leq X_1$, i.e., $X_2 \subseteq X_1$. From the definition, we get easily

Lemma 5. With the above order, SH_n is a poset. For $f_1, f_2 \in PF_n$, $f_1 \leq f_1$ if and only if $H_{f_1} \leq H_{f_2}$.

Let $H_1 = \{X_1, X_2, \cdots, X_k\}$ and $H_2 = \{Y_1, Y_2, \cdots, Y_\ell\}$ be in SH_n. A *join* $H_1 \vee H_2$ and a *meet* $H_1 \wedge H_2$ are defined by

$$H_1 \vee H_2 = min\{X_1, X_2, \cdots, X_k, Y_1, Y_2, \cdots, Y_\ell\},$$

$$H_1 \wedge H_2 = min\{X_i \cup X_j; 1 \leq i \leq k, 1 \leq j \leq \ell\}$$

respectively(see Chapter 2 in [2]). We have

Lemma 6. Let f_1, f_2 be positive function. Then it holds

$$H_{f_1} \vee H_{f_2} = H_{f_1 \vee f_2}, \quad H_{f_1} \wedge H_{f_2} = H_{f_1 \wedge f_2}.$$

Proof. We may put $H_{f_1} = \{X_1, X_2, \cdots, X_k\}$ and $H_{f_2} = \{Y_1, Y_2, \cdots, Y_\ell\}$. If $X \in H_{f_1} \vee H_{f_2}$, then $X = X_{i_0}, 1 \leq i_0 \leq k$ or $X = Y_{j_0}, 1 \leq j_0 \leq \ell$, and $f_1(X) \vee f_2(X) = 1$. If there is an edge Y such that $Y < X$ and $f_1(Y) \vee f_2(Y) = 1$, then iy happens taht $X_i \leq Y < X$ or $Y_j \leq Y < X$. We have a contradiction. Hence we get $X \in H_{f_1 \vee f_2}$ and $H_{f_1} \vee H_{f_2} \subseteq H_{f_1 \vee f_2}$. Conversely, If $X \in H_{f_1 \vee f_2}$, then $f_1(X) \vee f_2(X) = 1$ and there is no edge Y such that $Y < X$ and $f_1(Y) \vee f_2(Y) = 1$. Now, it holds $X_{i_0} \leq X, 1 \leq i_0 \leq k$ or $Y_{j_0} \leq X, 1 \leq j_0 \leq \ell$. If $X_{i_0} < X$, then $Y = X_{i_0}$ satisfies that $Y < X$ and $f_1(Y) \vee f_2(Y) = 1$. When $Y = Y_{j_0}$, we get similarly a contradiction.

Let X be in $H_{f_1} \wedge H_{f_2}$. Then, there are some X_i and Y_j such that $X = X_i \cup Y_j$. This implies taht $X_i \leq X$, $Y_j \leq X$ and $f_1(X) \wedge f_2(X) = 1$. If there exists an edge Y such that $Y < X, f_1(Y) \wedge f_2(Y) = 1$, we have $X_{i'}$ and $Y_{j'}$ such that $X_{i'} \leq Y$ and $Y_{j'} \leq Y$. Hence, it holds that $Y' = X_{i'} \cup Y_{j'} \leq Y < X$, a contardiction. Conversely, take an edge $X \in H_{f_1 \wedge f_2}$. Then it is evident that $f_1(X) = f_2(X) = 1$. There exist some X_i and Y_j such that $X_i \leq X, Y_j \leq X$. Hence $X_i \cup Y_j \leq X$. As X is minimal, this implies $X = X_i \cup Y_j$.

It is well known that the set of all positive functins PF_n is lattice-isomorphic with the free distibutive lattice generated by n symbols, with the least element O and the greatest element I adjoined(see Chap.3, Thorem 5 in [4]). From Lemmas 4 and 5, we obtain

Theorem 7. The set of all simple hypergraphs SH_n with the order and the operations given above is lattice-isomorphic with the set of all positive Boolean functions. Hence, they are isomorphic with the free distibutive lattice generated by n symbols, with the least element O and the greatest element I adjoined.

4. Dual-minor functions, dual-major functions and coteries

In this section, We shall describe relations between properites of a positive function f and those of H_f. Main resluts were alredy given in [1],[3] and [8], but arrangements may be new. For a positive function, Ibaraki and Kameta noticed the result more stronger than Proposition 3.

Proposition 8[8]. For a positive function f, it holds

$$TrT(f) = minT(f^d) = H_{f^d}.$$

We define a mapping $Tr : SH_n \to SH_n$ by $Tr(H) = TrH$, for $H \in SH_n$. Then, it is dual-isomorphic and involutive. In fact, we can show easily

Proposition 9. The mapping Tr satisfies
(1) Tr is a bijection.
(2) For $H_1, H_2 \in SH_n$, $Tr(H_2) \leq H_1$ if and only if $H_1 \leq H_2$.
(3) For $H_1, H_2 \in SH_n$, $Tr(H_1 \vee H_2) = Tr(H_1) \wedge Tr(H_2)$.
(4)For $H_1, H_2 \in SH_n$, $Tr(H_1 \wedge H_2) = Tr(H_1) \vee Tr(H_2)$.
(5) $Tr^2 = Identity$.

A simple hypergraph $H \in SH_n$ is called a *coterie* if it is intersecting, i.e., $H \leq TrH$. The notion of coterie was introduced as a matematical abstraction to model mutual exclusin in distributed systems (see, for example [8]). If a coterie H is maximal in the poset SH_n, it is called *ND coterie(nondominated coterie).*

Proposition 10. A coterie H is ND coterie if and only if it is transversal, i.e.,$H = TrH$.

Proof. Assume that a coterie H satisfies the condition $TrH = H$. If there is a corerie H_1 with $H < H_1$, Then, it holds that $TrH_1 < TrH = H < H_1$. This contardicts to the fact H_1 is a coterie. Conversely, let H be a ND coterie. If $H \neq TrH$, there is an edge $X \in TrH$ with $X \notin H$. Put $H' = H \cup \{X\}$. Then $H < H'$ and $H' \leq TrH'$. Hence H is not ND coterie.

Now, the following is evident from Proposition 8.

Theorem 11 [8]. Let f be a positive function.

(1) It is dual-minor if and only if H_f is a coterie, i.e., $H_f \leq TrH_f$.

(2) It is dual-major if and only if $TrH_f \leq H_f$.

(3) It is self-dual if and only if H_f is a ND coterie, i.e.,$TrH_f = H_f$.

The conditions in (2) and (3) above are rerated to the *chromatic number* $\chi(H_f)$. Let H be a hypergraph and k be an integer ≥ 2. A *k-colouring* of the vertices is a partition $(S_1, S_2, \cdots, S_k)$ of the set of vertices into k classes such that every edge which is not a loop meets at least two clases of the partition. The *chromatic number* $\chi(H)$ is the smallest integer k for which H admits a k-colouring. It is known that a simple hypergraph H without loops satisfies $TrH \leq H$ if and only if $\chi(H) \geq 3$(see ,Lemma 2 in [1] or Chapter 2, Lemma 2 in [2]). Bebzaken [1] defined that H is a critical hypergraph if for every hypergraph H' with $H' < H$, it holds $\chi(H') < \chi(H)$, and shown that a hypergraph H which has more than one vertices is 3-colourling and critical if and only if it is transversal, i..e., $H = TrH$. Summing up, we have

Theorem 12[1]. Take $f \in PB_n$, $n \geq 2$, Assume that H_f has no loop.

(1) It is dual-major if and only if $\chi(H_f) \geq 3$.

(2) it is self-dual if and only if $\chi(H_f) = 3$ and H_f is critical.

5. The rank function of the distributive lattice SH_n

We shall discuss SH_n as a distributive lattice and use most of the terminology and notations in [9]. Let P be a finite poset. A subset C of P is called a *chain* if any two elements $X, Y \in P$ are comparable,i.e., $X \leq Y$ or $X \geq Y$. A chain C is said to be *saturated* if for any $X, Y \in P$, there is no element $Z \in P - C$ such that $X < Z < Y$ and $C \cup \{Z\}$ is a chain. For $X, Y \in P$, if $X < Y$ and there is no element Z such that $X < Z < Y$, it is said that Y *covers* X. The *length* of a finite chain C is defined to be $\ell(C) = |C| - 1$. If all maximal chains of P have the same length ℓ, P is calle a poset *graded of rank* ℓ. In this case, we can define the *rank function* $\rho : P \to \{0, 1, \cdots, \ell\}$ as follows. If $X \in P$ is minimal, we put $\rho(X) = 0$. When Y covers X, we define $\rho(Y) = \rho(X) + 1$. A ranked lattice L is *modular* if and only if for any $X, Y \in L$, $\rho(X) + \rho(Y) = \rho(X \wedge Y) + \rho(X \vee Y)$. Let L be a finite distributive lattice. If an element $X \in L$ can not be given by $X = Y \vee Z$ for some $Y, Z \in L$, it is called *join-irreducible*. Let P be the set of all join-irreducible elements in L. Then P is a finite poset. Put $J(P)$ be the distributve lattice of all ideals of P. Now we describe the very important theorem(*The fundamental theorem for finite distributive lattice*): A finite distributive lattice is isomorphic to $J(P)$(see, for example, Theorem3.4.1 in [9]). It is also known that If P is a poset of m-elements, then $J(P)$ is a poset graded of rank m, and that for an ideal $I \in J(P)$, $\rho(I) = |I|$(see Proposition 3.4.4 in [9]).

Now we shall apply the above results to the distributive lattice SH_n. Take a simple hypergraph $H = \{X_1, X_2, \cdots, X_k\} \in SH_n$, $k \geq 2$. Put $H_1 = \{X_1, \cdots, X_\ell\}$, $H_2 = \{X_{\ell+1}, \cdots, X_k\}$ $1 \leq \ell < k$. Then $H_1, H_2 \in SH_n$ and $H = H_1 \vee H_2$, i. e., H is not join-irreducible. Hence a simple hypergraph is join-irreducible if and only if it has only one edge. Hence, the poset P_n^* of all join-irreducible elements of SH_n is dual-isomorphic to P_n, and an ideal of P_n^* is considerd as a filter of P_n

Proposition 13. Let P_n^* be the poset of all join-irreducible elements in SH_n. Then P_n^* consists of hypergraphs with one ege and is dual-isomorphic to P_n. Let $J(P_n^*)$ be the distributive lattice of all ideals of P_n^*. Then SH_n is lattice-isomorphic to $J(P_n^*)$. Moreover, $J(P_n^*)$ is considerd as the set of all filters of P_n. For a simple hypergraph $H = \{X_1, X_2, \cdots, X_m\} \in SH_n$, the corresponding filter $J(H)$ is given by

$$J(H) = \cup_{i=1}^m \{X \in P_n ; X_i \subset X\}.$$

Since $|P_n^*| = |P_n| = 2^n$, SH_n is a lattcie of ranked 2^n. For $H = \{X_1, X_2, \cdots, X_m\}$, we have

$$\rho(H) = \rho(F(H)) = \text{the number of subsets} X \in P_n \text{ such that} X_i \subseteq X \text{ for some } X_i.$$

Let f be a positive function. Then it is evident that $J(H_f) = T(f)$ and $J(TrH_f) = J(H_{f^d}) = T(f^d)$, Hence, we have

Proposition 14. For a positive functin f, it holds that $\rho(H_f) = |T(f)|$ and $\rho(TrH_f) = |T(f^d)|$.

Using the above, we get

Proposition 15.

(1) For $H_1, H_2 \in SH_n$, if $H_1 \leq H_2$, then, $\rho(H_1) \leq \rho(H_2)$.

(2) For $H \in SH_n$, it holds $\rho(TrH) = 2^n - \rho(H)$.

Proof. (1) is evident, as we can put $H_1 = H_{f_1}$ and $H_2 = H_{f_2}$ with some positive functions f_1, f_2. We prove (2).Takeing a positive function f, we set $H = H_f$. Then, from Prposition 2, we get $\rho(TrH_f) = |T(f^d)| = 2^n - |T(f)|$. This shows (2).

For a hypergrah $H = \{X\}$ with one edge, set $\rho(H) = \rho(X)$. Then it is evident that $\rho(H) = \rho(X) = 2^{n-|X|}$. As SH_n is distributive, it is modular. Hence, for any $H_1, H_2 \in SH_n$, it holds $\rho(H_1 \vee H_2) = \rho(H_1) + \rho(H_2) - \rho(H_1 \wedge H_2)$. If

$H_1 = \{X_1\}, H_2 = \{X_2\}$, then $H_1 \vee H_2 = \{X_1, X_2\}$ and $H_1 \wedge H_2 = \{X_1 \cup X_2\}$. Hence, we have $\rho(\{X_1, X_2\}) = \rho(X_1) + \rho(X_2) - \rho(X_1 \cup X_2)$. In general, using an induction, we obtain

Proposition 16. For $H = \{X_1, X_2, \cdots, X_m\} \in SH_n$, it holds,

$$\rho(H) = \sum_{i=1}^{m} \rho(X_i) - \sum_{i_1<i_2} \rho(X_{i_1} \cup X_{i_2}) + \sum_{i_1<i_2<i_3} \rho(X_{i_1} \cup X_{i_2} \cup X_{i_3})$$

$$-(-1)^{\ell} \sum_{i_1<i_2<\cdots<i_\ell} \rho(X_{i_1} \cup X_{i_2} \cup \cdots, \cup X_{i_\ell}) - \cdots - (-)^m \rho(X_1 \cup X_2 \cup \cdots \cup X_m),$$

where $\rho(X) = 2^{n-|X|}$, for any edge X.

6. Decomposition of dual-minor functions

For a function f, g, the *extensin of f with respect to g* is defined by $f \uparrow g = f + f^d g$. It is known that if g is self-dual and f is dual-minor then $f \uparrow g$ is self-dual. Bioch and Ibaraki [3] obtainded a condition when a given dual-minor function is deconposed into a conjunction of self-dual functins. In fact, they gave:

Let f be a dual-minor function. Then f can be decomposed into k self-dual functions $f \uparrow g_i$, $i = 1, 2, \cdots, k$;

$$f = (f \uparrow g_1)(f \uparrow g_2) \cdots (f \uparrow g_k),$$

defined by self-dual functions $g_1, g_2, \cdots, g_k$, if and only if

$$g_1 g_2 \cdots g_k \leq f + f^*.$$

The conditon in the above theorem, $g_1 g_2 \cdots g_k \leq f + f^*$ is equivalent to

$$g_1 + g_1 + \cdots + g_k \geq \bar{f} f^d,$$

i.e., $\cup_{i=1}^{k} T(g_i) \supseteq FT(f) = T(\bar{f} f^d)$. Firstly, we shall show

Theorem 17. Let f be a dual-minor function.

(1) Asumme that f is docomposed into a conjunction of k self-dual functions, i.e., $f = f_1 f_2 \cdots f_k$, Put $T_i = T(\bar{f} f_i)$ for $1 \leq i \leq k$. Then it holds

(a) $FT(f) = \cup_{i=1}^{k} T(\bar{f} f_i)$,

(b) $X \in T_i$ if and only if $\bar{X} \in FT(f) \setminus T_i$.

(2) If there is a family of subsets $T_1, T_2, \cdots, T_k$ of $FT(f)$ such that

(a) $FT(f) = \cup_{i=1}^{k} T_i$,

(b) $X \in T_i$ if and only if $\bar{X} \in FT(f) \setminus T_i$.

Define functions f_i by $T(f_i) = T(f) \cup T_i$, for $1 \leq i \leq k$. Then f_i is self-dual and $f = f_1 f_2 \cdots f_k$.

Proof. (1) As $f^d = f_1 + f_2 + \cdots + f_k$, we have $\bar{f} f^d = \bar{f} f_1 + \bar{f} f_2 + \cdots + \bar{f} f_k$. This implies (a). If $X \in T_i$, then, $f(X) = 0$ and $f_i(X) = 1$. As $f_i = f_i^d$, we get $f_i(\bar{X}) = 0$. Hence $\bar{X} \notin T_i$. But from $X \in FT(f)$, we get $\bar{X} \in FT(f)$. Thus we have $\bar{X} \in FT(f) \setminus T_i$. Coversely if $\bar{X} \in FT(f) \setminus T_i$, then $f_i(\bar{X}) = 0$ and $\bar{X} \in FT(f)$. If $f_i(X) = f_i^d(X) = \bar{f}_i(\bar{X}) = 0$, we have $f_i(\bar{X}) = 1$, a contradiction. Hence, we get $X \in T_i$.

(2)From the definitin, we have $f \leq f_i$. Hence we get $f \leq f_1 f_2 \cdots f_k$. Next, we show that $\cap_{i=1}^k T_i = \emptyset$. Let an edge X be contained in $\cap_{i=1}^k T_i$. Then, $\bar{X} \in FT(f) \setminus T_i$ for all i, $1 \leq i \leq k$. Hence, we have $\bar{X} \notin T_i$ for all i. This implies that $\bar{X} \notin FT(f)$, a contradiction. Hence, $\cap_{i=1}^k T_i = \emptyset$. Since $T(f_1 f_2 \cdots f_k) = T(f) \cup (\cap_{i=1}^k T_i) = T(f)$, we get $f = f_1 f_2 \cdots f_k$. We shall show $T(f_i^d) = T(f_i)$. Take $X \in T(f_i)$. Assume that $f(X) = 1$. If $f(\bar{X}) = 1$, then $X \in T(f) \cap F(f^d) = TF(f) = \emptyset$, since f is dual-minor. Hence $f(\bar{X}) = 0$. As $X \notin FT(f)$, $\bar{X} \notin FT(f)$, Hence, $f(\bar{X}) = 0$, i.e., $f_i^d(X) = 1$. If $X \in T_i$, then $\bar{X} \in FT(f) \setminus T_i$ and $f(\bar{X}) = 0$. Hence $f_i(\bar{X}) = 0$. Thus we have $f_i^d(\bar{X}) = 1$. Conversely assume $X \in T(f_i^d)$. Then $f_i(\bar{X}) = 0$, i.e., $f(\bar{X}) = 0$, $\bar{X} \notin T_i$. Hence, $\bar{X} \in FT(f) \setminus T_i$. This yields $X \in T_i$, i.e., $X \in T(f_i)$.

A family $\{T_i,\ 1 \leq i \leq k\}$ in (2) of the above theorem is called a *generating system for a decomposition*. In this case, As T_i satisfies that $X \in T_i$ if and only if $\bar{X} \in FT(f) \setminus T_i$, It holds that $|T_i| = |FT(f) \setminus T_i|$. Hence, we have $|T_i| = |FT(f)|/2$. As it is shown in the above proof, it holds that

$$\cap_{i=1}^k T_i = \emptyset.$$

If we put $S_i = FT(f) \setminus T_i$ for $1 \leq i \leq k$. Then, it is evident that $X \in S_i$ if and only if $\bar{X} \in FT(f) \setminus S_i$. Take $X \in FT(f)$. As $\cap_{i=1}^k T_i = \emptyset$, there is some T_j such that $X \notin T_j$. Hence, $X \in S_j$. This implies that $\cup_{i=1}^k S_i = FT(f)$. If a dual-minor function f is docompopsed into $f = f_1 f_2 \cdots f_k$, where f_i for $1 \leq i \leq k$ are self-dual functins. Let T_i is a set given (1) in the above theorem. Then the set $G_i = \{g : T_i \subseteq T(g) \subseteq T_i \cup T(f),\ g_i \text{ are self-dual.}\}$ is nonempty, since it contains f_i. Then we can put $f_i = f + f^d g_i$ for any $g_i \in G_i$. Hence every dcomposition of a self-dual function is one given by Biochi and Ibaraki.

Proposition 18. Asumme that f is docomposed into a conjunction of k self-dual functions, i.e., $f = f_1 f_2 \cdots f_k$, Put $T_i = T(\bar{f} f_i)$ for $1 \leq i \leq k$.

(1) $|T_i| = |FT(f)|/2$.

(2) Set $S_i = FT(f) \setminus T_i$ for $1 \leq i \leq k$. Then $\{S_i, 1 \leq i \leq k\}$ is a complementary generating system for a decomposition of f.

(3)Set $G_i = \{g_i; T_i \subseteq T(g_i) \subseteq T_i \cup T(f), g_i \text{ are self-dual.}\}$ for $1 \leq i \leq k$. Then each G_i is not empty and for any $g_i \in G_i$, we can put $f_i = f + f^d g_i$.

For every variable x_i, the function $f(X) = x_i$, denoted simply by x_i, is self-dual. For a positive dual-minor function f, Let $x_{j_1} x_{j_2} \cdots x_{j_k}$ be one of its prime implicants. By putting $f_i = f + f^d g_i$, $g_i = x_{j_i}$, Bioch and Ibaraki [3] gave a decomposition $f = f_1 f_2 \cdots f_k$, which is called a *canonical decomposition* of f. For any variable x_i, $T(x_i) \cup T(\bar{x}_i) = P_n$ and $X \in T(x_i)$ if and only if $\bar{X} \in T(\bar{x}_i)$. If f is dual-minor and not self-dual, we put $T_1 = T(x_i) \cap FT(f)$ and $T_2 = T(\bar{x}_i) \cap FT(f)$. Then they give a generating system for a decomposition of f. In general, Let T_1 is a subset of $FT(f)$ such that $X \in T_1$ if and only if $\bar{X} \in FT(f) \setminus T_1$. Put $T_2 = FT(f) \setminus T_1$. Then $\{T_1, T_2\}$ is a generating system for a decomposition.

For a function f, We can put $FT(f) = \{X_1, \bar{X}_1, X_2, \bar{X}_2, \cdots, X_m, \bar{X}_m\}$. Set $X_i^1 = X_i$ and $X_i^0 = \bar{X}_i$ for any $1 \le i \le m$. For $W = (w_1, w_2, \cdots, w_m) \in \{0,1\}^m$, Put $T(W) = \{X_1^{w_1}, X_2^{w_2}, \cdots, X_m^{w_2}\}$. Then $T(\bar{W}) = FT(f) \setminus T(W)$ and $\{T(W), T(\bar{W})\}$ is a generating system for a decomposition.

Proposition 19. Let f be a dual-minor function. Put $FT(f) = \{X_1, \bar{X}_1, X_2, \bar{X}_2, \cdots, X_m, \bar{X}_m\}$. Take $T(W) = \{X_1^{w_1}, X_2^{w_2}, \cdots, X_m^{w_2}\}$. Then $T(\bar{W}) = FT(f) \setminus T(W)$ and $\{T(W), T(\bar{W})\}$ is a generating system for a decomposition. Hence , there are 2^{m-1} kinds of decompositions of f into two self-dual functions.

Let $t = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \cdots x_{j_k}^{\epsilon_k}$ be one of implicants of $f + f^*$, where $\epsilon_i \in \{0,1\}$, and $x_j^1 = x_j$, $x_j^0 = \bar{x}_j$. Put $T_i = T(x_{j_i}^{\epsilon_i}) \cap FT(f)$. As $t \le f + f^*$, we have $\cup_{i=1}^k T_i = FT(f)$. It is evident that $\bar{t} \le f + f^*$ and that $\bar{t}$ gives the complementary generating system for a decomposition of f. A generating system for a decomposition is said to be *minimal* if none of its subsets can be deleted. Let $t = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \cdots x_{j_k}^{\epsilon_k}$ be a prime implicant of $f + f^*$, the corresponding generating system for a decomposition is minimal.

Proposition 20. Let f be a dual-minor function. Let $t = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \cdots x_{j_k}^{\epsilon_k}$ be one of prime implicants of $f + f^*$, where $\epsilon_i \in \{0,1\}$. Put $T_i = T(x_{j_i}^{\epsilon_i}) \cap FT(f)$. Then T_i, $1 \le i \le k$ is a minimal generating system for decomposition. The complementary prime implicant $\bar{t}$ give the complementary generating system.

7. Decomposition of positive dual-minor functions

Let f be a positive dual-minor functon. Put $m(f) = minFT(f)$. Bioch and Ibaraki proved [3]

$$m(f) = min(T(f^d)) \setminus minT(f) = H_{f^d} \setminus H_f.$$

They also definded the positive closure of f by$\hat{f} = \wedge\{h | h \ge f^d \bar{f}\}$, and shown $m(f) = H_{\hat{f}} = minT(\hat{f})$. As $TrH_f = H_{f^d}$, it is evident that $m(f) = TrH_f \cap F(f)$. We put also $M(f) = maxFT(f)$. Then we have

$$M(f) = \{\bar{X};\, X \in m(f)\}.$$

Given a hypergraph, we define an *intersecting subset* to be a set of edges having non-empty pairwise intersection.

Lemma 21. Let f be a positive dual-minor function and let m_1 be a maximal intersecting family of $m(f)$. Put

$$T_1 = \{X \in FT(f);\ \text{there is some}\, Y \in m_1 \,\text{such that}\ \ Y \subset X\},$$

$$T_2 = \{X \in FT(f);\ \text{there is some}\, Y \in m_1 \,\text{such that}\ \ X \subset \bar{Y}\}.$$

Then $T_1 \cup T_2 = FT(f)$ and $T_1 \cap T_2 = \emptyset$. Hence. $minT_2 = m(f) \setminus M_1$. Moreove, $X \in T_1$ if and only if $\bar{X} \in T_2$.

Proof. Set $m_2 = m \setminus m_1$ and $M_1 = maxT_1$. Then $M_1 \subseteq M(f)$. Put $M_2 = M(f) \setminus M_1$. We shall show $M_1 = \{\bar{X}; X \in m_2\}$. Take $X \in m_2$. As m_1 is maximal, there is an edge $Y \in m_1$ with $X \cap Y = \emptyset$. Hence, $Y \subseteq \bar{X}$. This implies $\bar{X} \in M_1$. Conversely, take $X \in M_1$. Then, there is an edge $Y \in m_1$ with $Y \subseteq X$. If $\bar{X} \in m_1$, then Y, $\bar{X} \in m_1$ and $Y \cap \bar{X} = \emptyset$. We have a contradiction. Hence $X \in m_2$. Next, we show $M_2 = \{\bar{X};\ X \in m_1\}$. Let $X \in m_1$. If $\bar{X} \in M_1$, there is an edge $Y \in m_1$ with $Y \subseteq \bar{X}$. Hence, $X \in m_1$, $\in Y \in m_1$ and $X \cap Y = \emptyset$. Hence, $\bar{X} \in M_2$. Let $X \in M_2$. If $\bar{X} \in m_2$, then $X \in M_1$. Hence, $\bar{X} \in m_1$.Thus we have

$$M_1 = \{\bar{X};\ X \in m_2\},\ M_2 = \{\bar{X};\ X \in m_1\},$$

$$m_1 = \{\bar{X};\ X \in M_2\},\ m_2 = \{\bar{X};\ X \in M_1\}.$$

Now, $X \in M_2$ if and only if $\bar{X} \in m_1$. This implies that is $X \in M_2$ and only if $X \in maxT_2$. Hence, $M_2 = maxT_2$. Take $X \in m_2$. Then, as m_1 is maximal, there is $Y \in m_1$ with $X \cap Y = \emptyset$. Hence $X \subseteq \bar{Y}$. As $X \in m(f)$, we get $X \in minT_2$. Conversely, assume $X \in minT_2$. Then there is $Y \in m_1$ with $X \subseteq \bar{Y}$. Hence, $X \cap Y = \emptyset$ and $X \notin m_1$. This implies $X \in m_2$. Thus we obtain

$$M_2 = maxT_2,\ minT_2 = m_2 = m(f) \setminus m_1.$$

If $X \in FT(f) \setminus T_1$, there is a edge $Y \in M_2$ such that $X \subseteq Y$, As $Y = \bar{Z}, Z \in m_1$. Hence, $X \in T_2$. Next, take $X \in T_1 \cap T_2$. Then, there are Y_1, $Y_2 \in m_1$ with $Y_1 \subseteq X \subseteq \bar{Y}_2$. Hence, we have $Y_1 \cap Y_2 = \emptyset$, a contradiction. Thus $T_1 \cap T_2 = \emptyset$.

If $X \in T_1$, there is an edge $Y \in m_1$ with $Y \subseteq X$. Hence, $\bar{X} \subseteq \bar{Y}$, that is, $\bar{X} \in T_2$. Similarly, we can show the converse.

A family of maximal intersecting subsets $\{m_i,\ 1 \le i \le k\}$ of $m(f)$ with $\cup_{i=1}^{k} m_i = m(f)$ will be called a *generating system for a positive decomposition* of a positive dual-minor function f.

Theorem 22. Let f be a positive dual-minor function.

(1) Asumme that f is docomposed into a conjunction of k positive self-dual functions, i.e., $f = f_1 f_2 \cdots f_k$. Put $m_i = minT(\bar{f} f_i) \cap m(f)$ for $1 \le i \le k$. Then $\{m_i,\ 1 \le i \le k\}$ is a generating system for a positive decomposition of f.

(2) If $\{m_i,\ 1 \le i \le k\}$ is a generating system for a positive decomposition of f. Define positive functions f_i by $H_{f_i} = minT(f_i) = m_i \cup H_f$, for $1 \le i \le k$. Then each f_i is positive, self-dual and $f = f_1 f_2 \cdots f_k$.

Proof.(1) As $FT(f) = \cup_{i=1}^{k} T(\bar{f} f_i)$, $m(f) = min(\cup_{i=1}^{k} T(\bar{f} f_i)) \subseteq \cup_{i=1}^{k} minT(\bar{f} f_i)$. Hence, $m(f) = \cup_{i=1}^{k} m_i$. We show each m_i is a maximal intersecting subset. Take any X, $Y \in m_i$. Then $f_i(X) = f_i(Y) = 1$. As $f_i^d = f_i$, $f_i(\bar{X}) = f_i(\bar{Y}) = 0$. If $X \cap Y = \emptyset$, it holds that $X \subseteq \bar{Y}$. As f_i is positive, $1 = f_i(X) = f_i(\bar{Y})$. We get a contradiction. Let $X \in m(f) \setminus m_i$. Assume X intersects all edges in m_i. Then $\bar{X}$ contains no edge in m_i. Hence, $\bar{X} \in T(f^d \bar{f}) \setminus T(f_i \bar{f})$. Hence we have $X \in T(f_i \bar{f})$, a contradictionn. Thus m_i is maximal.

(2) From the definition, it is evident that each f_i is positive and satisfies $f \leq f_i$. We show that each f_i is self-dual. Put $T_i = \{X \in FT(f)$; there is $Y \in m_i$ such that $Y \subseteq X\}$. Take X with $f_i(X) = 1$. If $f(X) = 1$, then $X \notin FT(f)$ and $f(\bar{X}) = 0$. Hence, $f_i(\bar{X}) = 0$, i.e., $f_i^d(X) = 1$. If $X \in T_i$, from lemma 21, we get $\bar{X} \in FT(f) \setminus T_i$. Hence $f_i(\bar{X}) = 0$, i.e., $f_i^d(X) = 1$. Conversely, if $f_i(X) = 0$, then $f(X) = 0$ and $X \notin T_i$. Hence using lemma 21, we get $\bar{X} \in T_i$, i.e., $f_i(\bar{X}) = 1$. Hence, $f_i^d(X) = 0$.

It is evident that $f \leq f_1 f_2 \cdots f_k$. Since T_i, $1 \leq i \leq k$, is a generating system for a decomposition, we have $\cap_{i=1}^{k} T_i = \emptyset$. This implies that $f = f_1 f_2 \cdots f_k$.

As m_1 and m_2 are maximal intersecting subsets, it holds that $m_1 \cap m_2 = \emptyset$. Hence, we have

Corollary 23 [3]. Let f be a positive dual-minor function. Then, f is decomposed into a conjucttion of two positive self-dual functions if and only if there is a generating system $\{m_1, m_2\}$ for a positive decomposition with $m_2 = m(f) \setminus m_1$.

As similarly as (3) in Proposition 18, we have

Proposition 24. Asumme that f is docomposed into a conjunction of k positiveself-dual functions, i.e., $f = f_1 f_2 \cdots f_k$. Put $T_i = T(\bar{f} f_i)$ for $1 \leq i \leq k$. Set $G_i = \{g_i; T_i \subseteq T(g_i) \subseteq T_i \cup T(f), g_i$ are positive self-dual.$\}$ for $1 \leq i \leq k$. Then each G_i is not empty and for any $g_i \in G_i$, we can put $f_i = f + f^d g_i$.

Proposition 25. Let f be a positive dual-minor function. For each variable x_i, put $m(x_i) = T(x_i) \cap m(f)$. Then each $m(x_i)$ is intersecting and maximal. If $x_{i_1} x_{i_2} \cdots x_{i_k}$ is a prime implicant of $f + f^*$, then $\{m(x_{i_1}), m(x_{i_2}), \cdots, m(x_{i_k})\}$ is a minimal generating system for a positive decomposition of f.

Proof. It is evident that $m(x_i)$ is intersecting. If $X (\in m(f) \setminus m(x_i))$ intersects all edges of $m(x_i)$, as $\bar{X} \in T(x_i) \cap FT(f)$, there is $Y \in m(x_i)$ such that $Y \subseteq \bar{X}$. Then we have $Y \cap X = \emptyset$, a contradiction. Hence, $m(x_i)$ is a maximal intersecting subset of $m(f)$. The rest of the statemment can be shown as similarly as the coresponding statement of Proposition 20.

Given a hypergraph $H = \{E_1, E_2, \cdots, E_n\}$, its *representative graph* $L(H)$ is a graph whose vertices are points $E_1, E_2, \cdots, E_n$, the vertices E_i, E_j being adjacement if and only if $E_i \cap E_j \neq \emptyset$. Now, we can get easily,

Proposition 26. Let f be a positive dual-minor function. Let denote by $L(m(f))$ the representative graph of the simple hypergraph $m(f)$. Then $\{m_i, 1 \leq i \leq k\}$ is a generating system for a positive decomposition if and only if each m_i is a maximal clique of $L(m(f))$ and $\cup_{i=1}^{k} m_i = L(m(f))$.

For a positive self-dual function f, put $m(x_i) = T(x_i) \cap m(f)$, $1 \leq i \leq n$ as above. Assume $m(f) = \{X_1, X_2, \cdots, X_m\}$. Then, we obtain a hypergraph on $m(f)$ with edges $m(x_i)$, $1 \leq i \leq n$, denoted $m(x)$. A covering of $m(x)$, i.e., a partial hypergraph of $m(x)$ which covers all vetrices of $m(x)$ is a generating system for a positive decomposition. Let $A = (a_{ij})$ be the incident matrix, i.e., $A = (a_{ij})$ given by $a_{ij} = 1$ if $m(x_i) \ni X_j$ and $a_{ij} = 0$ otherwise for $1 \leq i \leq n$, $1 \leq j \leq m$. The problem of finding a genereting system for a positive decomposition is reduced to the problem of finding rows $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ which cover all columns, i.e., for each j, $1 \leq j \leq m$, there is at least one row a_{i_ℓ} with $a_{i_\ell j} = 1$, the*Set Covering Problem.*

References

[1] C. Benzakken, Critical hypergraphs for the weak chromatic number, J. Combinatorial Theory Ser. B 29, 328-338,1989.

[2] C. Berge, Hypergrphs (North-Holland, Amsterdam, 1989).

[3] B. C. Bioch and T. Ibaraki, Decompositions of positive self-dual Boolean functions, Discrete Math. 140,23-46, 1995.

[4] G. Birkhoff, Lattice Theory (American Math. Soc. Colloquim Publications, 1940).

[5] Y. Crama, Dualization of regular Boolean functions, Discrete Appl. Math. 16, 79-85, 1987.

[6] H. Gracia-Molina and D. Barbara, How to assign votes in a distributed system, J. ACM.32, 841-860, 1985.

[7] L. Lamport, Time, clocks, and the ordering of events in a distributed system, Comm. ACM 21, 558-565, 1978.

[8] T. Ibaraki and T. Kameta, A theory of coteries: mutual exclusion in distributed system, IEEE Trans. Parallel and Distributed systems 4, 779-794, 1993.

[9] R. T. Stanley, Enumerative Combinatorics Vol. 1(Wadworth and Brooks, California, 1989).