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On the Normal Generation of Ample Line Bundles on Abelian Varieties Defined Over Some Special Field

By

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Abstract

Let $\mathcal{L}$ be an ample line bundle on an abelian variety A defined over an algebraically closed field k. We already know that $\mathcal{L}$ is normally generated if $\mathcal{L}$ is base point free and $\mathrm{char}(k) \neq 2$. In this article, we prove that the above result is also true if $\mathrm{char}(k) = 2$.

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0 Introduction

Let $\mathcal{L}$ be an ample line bundle on an abelian variety A defined over an algebraically closed field k. It is known that $\mathcal{L}^{\otimes n}$ is very ample and normally generated if $n \geq 3$, but in general, it is not very ample, not normally generated if $n = 2$. In Ohbuchi [7], we have the following result.

Theorem A *$\mathcal{L}^{\otimes 2}$ is not very ample if and only if the polarized variety $(A, \mathcal{L})$ is isomorphic to $(A_1 \times A_2, \mathcal{O}(D_1 \times A_2 + A_1 \times D_2))$ where A_1, A_2 are abelian varieties,* $\mathrm{dim} A_1 > 0$, $\mathrm{dim}\Gamma(A_1, \mathcal{O}(D_1)) = 1$ *and* $\mathrm{dim} A_2 \geq 0$.

And in Ohbuchi [8], we have the following result.

Theorem B *We assume that $\mathcal{L}$ is a symmetric ample line bundle and* $\mathrm{char}(k) \neq 2$. *Then $\mathcal{L}^{\otimes 2}$ is normally generated if and only if* $\phi_{\mathcal{L}}(\mathrm{Bs}|\mathcal{L}|) \cap \hat{A}[2] = \emptyset$.

The main purpose of this paper is to prove Theorem B under the assumption of $\mathrm{char}(k) = 2$. The main theorem is the following;

Theorem C *We assume that $\mathcal{L}$ is a symmetric ample line bundle. Then $\mathcal{L}^{\otimes 2}$ is normally generated if and only if $\phi_{\mathcal{L}}(\mathrm{Bs}|\mathcal{L}|) \cap \hat{A}[2] = \emptyset$.*

NOTATIONS
$\mathcal{O}_A$: The structure sheaf of a variety A
f^*: The pull back defined by a morphism f
f_*: The direct image defined by a morphism f
$\underline{L}$: The invertible sheaf associated with a line bundle L
$\Gamma(A, \mathcal{F})$: The global sections of a sheaf $\mathcal{F}$
$H^i(C, \mathcal{F})$: The i-th cohomology group of a sheaf $\mathcal{F}$
$\underline{G}$: The functor defined by a group scheme G
$i(L)$: The index of an non-degenerate line bundle L on an abelien variety A defined by the unique integer $i = i(L)$, $0 \leq i(L) \leq \dim(A)$ such that $H^j(A, \underline{L}) = \{0\}$ for $j \neq i$ and $H^i(A, \underline{L}) \neq 0$
$\hat{A}$: The dual abelian variety of an abelian variety A
$A[n]$: The n-torsion points of an abelian variety A
$\mathcal{G}(L)$: A theta group of a line bundle L on an abelian variety
ϕ_L: The dualizing map on an abelian variety A defined by a line bundle L
$K(L)$: The kernel of ϕ_L
$T_f : A \times S \to A \times S$: The translation morphism on an abelian variety A defined by an $f \in \underline{A}(S)$
$n_A : A \to A$: The morphism on an abelian variety defined by $n_A(x) = nx$ where n is an integer

1 General Theory of Theta Group

Let k be an algebraically closed field. The following definition is found in Mumford [5] p.221.

Definition 1 *A theta group is a system of group schemes and homomorphism*

$$0 \to \mathbb{G}_m \xrightarrow{i} G \xrightarrow{\pi} K \to 1$$

such that

(a) *K is a commutative group scheme*
(b) *there is an open covering $\{U_i\}_{i\in I}$ and sections $\sigma_i : U_i \to G (i \in I)$ of π i.e. $\pi\sigma_i = \mathrm{id}_{U_i}$*
(c)*i is a closed immersion, making $\mathbb{G}_m$ into the kernel of π*
(d)*$\mathbb{G} \subset$ the center of $\mathcal{G}$*

Let A be an abelian variety defined over k and let L be a line bundle on A. The following theorem is found in Mumford [5] p.225 Theorem 1).

Theorem 1 *For any scheme S, let $\underline{\mathrm{Aut}}(L/A)(S)$ be the group of automorphisms of $L_S = L \times S$ on $A \times S$. Then $\underline{\mathrm{Aut}}(L/A)$ is a contravariant functor from (Sch) to (Gr) and there* *is a group scheme* $\mathcal{G}(L)$ *and an isomorphism of group functors*

$$\underline{Aut}(L/S) \cong \underline{\mathcal{G}(L)}.$$

And the natural homomorphism of groups

$$1 \to H^0(S, \mathcal{O}_S^*) \xrightarrow{i(S)} \underline{Aut}(L/A)(S) \xrightarrow{j(S)} \{f \in \underline{A}(S); T_f^*(L_S) \cong L_S\} \to 1$$

induces homomorphisms of groups

$$1 \to \mathbb{G}_m \xrightarrow{i} \mathcal{G}(L) \xrightarrow{j} K(L) \to 1$$

making $\mathcal{G}(L)$ *into a theta group and* $\mathbb{G}_m =$ *the center of* $\mathcal{G}(L)$.

Proof. See Mumford [5], p.225 Theorem 1.

Q.E.D.

Lemma 1 *There is a cross section* $\tau : K(L) \to \mathcal{G}(L)$ *such that* $j\tau = \mathrm{id}$ *and* (i, τ) : $\mathbb{G}_m \times K(L) \to \mathcal{G}(L)$ *is an isomorphism as a scheme.*

Proof. See Mumford [5], p.221.

Q.E.D.

Definition 2 *Let* V *be a finite dimensional vector space and let* l *be an integer. A representation of* $\mathcal{G}(L)$ *on* V *of weight* l *is a group scheme homomorphism* $\sigma : \mathcal{G}(L) \to GL(V)$ *such that* $\sigma(\lambda) = \lambda^l \mathrm{id}_V$ *for any* $\lambda \in \underline{\mathbb{G}}_m(S)$.

Theorem 2 *Let* $R = \mathrm{Spec}(R)$ *be an affine scheme defined over* k. *Then there is a functorial isomorphism* $\underline{h}(S)$ *such that*

$$\underline{h}(S) : \underline{\mathcal{G}}(L)(S) \to \{(x, \psi); x \in \underline{K(L)}(S), \psi : L_S \xrightarrow{\sim} T_x^* L_S\}.$$

Proof. By the definition, $\underline{Aut}(L/A) \cong \{(x, \psi); x \in \underline{K(L)}(S), \psi : L_S \xrightarrow{\sim} T_x^* L_S\}$. So Theorem is given by Theorem 1.

Q.E.D.

Definition 3 *Let* $S = \mathrm{Spec}(R)$ *be an affine scheme defined over* k *and let* $i = i(L)$ *be the index of* L. *Let* $(z, \psi) \in \underline{\mathcal{G}(L)}(S)$. *We put* $U_z : H^i(A \times S, \underline{L}_S) \xrightarrow{T_z^*} H^i(A \times S, T_x^*(\underline{L}_S)) \xleftarrow{H^i(\psi)} H^i(A \times S, \underline{L}_S)$.

By Definition 3, we have a representation of $\mathcal{G}(L)$ on $H^i(A, \underline{L})$ of weight 1.

Theorem 3 (Theta Structure Theorem) *The representation in Definition 3 is the only irreducible representation of* $\mathcal{G}(L)$ *on* $H^i(A, \underline{L})$ *of weight 1. And any representation of* $\mathcal{G}(L)$ *of weight 1 is completely reducible.*

Proof. See Sekiguchi [12] p.726 Theorem A.6.

Q.E.D.

Theorem 4 *Let $f : A \to B$ be an isogeny of abelian varieties A, B with kernel K, let L be a non-degenerate line bundle on B and let M is a non-degenerate line bundle on A. We assume that there is an isomorphism $\alpha : f^*L \xrightarrow{\sim} M$. Let K^* be the level subgroup corresponding to α, let $\mathcal{G}(M)^*$ be the centeralizer of K^* in $\mathcal{G}(L)$ and let $\overline{f} : \mathcal{G}(M)^* \to \mathcal{G}(M)$ be a canonical map. If R is a k-algebra and $z \in \mathcal{G}(M)^*(\mathrm{Spec}(R))$, then we have the following commutative diagram:*

$$\begin{array}{ccc} H^i(B,\underline{M}) \otimes R & \xrightarrow{f^*} & H^i(A,\underline{L}) \otimes R \\ U_{\overline{f}(z)} \downarrow & & \downarrow U_z \\ H^i(B,\underline{M}) \otimes R & \xrightarrow{f^*} & H^i(A,\underline{L}) \otimes R. \end{array}$$

Proof. See Mumford [6] p.297 Theorem 2 and Sekiguchi [12] p.710 Proposition 0.1.

Q.E.D.

Theorem 5 *Let L be a non-degenerate line bundle with index i on A and let M be a non-degenerate line bundle with index j on B. Let R be a k-algebra and let $z = (z_1, z_2) \in \underline{\mathcal{G}(L)}(\mathrm{Spec}(R)) \times \underline{\mathcal{G}(M)}(\mathrm{Spec}(R))$ and $\overline{z}$ is a canonical image of z in $\underline{\mathcal{G}(p_1^*L \otimes p_2^*M)}(\mathrm{Spec}(R))$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} H^i(A,\underline{L}) \otimes H^j(B,\underline{M}) \otimes R & \to & H^{i+j}(A \times B, p_1^*\underline{L} \otimes p_2^*\underline{M}) \otimes R \\ U_{z_1} \otimes U_{z_2} \downarrow & & \downarrow U_{\overline{z}} \\ H^i(A,\underline{L}) \otimes H^j(B,\underline{M}) \otimes R & \to & H^{i+j}(A \times B, p_1^*\underline{L} \otimes p_2^*\underline{M}) \otimes R \end{array}$$

where p_1 and p_2 are projections on $A \times B$.

Proof. See Sekiguchi [12] p.710 Proposition 0.4.

Q.E.D.

Theorem 6 *Let $\pi : A \to B$ be an isogeny of abelian varieties A, B and let L be a line bundle on A. Then there is a natural one-to-one correspondence between*

a) isomorphism classe of line bundle M on B such that $\pi^*M \cong L$

b) homomorphism $ker(\pi) \to \mathcal{G}(L)$ lifting the inclusion $ker(\pi) \subset A$.

Proof. See Mumford [5] p.231 Theorem 2.

Q.E.D.

Definition 4 *The subgroup scheme $ker(\pi) \subset \mathcal{G}(L)$ is called a level subgroup of $\mathcal{G}(L)$.*

2 Normal Generation of $L^{\otimes 2}$

Let L be an ample line bundle on an abelian variety A defined over an algebraically closed field k. We first recall the definition of normal generation of L.

Definition 5 *L is normally generated if*
$$\Gamma(A,\underline{L})^{\otimes n} \to \Gamma(A,\underline{L}^{\otimes n})$$
is surjective for any positive integer n.

In this section, we prove Theorem C. First we prepare several lemmas and definitions.

Lemma 2 *Let L be a symmetric (i.e. $(-1)_A^*L \cong L$) ample line bundle on A. We put $\xi : A \times A \to A \times A$ by $\xi(x,y) = (x+y, x-y)$. Then $\xi^*(p_1^*L \otimes p_2^*L) \cong p_1^*L^{\otimes 2} \otimes p_2^*L^{\otimes 2}$.*

Proof. See Mumford [6] p.320 Proposition 1.

Q.E.D.

Lemma 3 *Assume that M is an ample line bundle on A, then*
$$\Gamma(A,\underline{L}^{\otimes m}) \otimes \Gamma(A,\underline{L}^{\otimes n}) \to \Gamma(A,\underline{L}^{\otimes m+n})$$
is surjective if $n \geq 2$ and $m \geq 3$.

Proof. See Koizumi [4] p.882 Theorem 4.6., Sekiguchi [11] p.321 Theorem 2.3. and Sekiguchi [12] p.723 Theorem 2.4.

Q.E.D.

The following definitions are found in Mumford [5] pp.104-108.

Definition 6 *Let $x \in A$. We put $\mathbb{H}_x = Hom_{\text{cont}}(\mathcal{O}_{x,A}, k)$ and $\mathbb{H} = \bigoplus_{x \in A} \mathbb{H}_x$ where Hom_{cont} means the linear map*
$$L : \mathcal{O}_{A,x} \to K \text{ with } L(\mathfrak{m}_x^s) = \{0\}$$
for some integer $s \in \mathbb{N}$ where $\mathfrak{m}_x$ is a unique maximal ideal of $\mathcal{O}_{A,x}$.

If $L \in \mathbb{H}$, then $L \in \Gamma(Z, \mathcal{O}_Z)^*$ for some Z where Z is a zero-dimensional subscheme of A. We call L is supported by Z if $L \in \Gamma(Z,\mathcal{O}_Z)^*$. Let $m : A \times A \to A$ be a multiplication morphism on A and $m^* : \mathcal{O}_A \otimes \mathcal{O}_A \to \mathcal{O}_A$ be its dual. Let $L \in \mathbb{H}$ and we assume that L is supported by $x \in A$ for some closed point A.

Definition 7 *A differential operator $D_L : \mathcal{O}_A \to \mathcal{O}_A$ which consists of maps $D_L : \Gamma(U,\mathcal{O}_A) \to \Gamma(T_x^*U, \mathcal{O}_A)$ is defined by the composition:*
$$D_L = (\mathrm{id} \otimes L)m^* : \mathcal{O}_A \to \mathcal{O}_A \otimes \mathcal{O}_A \to \mathcal{O}_A \times \mathcal{O}_{A,x} \to \mathcal{O}_A.$$

Let $\Lambda_n = k[\epsilon]/(\epsilon^n)$. Let $t_n^{(i)} : \mathrm{Spec}(\Lambda_n) \to A$ be a morphism defined by $(t_n^{(i)})^*(t_s) = 0$ if $s \neq i$ and $(t_n^{(i)})^*(t_s) = \epsilon$ if $s = i$ and support of $t_n^{(i)} = 0 \in \underline{A}(\mathrm{Spec}(k))$. We assume that $t_n^{(i)} \in \underline{K(L)}(\mathrm{Spec}(\Lambda_n))$. Then we have a map

$$U_{t_n^{(i)}} : \Gamma(A \times \mathrm{Spec}(\Lambda_n), \underline{L} \otimes \Lambda_n) \to \Gamma(A \times \mathrm{Spec}(\Lambda_n), \underline{L} \otimes \Lambda_n).$$

We put $V_n^{(i)} = U_{t_n^{(i)}}$. Let $s \in \Gamma(A, \underline{L})$. We set $V_n^{(i)}(s) = \Sigma_{l=0}^n (D_{l,n}^{(i)} s)\epsilon^l$. Then $D_{l,n}^{(i)}$ is a differential operator.

Lemma 4 *If $t_n^{(i)}, t_m^{(i)} \in K(L)$, then $D_{l,n} = D_{l,m}$.*

Proof. We may assume that $m = n - 1$. A canonical inclusion

$$\iota : \mathrm{Spec}(\Lambda_{n-1}) \to \mathrm{Spec}(\Lambda_n)$$

induces that

$$\iota^* : \underline{K(L)}(\mathrm{Spec}(\Lambda_n)) \to \underline{K(L)}(\mathrm{Spec}(\Lambda_{n-1})$$

and $\iota^*(t_{n-1}^{(i)}) = t_{n-1}^{(i)}$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} T^*_{t_n^{(i)}}(L \otimes \Lambda_n) & \overset{\sim}{\to} & L \otimes \Lambda_n \\ \downarrow & & \downarrow \\ T^*_{t_{n-1}^{(i)}}(L \otimes \Lambda_{n-1}) & \overset{\sim}{\to} & L \otimes \Lambda_{n-1}. \end{array}$$

Therefore we get the following diagram:

$$\begin{array}{ccc} \Gamma(A \times \mathrm{Spec}(\Lambda_n), \underline{L} \otimes \Lambda_n) & \to & \Gamma(A \times \mathrm{Spec}(\Lambda_n), \underline{L} \otimes \Lambda_n) \\ \downarrow & & \downarrow \\ \Gamma(A \times \mathrm{Spec}(\Lambda_{n-1}), \underline{L} \otimes \Lambda_{n-1}) & \to & \Gamma(A \times \mathrm{Spec}(\Lambda_{n-1}), \underline{L} \otimes \Lambda_{n-1}). \end{array}$$

Hence we have Lemma.

Q.E.D.

By Lemma 4, $D_{l,n}^{(i)}$ is independent of the choice of n. We put $D_l^{(i)} = D_{l,n}^{(i)}$.

Proposition 1 *Let $s \in \Gamma(A, \underline{L})$. There are integers N, $i_1, \cdots, i_j$ and $t_j^{(i_1)}, \cdots t_j^{(i_j)} \in \underline{K(L)}(\mathrm{Spec}(\Lambda_j))$ for every $j = 0, \cdots, N$ such that $\Gamma(A, \underline{L})$ is spanned by $D_j^{(l)} U_x s = U_x D_j^{(l)} s$ where $x \in \underline{K(L)}(\mathrm{Spec}(k))$.*

Proof. By Theorem 3, Proposition is clear.

Q.E.D.

Let $V = 2_A^* \Gamma(A, \underline{L}) \subset \Gamma(A, 2_A^* \underline{L})$ where $2_A : A \to A$ is given by $2_A(x) = 2x$. Let $V_x = U_x(V)$ where $x \in \tau(K(2_A^* L))$ is a closed point and τ is a cross section given in Lemma 1 and let $V_{x, D_j^{(l)}} = V_{x,j}^{(l)} = D_j^{(l)}(V_x)$.

Lemma 5 *There are closed points $x_1 \cdots, x_t \in \tau(K(2_A^* L))$ such that $\Gamma(A, 2_A^* \underline{L}) = \sum_{i=1}^{t} \sum_{j,l} V_{x_i,j}^{(l)}$.*

Proof. Lemma is given by Proposition 1.

Q.E.D.

Theorem 7 *We have the following commutative diagram:*

$$\begin{array}{ccc} \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}) & \overset{\xi^*}{\to} & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2}))\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}) \\ 2_{A\times A}^* \uparrow & & \uparrow 2_{A\times A}^* \\ V_{x,D_j^{(l)}}\otimes V_{x,D_j^{(l)}} & \overset{\xi^*}{\to} & \Gamma(A,\underline{L}^{\otimes 2}))\otimes\Gamma(A,\underline{L}^{\otimes 2}) \end{array}$$

Proof. Let $t\in \underline{\mathcal{G}(2_A^*L)}(\mathrm{Spec}(R))$. By Theorem 5, we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma(A,2_A^*\underline{L})\otimes\Gamma(A,2_A^*\underline{L})\otimes R & \to & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R \\ U_t\otimes U_t\downarrow & & \downarrow U_{(t,t)} \\ \Gamma(A,2_A^*\underline{L})\otimes\Gamma(A,2_A^*\underline{L})\otimes R & \to & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R. \end{array}$$

Moreover we have the following commutative diagram by Theorem 4:

$$\begin{array}{ccc} \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R & \overset{\xi^*}{\to} & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R \\ U_{\overline{\xi}(z)}\downarrow & & \downarrow U_z \\ \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R & \overset{\xi^*}{\to} & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R \end{array}$$

where $z\in\underline{\mathcal{G}(p_1^*(2_A^*L)\otimes p_2^*(2_A^*L)^*}(\mathrm{Spec}(R))$. As $A[2]\times A[2]$ is a level subgroup of $\mathcal{G}(p_1^*(2_A^*L^{\otimes 2})\otimes p_2^*(2_A^*L^{\otimes 2})$ and $\mathcal{G}(p_1^*(2_A^*L)\otimes p_2^*(2_A^*L)$. We take $z\in\underline{A[2]\times A[2]}(\mathrm{Spec}(R))$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R & \overset{\xi^*}{\to} & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R \\ U_{(t,t)}\downarrow & & \downarrow U_z \\ \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R & \overset{\xi^*}{\to} & \Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))\otimes R \end{array}$$

where $\xi(z)=(t,t)$ because $2z=0$. By the definition of V, we have that

$$V\otimes V=\Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))^{A[2]\times A[2]}.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccc} V\otimes V\otimes R & \to & \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R \\ \mathrm{id}\downarrow & & \downarrow U_{(t,t)} \\ V\otimes V\otimes R & \to & \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))\otimes R. \end{array}$$

As $\Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L}))^{\text{diagonal image of } A[2] \text{ in } A[2]\times A[2]}=\Gamma(A\times A, p_1^*(\underline{L}^{\otimes 2})\otimes p_2^*(\underline{L}^{\otimes 2}))$ and a diagonal image of $A[2]$ in $A[2]\times A[2]$ is a level subgroup of $\mathcal{G}(p_1^*(2_A^*L)\otimes p_2^*(2_A^*L)$, we have $U_{(t,t)}U_{(s,s)}=U_{(s,s)}U_{(t,t)}$ for any $s,t\in\underline{A[2]}(\mathrm{Spec}(R))$ and the following commutative diagram:

$$\begin{array}{ccc} U_{(s,s)}V\otimes V\otimes R & \to & \Gamma(A\times A, p_1^*(\underline{L}^{\otimes 2})\otimes p_2^*(\underline{L}^{\otimes 2}))\otimes R \\ \mathrm{id}\downarrow & & \downarrow U_{(t,t)} \\ U_{(s,s)}V\otimes V\otimes R & \to & \Gamma(A\times A, p_1^*(\underline{L}^{\otimes 2})\otimes p_2^*(\underline{L}^{\otimes 2}))\otimes R \end{array}$$

for any $s,t\in\underline{A[2]}(\mathrm{Spec}(R))$. Hence we have Theorem.

Q.E.D.

Corollary 1 *We have the following commutative diagram:*

$$\begin{array}{ccc} \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L})) & \xrightarrow{\sim} & \Gamma(A\times A, p_1^*(\underline{L}^{\otimes 4})\otimes p_2^*(\underline{L}^{\otimes 4})) \\ 2_{A\times A}^*\uparrow & & \uparrow \xi^* \\ V_{x,D_j^{(l)}}\otimes V_{x,D_j^{(l)}} & \xrightarrow{\xi^*} & \Gamma(A,\underline{L}^{\otimes 2})\otimes\Gamma(A,\underline{L}^{\otimes 2}) \end{array}$$

Proof. As $\Gamma(A\times A, p_1^*(2_A^*\underline{L}^{\otimes 2})\otimes p_2^*(2_A^*\underline{L}^{\otimes 2}))^{\text{diagonal image of } A[2] \text{ in } A[2]\times A[2]} = \Gamma(A\times A, p_1^*(\underline{L}^{\otimes 4})\otimes p_2^*(\underline{L}^{\otimes 4}))$, the diagonal image of $A[2]$ in $A[2]\times A[2]$ is a level subgroup of $\mathcal{G}(p_1^*(2_A^*L^{\otimes 2})\otimes p_2^*(2_A^*L^{\otimes 2}))$ and $\xi\xi = 2_{A\times A}$, We have Corollary.

Q.E.D.

Proof of Theorem C.

By Theorem 7, we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma(A\times A, p_1^*(2_A^*\underline{L})\otimes p_2^*(2_A^*\underline{L})) & \xrightarrow{\sim} & \Gamma(A\times A, p_1^*(\underline{L}^{\otimes 4})\otimes p_2^*(\underline{L}^{\otimes 4})) \\ 2_{A\times A}^*\uparrow & & \uparrow 2_{A\times A}^* \\ V_{x,D_j^{(l)}}\otimes V_{x,D_j^{(l)}} & \xrightarrow{\xi^*} & \Gamma(A,\underline{L}^{\otimes 2})\otimes\Gamma(A,\underline{L}^{\otimes 2}). \end{array}$$

Let $W_{x,D_j^{(l)}} \subset \Gamma(A, 2_A^*\underline{L})$ be a subspace generated by $e^*(s)2_A^*(s')$ where $s, s' \in V_{x,D_j^{(l)}}$ and $e^* : V_{x,D_j^{(l)}} \to k$ be an evaluation map defined by $0 \in A$. As a cup product map $\mu : \Gamma(A,\underline{L}^{\otimes 2})\otimes\Gamma(A,\underline{L}^{\otimes 2}) \to \Gamma(A,\underline{L}^{\otimes 4})$ is given by

$$\Gamma(A,\underline{L}^{\otimes 2})\otimes\Gamma(A,\underline{L}^{\otimes 2}) \xrightarrow{\xi^*} \Gamma(A,\underline{L}^{\otimes 4})\otimes\Gamma(A,\underline{L}^{\otimes 4}) \xrightarrow{e^*\otimes \mathrm{id}} \Gamma(A,\underline{L}^{\otimes 4})$$

so we have that the image of $\Gamma(A,\underline{L}^{\otimes 2})\otimes\Gamma(A,\underline{L}^{\otimes 2}) \to \Gamma(A,\underline{L}^{\otimes 4}) \cong \Gamma(A, 2_A^*\underline{L})$ is equal to $\sum_{i=1}^{t}\sum_{j,l} W_{x_i,j}^{(l)}$ and $\Gamma(A, 2_A^*\underline{L}) = \sum_{i=1}^{t}\sum_{j,l} V_{x_i,j}^{(l)}$ by Lemma 5. Therefore μ is surjective if and only if $V_{x_i,j}^{(l)} = W_{x_i,j}^{(l)}$ for any x_i, j, l. By Lemma 3, μ is surjective if and only $L^{\otimes 2}$ is normally generated. Hence $L^{\otimes 2}$ is normally generated if and only if $V_{x_i,j}^{(l)} = W_{x_i,j}^{(l)}$ for any x_i, j, l. If there is an $s \in V_{x_i,j}^{(l)}$ such that $e^*(s) \neq 0$, then $V_{x_i,j}^{(l)} = W_{x_i,j}^{(l)}$ and if any $s \in V_{x_i,j}^{(l)}$ satisfies $e^*(s) = 0$, then $W_{x_i,j}^{(l)} = \{0\}$. Therefore $V_{x_i,j}^{(l)} = W_{x_i,j}^{(l)}$ for any $W_{x_i,j}^{(l)}$ if and only if $\phi_{\mathcal{L}}(\mathrm{Bs}|\mathcal{L}|) \cap \hat{A}[2] = \emptyset$. Hence Theorem is proved.

Q.E.D.

Corollary 2 *Let L be an ample line bundle on an abelian variety A. If L is base point free, then $L^{\otimes 2}$ is normally generated.*

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