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A Topic of Quadratic First Integral of Linear Symplectic System

By

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Abstract

It is shown that every quadratic first integral of a linear symplectic system is expressed as a linear combination of quadratic forms constructed from generalized eigenvectors corresponding to four coupled eigenvalues.

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1. Problem and preliminaries

In this introductory section, the problem to be treated is presented and a few preliminaries are arranged.

Let T denote a $2N$-dimensional real symplectic matrix, that is,

$$T'JT = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1}$$

where the dash means matrix transposed. Throughout the paper, we suppose that T does not have eigenvalues ± 1, and W_a and $\tilde{W}_a$ denote the eigenspace and the generalized eigenspace of T corresponding to an eigenvalue a, respectively. We consider the linear recurrence on R^{2N} which is a discrete version of a linear Hamiltonian system:

$$x_{n+1} = Tx_n \qquad (n = 0, 1, \dots). \tag{2}$$

By supposition, the recurrence has no 2-periodic point except for the origin. A function f on R^{2N} is called *an invariant* of (2) if its value remains constant along every solution $\{x_n\}$ of (2). For any real symmetric matrix S, the quadratic form

$x'Sx/2$ is denoted by $S[x]$, and an invariant $S[x]$ is called *a quadratic invariant.* Our problem is to make clear the structure of quadratic invariants of a discrete Hamiltonian system. Let us start by showing an obvious but basic lemma.

Lemma 1. *$S[x]$ is a quadratic invariant of* (2) *if and only if*

$$T'ST = S, \quad or \quad (T')^{-1}ST^{-1} = S. \tag{3}$$

The symbol Ω is used to mean the set of all coefficient matrices of quadratic invariants:

$$\Omega = \{S \in M(2N, R) \mid T'ST = S, \ S' = S\}.$$

For the time being, we assume that matrices and vectors are complex-valued, the field of scalars being C, and L and $\langle , \rangle$ mean C^{2N} and a symplectic inner product $\langle x, y \rangle = x'Jy$ on L.

Lemma 2. *Let $\xi_1, \xi_2, \ldots, \xi_u$ be linearly independent vectors. Then, the u^2 matrices defined by $(J\xi_i)(J\xi_j)'$ are linearly independent, and further, it holds that $(T')^{-1}\{(J\xi_i)(J\xi_j)'\}T^{-1} = \{J(T\xi_i)\}\{J(T\xi_j)\}'$.*

Proof. Choose a vector η subject to $(J\xi_j)'\eta \neq 0$ and $(J\xi_i)'\eta = 0$ $(i \neq j)$. Put $\Sigma c_{ij}(J\xi_i)(J\xi_j)' = 0$ and multiply η from the right, and it follows that $\Sigma_i((J\xi_j)'\eta) \cdot c_{ij} J\xi_i = 0$. Since $J\xi_i$ are linearly independent, we have $c_{ij} = 0$. The second assertion is obvious due to $T'JT = J$.

2. Quadratic invariant

This section is devoted to determination of all quadratic invariants.

Let $\{\xi_1, \xi_2, \ldots, \xi_u\}$ and $\{\eta_1, \eta_2, \ldots, \eta_v\}$ be generalized eigenvectors of T which form two distinct Jordan blocks with eigenvalues a and b, respectively. That is,

$$T\xi_1 = a\xi_1, \quad T\xi_i = a\xi_i + \xi_{i-1} \qquad (2 \leqq i \leqq u),$$

$$T\eta_1 = b\eta_1, \quad T\eta_i = b\eta_i + \eta_{i-1} \qquad (2 \leqq j \leqq v),$$

For every vector ζ, $\xi_u \neq (T - aI)\zeta$ and $\eta_v \neq (T - bI)\zeta$.

We define uv matrices M_{ij} and introduce a linear combination S of them:

$$M_{ij} = (J\xi_i)(J\eta_j)' \quad (1 \leqq i \leqq u, \ 1 \leqq j \leqq v), \quad S = \sum_{1 \leqq i \leqq u, 1 \leqq j \leqq v} c_{ij} M_{ij}. \tag{4}$$

where c_{ij} are complex constants. Our first schedule is to obtain a condition that S satisfies (3) by disregarding a restriction that S is either real or symmetric.

Owing to Lemma 2, $(T^{-1})'ST^{-1}$ is equal to S, if and only if all of the following equations hold good.

$$(1-ab)\,c_{uv}=0,$$
$$(1-ab)\,c_{iv}=bc_{i+1,v}\quad (1\leqq i\leqq u-1),$$
$$(1-ab)\,c_{uj}=ac_{u,j+1}\quad (1\leqq j\leqq v-1),$$
$$(1-ab)\,c_{ij}=ac_{i,j+1}+bc_{i+1,j}+c_{i+1,j+1}\quad (1\leqq i\leqq u-1,\ 1\leqq j\leqq v-1).$$

By m is denoted $\min(i,j)$. When $ab\neq 1$, it is easy to prove that all of c_{ij} vanish. When $ab=1$, c_{ij} vanishes for $i+j>m+1$ and the remaining coefficients are subject to $ac_{i,j+1}+bc_{i+1,j}+c_{i+1,j+1}=0$. Concerning the latter ones, we put

$$c_{ij}=(-a/b)^{i-1}f_{(m+1)-(i+j)}(i)\qquad (2\leqq i+j\leqq m+1).$$

Then, the equations among the remaining c_{ij} turn out to be equivalent to

$$\text{(5)}\quad \begin{aligned} &f_0(i)=\text{arb. const.},\quad f_k(1)=\text{arb. const.}\qquad (0\leqq k\leqq m-1),\\ &f_{k+1}(i)=f_{k+1}(1)-\frac{1}{b}\,(f_k(2)+f_k(3)+\cdots f_k(i))\\ &\qquad\qquad\qquad (1\leqq k\leqq m-2,\ 2\leqq i\leqq m+1-k). \end{aligned}$$

Thus, we have attained the following lemma.

Lemma 3. *With respect to matrices which are linear combinations of* M_{ij} *and satisfy* (3), *the followings hold good.*

(a) *When* $ab\neq 1$, *there exists no non-zero matrix.*

(b) *When* $ab=1$, *there are* m *linearly independent matrices* S_k *such that*

$$\text{(6)}\quad \begin{aligned} S_k=&f_{k-1}(1)M_{1,m+1-k}+(-a^2)f_{k-1}(2)M_{2,m-k}\\ &+\cdots+(-a^2)^{m-k}f_{k-1}(m+1-k)M_{m+1-k,1}\qquad (1\leqq k\leqq m), \end{aligned}$$

where $f_k(i)$ *are constants defined by* (5).

According to Lemma 2, an arbitrary matrix S is expressed as a linear combination of $(J\xi_i)(J\xi_j)'$, where $\{\xi_1,\xi_2,\ldots\}$ denotes a whole of eigenvectors and generalized ones and is a basis of L. By use of Lemmas 2 and 3, we have the following immediately.

Lemma 4. *A matrix* S *satisfies* (3), *if and only if* S *is expressed as a linear combination of matrices* S_k *which are constructed from all combinations of generalized eigenvectors with eigenvalues* a *and* $1/a$ *in a manner as in* (6).

Now, we are in a position to study quadratic invariants. Since the coefficient matrix of a quadratic invariant is real and symmetric, two conditions $S'=S$ and

$\bar{S}=S$ must be satisfied besides (3). Define S_k under the condition $f_{k-1}(1)\neq 0$ $(k=1,\dots,m)$. Then, according to Lemma 2, $\bar{S}_k \neq S_k$ and $S_k+\bar{S}_k$ are linearly independent. Furthermore, put, for each S_k,

$$(7)\qquad U_k=\frac{1}{4}(S_k+\bar{S}_k+S'_k+\bar{S}'_k)\quad V_k=\frac{\sqrt{-1}}{4}(S_k-\bar{S}_k+S'_k-\bar{S}'_k)\qquad (1\leqq k\leqq m),$$

and the matrices U_k and V_k belong to Ω, since $T'S'T=S'$ and $T'\bar{S}T=\bar{S}$ follow automatically from $T'ST=S$. Due to Lemma 4, every element of Ω is expressed as a linear combination of U_k and V_k with real coefficients, though they are not necessarily linearly independent.

It is well known that if a is an eigenvalue of a real symplectic matrix, so are $1/a$, $\bar{a}$, and $1/\bar{a}$ [1]. The above matrices U_k and V_k are constructed from the vectors selected from the four generalized eigenspaces, in other words, they depend on a quartette of $\tilde{W}_a$, $\tilde{W}_{1/a}$, $\tilde{W}_{\bar{a}}$, and $\tilde{W}_{1/\bar{a}}$. Hereafter, we adopt a convention that a means an eigenvalue subject to

$$(8)\qquad |a|\geqq 1,\qquad 0\leqq \arg(a)\leqq \pi.$$

By means of this, only one eigenvalue is selected among the four. Next, suppose that $\tilde{W}_a$ is a direct sum of u subspaces B_i corresponding to respective Jordan blocks. We choose a set of generalized eigenvectors as follows.

$$(9)\qquad \begin{aligned} &B_i=\operatorname{span}\{\xi_1^{(i)},\dots,\xi_{j(i)}^{(i)}\},\quad i=1,\dots,u, && s.\ t.\\ &T\xi_j^{(i)}=a\xi_j^{(i)}+(1-\delta_{j1})\xi_{j-1}^{(i)},\ j=1,\dots,j(i), && j(1)\geqq j(2)\geqq\cdots\geqq j(u),\end{aligned}$$

where $\dim\tilde{W}_a=j(1)+\cdots+j(u)$. In this case, there exists a unique basis $\{\eta_j^{(\beta)}\}$ of $\tilde{W}_{1/a}$ which satisfy $\langle\xi_i^{(\alpha)},\eta_j^{(\beta)}\rangle=\delta_{ij}\delta_{\alpha\beta}$, and $\tilde{W}_{1/a}$ is a direct sum of the following u subspaces (Appendix 2).

$$C_i=\operatorname{span}\{(T-I/a)^{j(i)-1}\eta_1^{(i)},\ (T-I/a)^{j(i)-2}\eta_1^{(i)},\dots,\eta_1^{(i)}\}\qquad (i=1,\dots,u).$$

Here, $(T-I/a)^{j(i)-1}\eta_1^{(i)}$ is an eigenvector. As is easily seen, B_i and C_j are skew-orthogonal to each other when $i\neq j$. With respect to the remaining two generalized eigenspaces, we may define subspaces $\bar{B}_i$ and $\bar{C}_j$ similarly.

Now, under the condition that ± 1 are not eigenvalues, the four values may be reduced. To be concrete, there are three cases with respect to eigenvalues.

(a) Case of $a\in R$ and $a>1$: $\bar{a}=a$.

(b) Case of $|a|=1$ and $0<\arg(a)<\pi$: $a=1/\bar{a}$.

(c) Case of $a\in R$ and $|a|>1$: Four eigenvalues.

Case (a). Since all generalized eigenvectors can be selected as real-valued, V_k vanish in (7). Then, according to Lemma 3, we obtain $j(1)+3j(2)+\cdots+(2u-1)j(u)$ linearly independent quadratic invariants from combination of B_i and C_j.

Case (b). In this case, B_i equals to $\bar{C}_i$. Then, from combination of B_i and C_i, only U_k are obtained, whereas neither U_k nor V_k vanish when $i \neq j$. Then, we have $j(1) + 5j(2) + \cdots + (4u-3)j(u)$ linearly independent quadratic invariants related to an eigenvalue a.

Case (c). For all combinations of B_i and C_j, both U_k and V_k are obtained. Then, $2j(1) + 6j(2) + \cdots + (4u-2)j(u)$ linearly independent quadratic invariants are obtained.

For an eigenvalue a, we put

$$\text{num}\,(a) = \begin{cases} j(1) + 3j(2) + \cdots + (2u-1)j(u) & \text{Case (a),} \\ j(1) + 5j(2) + \cdots + (4u-3)j(u) & \text{Case (b),} \\ 2j(1) + 6j(2) + \cdots + (4u-2)j(u) & \text{Case (c).} \end{cases}$$

Then, the following theorem holds good.

Theorem 1. *The discrete system* (2) *admits Σ' num* (*a*) *linearly independent quadratic invariants, where Σ' means summation over eigenvalues subject to* (8).

Let us pay attention to the fact that if V_k does not vanish, its rank equals to 4, and if U_k does not vanish, its rank equals to 4 or 2. Furthermore, because of Lemma 4, rank U_k is 2, if and only if there exist non-zero vectors ξ and η such that $T\xi = a\xi$ and $T\eta = (1/a)\eta$, where a is real, or complex with the absolute value one. In the former case, the signature of U_k is (1, 1), while in the latter case U_k is (semi) positive-definite. This leads to the following corollary.

Corollary 1. *The discrete system* (2) *leaves a* 2-*dimensional plane Γ invariant, and every solution on Γ lies on an elliptic curve, if and only if* (2) *admits a quadratic invariant $S[x]$ such that S is* (*semi*) *positive-definite and is of rank* 2. *In this case, Γ is characterized as $J \cdot Im\,(S)$.*

3. Remarks on symmetry generated by quadratic invariant

According to [2], the whole of quadratic invariants is closed with respect to the Poisson bracket [1], and forms a Lie algebra Θ. The Poisson bracket is represented on Ω as follows.

$$\{S, T\} = SJT - TJS.$$

Returning to M_{ij} defined by (4), we consider $M_1 = (J\xi_1)(j\eta_1)'$ and $M_2 = (J\xi_2)(J\eta_2)'$. Then, it follows that

$$\{M_1, M_2\} = \langle \eta_1, \xi_2 \rangle (J\xi_1)(J\eta_2)' - \langle \eta_2, \xi_1 \rangle (J\xi_2)(J\eta_1)'.$$

Due to Appendix 1, we can see that $\{M_1, M_2\}$ vanishes, if $\{\xi_1, \eta_1\}$ and $\{\xi_2, \eta_2\}$ belong to different quartettes of the four generalized eigenspaces mentioned in the previous section. Therefore, U_k and V_k in §3 form a closed subalgebra and Θ is a direct sum of these subalgebras. In a word, the symmetry group generated by quadratic invariants is determined only by the structure of respective generalized eigenspaces.

Appendix 1. *Skew-orthogonality of* $\tilde{W}_a$ *and* $\tilde{W}_b$ $(ab \neq 1)$.

Let $\{\xi_1, \xi_2, \dots, \xi_u\}$ and $\{\eta_1, \eta_2, \dots, \eta_v\}$ be generalized eigenvectors of T which form two distinct Jordan blocks subject to (9). Then, since T is symplectic, it holds that

$$\langle \xi_i, \eta_j \rangle = ab \langle \xi_i, \eta_j \rangle + a(1 - \delta_{j1})\langle \xi_i, \eta_{j-1} \rangle + b(1 - \delta_{i1})\langle \xi_{i-1}, \eta_j \rangle + (1 - \delta_{j1})(1 - \delta_{i1})\langle \xi_{i-1}, \eta_{j-1} \rangle.$$

When $ab \neq 1$, we find that every $\langle \xi_i, \eta_j \rangle$ vanishes, starting from $\langle \xi_1, \eta_1 \rangle = 0$. Then, by considering all combinations of Jordan blocks, $\tilde{W}_a$ proves to be skew-orthogonal to $\tilde{W}_b$. In particular, when $a \neq \pm 1$, $\tilde{W}_a$ is skew-orthogonal to itself, that is, null.

Appendix 2. *Commutation relation between* $\tilde{W}_a$ *and* $\tilde{W}_{1/a}$ $(a \neq 1)$.

Suppose that $\tilde{W}_a$ is a direct sum of u Jordan blocks $K_1, \dots, K_u$, and K_i is spanned by a set of generalized eigenvectors $\{\xi_1^{(i)}, \xi_2^{(i)}, \dots, \xi_{j(i)}^{(i)}\}$ subject to $T\xi_j^{(i)} = a\xi_j^{(i)} + (1 - \delta_{j1})\xi_{j-1}^{(i)}$. Then, the followings hold good.

(1) There is a unique basis $\{\eta_1^{(1)}, \dots, \eta_{j(1)}^{(1)}, \dots, \eta_1^{(u)}, \dots, \eta_{j(u)}^{(u)}\}$ of $\tilde{W}_{1/a}$ such that $\langle \xi_i^{(\alpha)}, \eta_j^{(\beta)} \rangle = \delta_{ij}\delta_{\alpha\beta}$.

(2) For any i subject to $1 \leqq i \leqq u$, it holds that $(T - I/a)^k \eta_1^{(i)} \neq 0$ $(0 \leqq k \leqq j(i) - 1)$ and $(T - I/a)^{j(i)} \eta_1^{(i)} = 0$. In other words, $\{(T - I/a)^k \eta_1^{(i)}\}_{k = j(i)-1, \dots, 0}$ constructs a Jordan block in $\tilde{W}_{1/a}$.

(3) $(T - I/a)^k \eta_1^{(i)}$ $(0 \leqq k \leqq j(i) - 1)$ is expressed as a linear combination of $\eta_{j(i)}^{(i)}, \dots, \eta_{k+1}^{(i)}$.

First, we intend to verify the following proposition. When L is a direct sum of three subspaces: $L = L_1 + L_2 + L_3$ such that L_1 and L_2 are null, and L_3 is skew-orthogonal to L_1 and L_2, then it follows that

(a) $\dim L_1 = \dim L_2$.

(b) For an arbitrary basis $\{\xi_i\}$ of L_1 there is a unique basis $\{\eta_i\}$ of L_2 subject to $\langle \xi_i, \eta_j \rangle = \delta_{ij}$.

It is to be noted that for any non-zero vector $\xi \in L_1$ there is a vector η subject to $\langle \xi, \eta \rangle = 1$ because of nondegeneracy, and η can be chosen as a vector in L_2 by supposition. By using the fact, induction with respect to k can prove that there are vectors $\xi_1, \dots, \xi_k \in L_1$ and $\eta_1, \dots, \eta_k \in L_2$ subject to $\langle \xi_i, \eta_j \rangle = \delta_{ij}$, as far as $k \leqq \dim L_1$. Exchanging the roles of L_1 and L_2, we have (a) and have shown that there are bases of L_1 and L_2 subject to $\langle\xi_i, \eta_j\rangle = \delta_{ij}$. Next, fix a pair of bases mentioned above, and choose an arbitrary basis $\{\xi_i^c\}$ of L_1 such that $\xi_i^c = \Sigma_j a_{ji}\xi_j$. Then, $\eta_i^c = \Sigma_j b_{ji}\eta_j$ satisfies $\langle\xi_i^c, \eta_j^c\rangle = \delta_{ij}$, if and only if $\Sigma_k a_{ki}b_{kj} = \delta_{ij}$. This proves (b).

If we put $L_1 = \tilde{W}_a$ and $L_2 = \tilde{W}_{1/a}$, and let L_3 be a direct sum of the remaining generalized eigenspaces, the assumption is satisfied because of Appendix 1. Then, Assertion (1) is verified.

Assertion (1) means that there is a symplectic matrix X such that $X^{-1}TX$ is block-diagonal, and that a half of its blocks are Jordan ones. For each Jordan block A, another block A'^{-1} exists, for $X^{-1}TX$ is also symplectic. That is, with respect to the vectors listed in (1), it holds that

$$T(\xi_1^{(i)},\dots,\xi_{j(i)}^{(i)}) = (\xi_1^{(i)},\dots,\xi_{j(i)}^{(i)}) \begin{pmatrix} a & 1 & 0 & . & .. \\ 0 & a & 1 & 0 & .. \\ 0 & 0 & a & 1 & .. \\ . & . & . & . & .. \end{pmatrix},$$

$$T(\eta_1^{(i)},\dots,\eta_{j(i)}^{(i)}) = (\eta_1^{(i)},\dots,\eta_{j(i)}^{(i)}) \begin{pmatrix} 1/a & 0 & 0 & 0 & .. \\ -1/a^2 & 1/a & 0 & 0 & .. \\ 1/a^3 & -1/a^2 & 1/a & 0 & .. \\ . & . & . & . & .. \end{pmatrix}.$$

This proves (2) and (3).

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