J. Math. Tokushima Univ.
Vol. 29 (1995), 9–22

On the Classical and the Quantum Mechanics in a Magnetic Field

Dedicated to Professor Yoshihiro Ichijyō on his 65th birthday

By

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(Received September 14, 1995)

Abstract

This paper studies the relationship between the classical mechanics in a magnetic field and its quantized system. The phase space for classical motion in a magnetic field is derived through the Marsden-Weinstein reduction starting with a principal $U(1)$-bundle . The quantized system associated to the classical system is defined as the Bochner-Laplacian on a line bundle associated to the principal bundle. In this context we obtain a generalization of Helton's theorem which gives a characterization of the periodicity of the classical trajectories by the spectrum of the associated quantum Hamiltonian (the Bochner-Laplacian).

1991 *Mathematics Subject Classification.* Primary 58G25: Secondary 58F05

1. Introduction

The purpose of this paper is to consider from the viewpoint of "Spectral Geometry" the relationship between a quantum mechanics and the associated classical mechanics in a magnetic field. In "Spectral Geometry" the spectrum of the Laplacian on a Riemannian manifold has been investigated from various viewpoint. Among others it has been clarified by many authors ([2], [3], for instance) that the spectrum of the Laplacian is closely

*Partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, Science, Sports and Culture

related to dynamical properties of the geodesic flow (the associated classical system to the Laplacian). On the other hand, geometrical definitions of the classical phase space for the motion in a gauge field were presented by Sternberg [13], Guillemin and Sternberg [5], Weinstein [15], and so on. On the basis of the formulation in [5] R. Schrader and M. Taylor [12] considered the eigenfunctions of a quantum Hamiltonian for a particle in a gauge field. They clarified that eigenfunctions are "uniformly distributed" in a certain region of the phase space if the classical flow is ergodic (see also [16]). In this paper we pay attention to the opposite case where every orbit of the classical flow is closed, and give a generalization of Helton's theorem [6].

Section 2 is devoted to a geometrical formulation of the classical motion of a particle in a gauge field through the reduction-procedure by Marsden and Weinstein [11] starting with a principal bundle over the configuration space. A magnetic field is defined as a connection (or its curvature) on a principal $U(1)$-bundle, which is related to the symplectic structure in the Hamiltonian dynamical system. In §3 we introduce a line bundle with a connection and the Bochner-Laplacian on it, which are regarded as quantum objects associated to the classical mechanics in the magnetic field. Motivated by Schrader and Taylor [12] we consider in §4 the $U(1)$-invariant pseudo-differential operators on the principal $U(1)$-bundle, which are found to be natural and useful for the spectral analysis on the associated line bundle. It should be noticed that a principal symbol in our sense has more information of the pseudo-differential operator than the usually defined principal symbol. Finally in §5 we present and prove Helton's theorem for a quantum Hamiltonian (the Bochner-Laplacian) for a particle in a magnetic field.

2. Classical mechanics in a magnetic field

Consider a principal G-bundle $\pi : P \to M$ over a compact C^∞ manifold M, where G is a connected compact Lie group acting freely on P on the right. The action of G on P is naturally lifted to the cotangent bundle T^*P, and the lifted action preserves the natural symplectic 2-form Ω on T^*P.

The Marsden-Weinstein reduction. The above symplectic action on (T^*P, Ω) defines the momentum map $J : T^*P \to \mathfrak{g}^*$ (the dual of the Lie algebra $\mathfrak{g}$ of G) as

$$\langle J(\alpha), v\rangle = \langle \alpha, v_P(p)\rangle \quad (\alpha \in T_p^*P,\ p \in P,\ v \in \mathfrak{g}),$$

where v_P is the vector filed on P induced from v by the action of G. Then, the following is easily checked.

Lemma 2.1. (1) *Each $\mu \in \mathfrak{g}^*$ is a regular value of J.*
(2) *J is Ad^*-equivariant, i.e.,*

$$J \circ R_g^* = Ad_{g^{-1}}^* \circ J$$

holds for any $g \in G$, where R_g denotes the (right) action: $p \mapsto p \cdot g$ on P by g and Ad_g^* is the co-adjoint action on $\mathfrak{g}^*$ by g.

For each $\mu \in \mathfrak{g}^*$, $J^{-1}(\mu)$ is a C^∞ submanifold of T^*P because of (1) of the above lemma. Let $G_\mu = \{g \in G; Ad^*_{g^{-1}}\mu = \mu\}$. Then, G_μ is a closed subgroup of G, and $J^{-1}(\mu)$ is left invariant under the action of G_μ. Moreover, it is easily seen that the action of G_μ on $J^{-1}(\mu)$ is free and proper (because G_μ is compact). Hence we get the C^∞ manifold $P_\mu = J^{-1}(\mu)/G_\mu$. J. Marsden and A. Weinstein [11] showed the existence of such a symplectic structure Ω_μ on P_μ that

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega,$$

where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the natural projection, and $i_\mu : J^{-1}(\mu) \to T^*P$ is the inclusion map. Thus we get the symplectic manifold (P_μ, Ω_μ) associated to $\mu \in \mathfrak{g}^*$, which is called the *reduced phase space* of (T^*P, Ω).

Connections on P. Suppose the principal G-bundle P is endowed with a connection $\widetilde{\nabla}$, namely, for each $p \in P$ there smoothly attached a linear subspace H_p (called the *horizontal space*) of the tangent space T_pP which satisfies

$$T_pP = H_p \oplus V_p, \tag{2.1}$$

and

$$H_{p\cdot g} = R_{g*}(H_p) \quad (g \in G), \tag{2.2}$$

where

$$V_p = \{v_P(p);\ v \in \mathfrak{g}\} = \text{kernel of } (\pi_*)_p : T_pP \to T_{\pi(p)}M$$

(which is called the *vertical space*) (cf. [8]). Set

$$V_p^\perp = \{\alpha \in T_p^*P; \langle \alpha, v\rangle = 0 \text{ for } \forall v \in V_p\},$$

$$H_p^\perp = \{\alpha \in T_p^*P; \langle \alpha, v\rangle = 0 \text{ for } \forall v \in H_p\}.$$

Then, from (2.1) we have the decomposition

$$T_p^*P = H_p^\perp \oplus V_p^\perp. \tag{2.3}$$

For $v \in T_{\pi(p)}M$ there uniquely exists $\tilde{v} \in H_p$ such that $\pi_*(\tilde{v}) = v$, and we can define the smooth surjective map $\Phi : T^*P \to T^*M$ given by

$$\langle \Phi(\alpha), v\rangle = \langle \alpha, \tilde{v}\rangle \qquad (\alpha \in T_p^*P, v \in T_{\pi(p)}M),$$

(which induce the onto linear map $\Phi_p : T_p^*P \to T^*_{\pi(p)}M$).

Lemma 2.2. (1) *Put* $\alpha = \beta + \gamma$ $(\beta \in H_p^\perp, \gamma \in V_p^\perp)$ *for* $\alpha \in T_p^*P$ *according to* (2.3). *Then,* $\Phi_p(\alpha) = \Phi_p(\gamma)$.

(2) Φ_p *induces an isomorphism:* $V_p^\perp \cong T^*_{\pi(p)}M$.

(3) *For each* $g \in G$, $\Phi_{p\cdot g} = \Phi_p \circ R_g^*$ *holds on* $T^*_{p\cdot g}P$.

Proof. (1) and (2) is easy to check. (3) Take $\alpha \in T^*_{p\cdot g}$. Then, for $\forall v \in T_{\pi(p)}M$ we have by (2.2)

$$\langle \Phi_p \circ R^*_g(\alpha), v\rangle = \langle R^*_g(\alpha), \tilde{v}_p\rangle = \langle \alpha, R_{g*}(\tilde{v}_p)\rangle = \langle \alpha, \tilde{v}_{p\cdot g}\rangle = \langle \Phi_{p\cdot g}(\alpha), v\rangle. \quad \square$$

Suppose $\alpha \in J^{-1}(\mu) \cap T^*_pP$ $(\mu \in \mathfrak{g}^*, p \in P)$. Then,

$$\langle J(\alpha), v\rangle \equiv \langle \alpha, v_P(p)\rangle = \langle \mu, v\rangle$$

for any $v \in \mathfrak{g}$. Hence, we have

$$J^{-1}(\mu) = \bigcup_{p\in P} (V^\perp_\mu)_p$$

where

$$(V^\perp_\mu)_p = \{\alpha \in T^*_pP;\ \langle \alpha, v_P(p)\rangle = \langle \mu, v\rangle \text{ for } \forall v \in \mathfrak{g}\}.$$

Since there exists unique $\hat{\mu} \in H^\perp_p$ such that $\langle \hat{\mu}, v_P(p)\rangle = \langle \mu, v\rangle$ for any $v \in \mathfrak{g}$, we have

$$(V^\perp_\mu)_p = \{\alpha = \hat{\mu} + \gamma;\ \gamma \in V^\perp_p\} = \hat{\mu} + V^\perp_p$$

(see Figure 1).

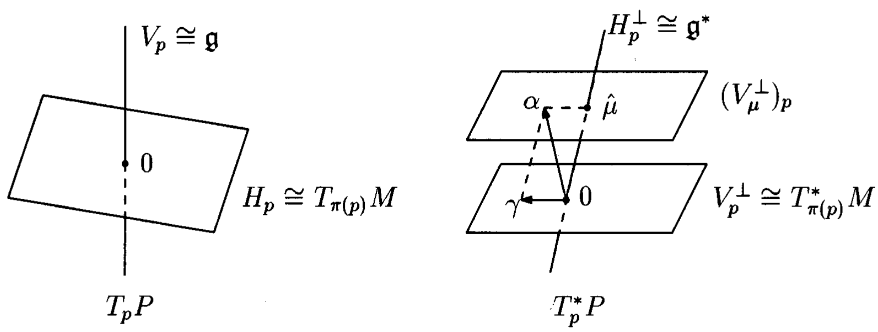

Figure 1

Lemma 2.3. *For each $g \in G_\mu$, and each $p \in P$, we have*

$$(V^\perp_\mu)_{p\cdot g} = R^*_{g^{-1}}((V^\perp_\mu)_p).$$

Proof. Noticing for $v \in \mathfrak{g}$ that $(R_{g^{-1}*}v_P)(p) = (Ad_g v)_P(p)$, we have for $\alpha \in (V^\perp_\mu)_p$

$$\langle R^*_{g^{-1}}(\alpha), v_P(p\cdot g)\rangle = \langle \alpha, (Ad_g v)_P\rangle = \langle J(\alpha), Ad_g v\rangle = \langle \mu, Ad_g v\rangle = \langle \mu, v\rangle.$$

Thus, $R^*_{g^{-1}}(\alpha)$ belongs to $(V^\perp_\mu)_{p\cdot g}$. $\square$

Let $\Phi_p^{(\mu)}$ denote the restriction of Φ_p to the subset $(V_\mu^\perp)_p \subset T_p^*P$. Then, $\Phi_p^{(\mu)}$ is a bijective map of $(V_\mu^\perp)_p$ onto $T_{\pi(p)}^*M$. Noticing Lemma 2.2,(3) and lemma 2.3, we obtain from Φ the smooth surjective map

$$\Psi_\mu : P_\mu = J^{-1}(\mu)/G_\mu \to T^*M,$$

which is a fibre bundle with the fibre G/G_μ. Here the manifold G/G_μ is identified with the co-adjoint orbit $\mathcal{O}_\mu = \{Ad_{g^{-1}}^*\mu;\ g \in G\}$ in $\mathfrak{g}^*$. Note that Ψ_μ is a diffeomorphism if $G_\mu = G$.

Classical dynamical systems in a magnetic field. From now on we consider the case where $G = U(1) = \{e^{it}; 0 \le t < 2\pi\}$. Then, the Lie algebra $\mathfrak{u}(1)$ of $U(1)$ and its dual $\mathfrak{u}(1)^*$ are both isomorphic to $\mathbf{R}$. In this case Ψ_μ is a bijection, and P_μ is diffeomorphic with T^*M. Let $\tilde{\theta}$ be the connection form of the connection $\widetilde{\nabla}$ on P, that is the $\mathfrak{u}(1)$-valued one-form on P assigning each vector $X = X_H + v_P(p)$ $(X_H \in H_p, v_P(p) \in V_p)$ in T_pP to $v \in \mathfrak{u}(1)$. The $\mathfrak{u}(1)$-valued two form $\tilde{\Theta} = d\tilde{\theta}$ on P is called the curvature form of $\widetilde{\nabla}$. Let $\{U_\alpha\}$ be a open covering of M, and $\{s_\alpha\}$ be a system of local sections of P associated to $\{U_\alpha\}$. Set $\Theta_\alpha = s_\alpha^*\tilde{\Theta}$. Then, on $U_\alpha \cap U_\beta (\neq \phi)$ we have

$$\Theta_\beta = Ad_{\varphi_{\alpha\beta}}\Theta_\alpha = \Theta_\alpha,$$

where $\varphi_{\alpha\beta}$ is the transition function: $U_\alpha \cap U_\beta \to U(1)$ satisfying $s_\alpha = s_\beta \cdot \varphi_{\alpha\beta}$. Thus $\Theta = \{\Theta_\alpha\}$ is a globally defined $\mathfrak{u}(1)$-valued two-form on M, which we call a *magnetic field.*

The following proposition describes the relationship between the magnetic field and the symplectic structure Ω_μ on the reduced phase space P_μ for $\mu \in \mathfrak{u}(1)^*$.

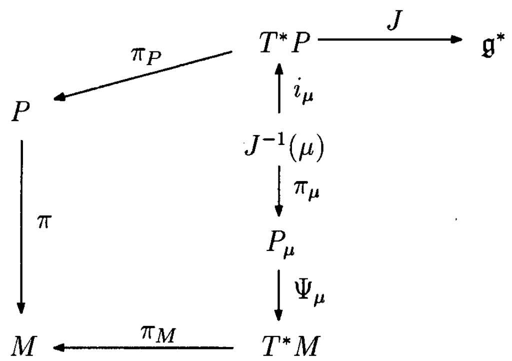

Figure 2

Proposition 2.4. *Let $\Theta_\mu = \langle \mu, \Theta \rangle$, which is a* $\mathbf{R}$*-valued two-form on M, and let Ω_0 be the natural symplectic form on T^*M. Then,*

$$\Omega_\mu = \Psi_\mu^*(\Omega_0 + \pi_M^*\Theta_\mu),$$

where π_M is the projection: $T^*M \to M$, that is, the symplectic manifold (P_μ, Ω_μ) is isomorphic with $(T^*M, \Omega_0 + \pi_M^*\Theta_\mu)$.

Proof. Let X be a vector in $T_{(p,\alpha)}T^*P$ ($p \in P, \alpha \in T_p^*P$). Then, X is expressed as

$$X = X_P + X^* \quad \text{with } X_P \in T_pP,\ X^* \in T_p^*P(= T_\alpha T_p^*P),$$

and X belongs to $T_{(p,\alpha)}J^{-1}(\mu)$ if and only if $X^* \in V_p^\perp$. Set $\Omega = d\omega$ (ω being the canonical one form on T^*P). Then, for two vector fields $X = X(p,\alpha)$, $Y = Y(p,\alpha)$ on a neighborhood of (p_0, α_0) in $J^{-1}(\mu)$ we have

$$\begin{aligned}\Omega(X,Y) &= \frac{1}{2}(X\langle\omega, Y\rangle - Y\langle\omega, X\rangle - \langle\omega, [X,Y]\rangle) \\ &= \frac{1}{2}(X\langle\alpha, Y_P\rangle - Y\langle\alpha, X_P\rangle - \langle\alpha, [X_P, Y_P]\rangle).\end{aligned}$$

Put $\alpha = \hat{\mu} + \gamma$ ($\gamma \in V_p^\perp$) and $X_P = X_H + X_V$ according to (2.1) (cf. Figure 1). Then, $\langle\hat{\mu}, (\cdot)_H\rangle = \langle\gamma, (\cdot)_V\rangle = 0$. X and Y are regarded as vector fields on P_μ if they are invariant under $U(1)$-action, i.e., X_H, X_V, Y_H and Y_V are $U(1)$-invariant. Hence, we find that $\langle\hat{\mu}, (\cdot)_V\rangle$ is a constant function and $[X_H, Y_V] = [X_V, Y_H] = [X_V, Y_V] = 0$. As a consequence, $\Omega(X,Y)$ is written as

$$(2.4)\qquad \frac{1}{2}\{(X_H + X^*)\langle\gamma, Y_H\rangle - (Y_H + Y^*)\langle\gamma, X_H\rangle - \langle\gamma, [X_H, Y_H]\rangle\} - \frac{1}{2}\langle\hat{\mu}, [X_H, Y_H]\rangle.$$

Note that $X_H + X^*$ is identified with $(\Psi_\mu \circ \pi_\mu)_*(X)$, and we see that the first term in (2.4) is nothing but $(\Psi_\mu^*\Omega_0)(\pi_\mu^*(X), \pi_\mu^*(Y))$. The second term is written using the connection form $\tilde{\theta}$ as

$$-\frac{1}{2}\langle\mu, \tilde{\theta}([X_H, Y_H])\rangle,$$

which is equal to $\Theta_\mu((\pi_M \circ \Psi_\mu \circ \pi_\mu)_*(X), (\pi_M \circ \Psi_\mu \circ \pi_\mu)_*(Y))$ (see [8, p.78]). □

Suppose M is endowed with a Riemannian metric m, which defines the inner products $(\cdot,\cdot)_x$ on T_xM and $(\cdot,\cdot)_x^*$ on T_x^*M, respectively for each $x \in M$. Let H be the function on T^*M defined by $H(\xi) = (\xi, \xi)_x^*$ ($x = \pi_M(\xi)$), and set $H_\mu = \Psi_\mu^*H$. The Hamiltonian dynamical system on (P_μ, Ω_μ) with the Hamiltonian function (the "kinetic energy") H_μ describes the motion of a (classical) particle with the "charge" $\mu (\in \mathfrak{u}(1)^*)$ in the magnetic field Θ.

3. Quantum mechanics in a magnetic field

We identify the Lie algebra $\mathfrak{u}(1)$ with $\mathbf{R}$ by $\exp t = e^{it} \in U(1)$, where exp is the exponential map: $\mathfrak{u}(1) \to U(1)$. Define the subset

$$\Lambda^* = \{\lambda \in \mathfrak{u}(1)^*;\ \langle\lambda, m\rangle \in \mathbf{Z} \text{ for } \forall m \in \mathbf{Z} \subset \mathbf{R} \cong \mathfrak{u}(1)\} \ \subset\ \mathfrak{u}(1)^*.$$

For each $\lambda \in \Lambda^*$ we define the irreducible unitary representation ρ_λ of $U(1)$ by

$$\rho_\lambda(e^{it}) = e^{i\langle\lambda, t\rangle} \ \in S^1 \subset \mathbf{C}\backslash 0.$$

Let $\pi_\lambda : E_\lambda \to M$ be the complex line bundle associated with the principal $U(1)$-bundle $\pi : P \to M$ by the representation ρ_λ, that is, the quotient manifold of $P \times \mathbf{C}$ with respect to the equivalence relation:

$$(p, z) \overset{\lambda}{\sim} (p \cdot g, \rho_\lambda(g^{-1})z) \quad (g \in U(1), z \in \mathbf{C}).$$

Each fibre of E_λ is naturally endowed with a metric.

The connection $\widetilde{\nabla}$ on the principal $U(1)$-bundle induces the linear connection $\widetilde{\nabla}^{(\lambda)}$ on E_λ which is defined as the covariant derivative $\widetilde{\nabla}^{(\lambda)} : A^0(M, E_\lambda)(= C^\infty(E_\lambda)) \to A^1(M, E_\lambda)$. Here $A^p(M, E_\lambda)$ $(0 \le p \le \dim M)$ denotes the space of E_λ-valued smooth p-forms on M, that is, $A^p(M, E_\lambda) = A^p(M) \otimes C^\infty(E_\lambda)$. We can extent the covariant derivative $\widetilde{\nabla}^{(\lambda)}$ to the space of E_λ-valued smooth tensor fields on M as

$$\widetilde{\nabla}^{(\lambda)}(T \otimes s) = \nabla T \otimes s + T \cdot \widetilde{\nabla}^{(\lambda)} s,$$

where T is a usual tensor field on M, $s \in C^\infty(E_\lambda)$, and ∇ is the Levi-Civita connection on (M, m). Let us take a local trivialization of the bundle $\pi : P \to M$: $\pi^{-1}(V) \cong V \times U(1)$, V being an open set of M. Let F_0 be the local section defined by

$$F_0(x) = (x, 1) \quad (x \in V),$$

$1 = e^0$ being the identity of $U(1)$, and let s_0 be the local section of E_λ defined by

$$s_0(x) = [(F_0(x), 1)]_\lambda.$$

Let $\tilde{\theta}$ and Θ be the connection form on P and the curvature form on M, respectively, of $\widetilde{\nabla}$. Then the following is easy to check.

Lemma 3.1. (1) *The connection form $\theta^{(\lambda)}$ of $\widetilde{\nabla}^{(\lambda)}$ with respect to the section s_0, i.e., $\widetilde{\nabla}^{(\lambda)} s_0 = \theta^{(\lambda)} s_0$, is given by*

$$\theta^{(\lambda)} = \rho_{\lambda *} F_0^* \tilde{\theta} = i\langle \lambda, F_0^* \tilde{\theta} \rangle.$$

(2) *The curvature form of $\widetilde{\nabla}^{(\lambda)}$ is given by*

$$\Theta^{(\lambda)} = i\Theta_\lambda \, (= i\langle \lambda, \Theta \rangle).$$

(3) *The connection $\widetilde{\nabla}^{(\lambda)}$ is compatible with the Hermitian structure in E_λ.*

From the connection $\widetilde{\nabla}^{(\lambda)}$ on E_λ and the Riemannian metric m on M we can naturally define a differential operator $L^{(\lambda)}$ called the *Bochner-Laplacian*, which is a second order, non-negative, formally self-adjoint elliptic operator acting on $C^\infty(E_\lambda)$ (see [9], for example). The operator $L^{(\lambda)}$ is locally expressed as

$$\begin{aligned} L^{(\lambda)} s &= -\sum_{j,k} m^{jk} \widetilde{\nabla}_j^{(\lambda)} \widetilde{\nabla}_k^{(\lambda)} s \\ &= -\left[\sum_{j,k} m^{jk} (\nabla_j + i a_j^{(\lambda)})(\nabla_k + i a_k^{(\lambda)}) f \right] s_0 \end{aligned}$$

for $s = fs_0$ on $V(\subset M)$ with $f \in C^\infty(V)$, where $\theta^{(\lambda)} = i\sum a_j^{(\lambda)} dx^j$. As the quantum object corresponding to the Hamiltonian system $(P_\lambda, \Omega_\lambda, H_\lambda)$ we take the differential operator $L^{(\lambda)}$ on E_λ, which is called the *Schrödinger operator with a magnetic vector potential.* Note that the classical system $(P_\lambda, \Omega_\lambda, H_\lambda)$ is quantized only for $\lambda \in \Lambda^*$, and that $L^{(0)}$ is just the Laplace-Beltrami operator on (M, m).

4. Pseudo-differential operators on the line bundle

Let $\mathcal{O}$ be an open set of $\mathbf{R}^n$, and put $\bar{\mathcal{O}} = \mathcal{O} \times U(1)$ $(U(1) = \{e^{it}; 0 \le t < 2\pi\})$. A (classical) pseudo-differential operator $A = a(\bar{x}, D)$ of order $m \in \mathbf{R}$ on $\bar{\mathcal{O}}$ is a linear operator of $C_0^\infty(\bar{\mathcal{O}})$ into $C^\infty(\bar{\mathcal{O}})$ given by

$$Au(x,t) = \int_{\mathbf{R}^n} \left(\sum_{\tau \in \mathbf{Z}} e^{ix\cdot\xi} e^{it\tau} a(x,t,\xi,\tau) \hat{u}(\xi,\tau) \right) d\xi$$

$(\bar{x} = (x,t) \in \mathcal{O} \times U(1))$, where $\hat{u}(\xi,\tau)$ is the Fourier transform of u:

$$\hat{u}(\xi,\tau) = (2\pi)^{-(n+1)} \int_{\mathcal{O}\times[0,2\pi]} e^{-iy\cdot\xi} e^{-is\tau} u(y,s) dy ds,$$

and $a(x,t,\xi,\tau) = a(\bar{x},\bar{\xi})$ is a smooth function on $\bar{\mathcal{O}} \times \mathbf{R}^{n+1}$ admitting an asymptotic expansion

$$a(\bar{x},\bar{\xi}) \sim \sum_{j=0}^{\infty} a_{m-j}(\bar{x},\bar{\xi})$$

with $a_{m-j}(\bar{x},\bar{\xi})$ positively homogeneous of degree $m-j$ in $\bar{\xi}$. Here $a_m(\bar{x},\bar{\xi})$ is called the principal symbol of A, and denoted by $\sigma(A)$. We denote by $\psi DO^m(\bar{\mathcal{O}})$ the set of properly supported pseudo-differential operators of order m on $\bar{\mathcal{O}}$. Then, ψDO's form an algebra, namely, for $A = a(\bar{x}, D) \in \psi DO^p(\bar{\mathcal{O}})$ and $B = b(\bar{x}, D) \in \psi DO^q(\bar{\mathcal{O}})$ we can see that $AB = C$ belongs to $\psi DO^{p+q}(\bar{\mathcal{O}})$ and $C = c(\bar{x}, D)$ with

$$c(\bar{x},\bar{\xi}) \sim \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial\bar{\xi}}\right)^{\alpha} a(\bar{x},\bar{\xi}) \cdot \left(-i\frac{\partial}{\partial\bar{x}}\right)^{\alpha} b(\bar{x},\bar{\xi})\ .$$

Let $\pi : P \to M$ be the principal $U(1)$-bundle considered in §3. Let V be a coordinate neighborhood of M with $\varphi : V \to \mathcal{O}$ a diffeomorphism onto an open subset $\mathcal{O}$ of $\mathbf{R}^n$, and suppose we have a local triviality $\pi^{-1}(V) \cong V \times U(1)$ of P. Then, we have a local coordinate $\bar{\varphi} : \pi^{-1}(V) \to \bar{\mathcal{O}} = \mathcal{O} \times U(1)$ of P. For $u \in C_0^\infty(\bar{\mathcal{O}})$, define $u_P = u \circ \bar{\varphi}$ (on $\pi^{-1}(V)$), $= 0$ (on $P\backslash\pi^{-1}(V)$), which belongs to $C^\infty(P)$. A linear operator $A : C^\infty(P) \to C^\infty(P)$ belongs to $\psi DO^m(P)$ (the set of pseudo-differential operators of order m on P) if for any local coordinate $(\pi^{-1}(U), \varphi, \bar{\mathcal{O}})$ the map, $A_{\bar{\mathcal{O}}}$, of $C_0^\infty(\bar{\mathcal{O}})$ into $C^\infty(\bar{\mathcal{O}})$ defined by $u(x) \mapsto (Au_P)(\bar{\varphi}^{-1}(\bar{x}))$ $(\bar{x} \in \bar{\mathcal{O}})$ belongs to $\psi DO^m(\bar{\mathcal{O}})$. By considering the behavior of $\sigma(A_{\bar{\mathcal{O}}})$ under a change of coordinates, we see that the family $\{\sigma(A_{\bar{\mathcal{O}}})\}$ defines a smooth function on T^*P called the principal symbol of A (denoted by $\sigma(A)$) (cf. [7, p.81], [14, p.47]).

For $g \in U(1)$, define $T_g : C^\infty(P) \to C^\infty(P)$ by $(T_g f)(p) = f(p \cdot g)$ $(p \in P)$. A pseudo-differential operator A on P is said to be $U(1)$-*invariant* if $A \circ T_g = T_g \circ A$ holds for any $g \in U(1)$.

Lemma 4.1. *Let $a(\bar{x}, \bar{\xi})$ be the principal symbol of a $U(1)$-invariant pseudo-differential operator A. Then,*

$$a((\bar{x}, \bar{\xi}) \cdot g) = a(\bar{x}, \bar{\xi})$$

holds for any $g \in U(1)$, where $(\bar{x}, \bar{\xi}) \mapsto (\bar{x}, \bar{\xi}) \cdot g = (\bar{x} \cdot g, R^*_{g^{-1}} \bar{\xi})$ is the action of $U(1)$ on T^*P. Thus, $a(\bar{x}, \bar{\xi})$ is a $U(1)$-invariant function on T^*P.

Proof. The lemma follows from the fact that the operator $B = T_g^{-1} \circ A \circ T_g$ is a psuedo-differential operator whose principal symbol $b(\bar{x}, \bar{\xi})$ is given by $b((\bar{x}, \bar{\xi}) \cdot g) = a(\bar{x}, \bar{\xi})$. □

Let $\pi_\lambda : E_\lambda \to M$ $(\lambda \in \Lambda^* \cong \mathbf{Z})$ be the associated line bundle of $\pi : P \to M$ as in §3. For each $p \in P$, define $\chi_p : \mathbf{C} \to \pi_\lambda^{-1}(\pi(p)) \subset E_\lambda$ by $z \mapsto [(p, z)]_\lambda$, and χ_p is a surjective $\mathbf{C}$-linear isometry. Let $C^\infty_\lambda(P)$ denote the set consisting of every $\mathbf{C}$-valued smooth function f on P such that

$$f(p \cdot g) = \rho_\lambda(g) f(p)$$

for every $g \in U(1)$, which is called an *equivariant function* with respect to ρ_λ. For $s \in C^\infty(E_\lambda)$, define the smooth function $\tilde{s}$ on P by $\tilde{s}(p) = \chi_p^{-1}(s(\pi(p))$. Then, $\tilde{s}$ belongs to $C^\infty_\lambda(P)$, and $\chi_\lambda : s \mapsto \tilde{s}$ gives a one-to-one correspondence between $C^\infty(E_\lambda)$ and $C^\infty_\lambda(P)$. We easily see the following.

Lemma 4.2. *Let A be a $U(1)$-invariant pseudo-differential operator on P. Then, $A\tilde{s}$ belongs to $C^\infty_\lambda(P)$ if $\tilde{s}$ belongs to $C^\infty_\lambda(P)$.*

Given a $U(1)$-invariant pseudo-differential operator A on P. Then, by virtue of the above lemma we can define the operator $A_\lambda : C^\infty(E_\lambda) \to C^\infty(E_\lambda)$ by $A_\lambda = \chi_\lambda^{-1} \circ A \circ \chi_\lambda$, which we call a pseudo-differential operator on E_λ.

Let F_0 and s_0 be the local sections of P and E_λ, respectively, defined in §3. For a local section $s = f s_0$ $(f \in C^\infty_0(V))$ we have

$$\tilde{s}(x, t) = e^{i\ell t} f(x),$$

where (x, t) is local coordinates of $V \times U(1) \cong \pi^{-1}(V)$ $(U(1) = \{e^{it}\})$ and $\ell = \langle \lambda, 1 \rangle \in \mathbf{Z}$ $(1 \in \mathfrak{u}(1) \cong \mathbf{R})$. Let A be a $U(1)$-invariant pseudo-differential operator on P which is locally expressed as

$$Au(x, t) = \int_{\mathbf{R}^n} \left(\sum_{\tau \in \mathbf{Z}} e^{ix \cdot \xi} e^{it\tau} a(x, \xi, \tau) \hat{u}(\xi, \tau) \right) d\xi, \qquad u \in C^\infty_0(\pi^{-1}(V)).$$

Here we easily see that $a(\cdot)$ does not depend on t.

Lemma 4.3. *Let $s = f s_0$ be a local section of E_λ with $f \in C^\infty_0(V)$. Then, we have*

$$A_\lambda s(x) = \left(\int_{\mathbf{R}^n} e^{ix \cdot \xi} a_\lambda(x, \xi) \hat{f}(\xi) d\xi \right) s_0(x)$$

with

$$a_\lambda(x,\xi) = a(x,\xi,\ell).$$

Proof. We have

$$(A\tilde{s})(x,t) = (2\pi)^{-(n+1)} \sum_{\tau\in\mathbf{Z}} \int_{\bar{\mathcal{O}}\times\mathbf{R}^n} e^{i(x-y)\cdot\xi} e^{i(t-s)\tau} a(x,\xi,\tau) e^{i\ell s} f(y)\,dy\,ds\,d\xi$$

$$= (2\pi)^{-n} \int_{\mathcal{O}\times\mathbf{R}^n} e^{i(x-y)\cdot\xi} \left((2\pi)^{-1} \sum_{\tau\in\mathbf{Z}} \int_{[0,2\pi]} a(x,\xi,\tau) e^{i(t-s)\tau} e^{i\ell s} ds \right) f(y)\,dy\,d\xi.$$

Here we have

$$\begin{aligned} & (2\pi)^{-1} \sum_{\tau\in\mathbf{Z}} \int_{[0,2\pi]} a(x,\xi,\tau) e^{i(t-s)\tau} e^{i\ell s} ds \\ = \; & (2\pi)^{-1} e^{i\ell t} \sum_{\tau\in\mathbf{Z}} a(x,\xi,\tau) \int_{[0,2\pi]} e^{-i(s-t)\tau} e^{i\ell(s-t)} d(s-t) \\ = \; & e^{i\ell t} a(x,\xi,\ell). \end{aligned}$$

Thus the lemma is proved. □

By means of this lemma we have

Proposition 4.4. *Suppose $A \in \psi DO^m(P)$ is $U(1)$-invariant. Then, A_λ is a pseudo-differential operator in the usual sense* (*cf.* [7, pp.91-92]) *on the line bundle E_λ, and its order is equal or less than m.*

The principal symbol $a(\bar{x},\bar{\xi})$ of a $U(1)$-invariant pseudo-differential operator A is invariant under the action of $U(1)$ on T^*P. Hence, the restriction of $a(\bar{x},\bar{\xi})$ on $J^{-1}(\lambda)$ ($\lambda \in \Lambda^*$) defines a function on $P_\lambda = J^{-1}(\lambda)/U(1)$, and we call this function the *principal symbol* of the operator A_λ (denoted by $\sigma(A_\lambda)$).

Given a $U(1)$-invariant function h on T^*P. Let X_h be the Hamiltonian vector field associated to h, i.e., $i_{X_h}\Omega = -dh$ ($i_{X_h}\Omega$ being the interior product), and let ϕ_s be the flow on T^*P generated by X_h. Then, we have the following (see [11], [1, p.304]).

Lemma 4.5. *The flow ϕ_s leaves $J^{-1}(\lambda)$ invariant and commutes with the action of $U(1)$, so it induces a flow $\phi_s^{(\lambda)}$ on P_λ. This flow is generated by the Hamiltonian vector field X_{h_λ} on $(P_\lambda,\Omega_\lambda)$ associated to the induced function h_λ from h (cf. Figure 3).*

By virtue of the above lemma we can see properties of the symbol of A_λ from those of $U(1)$-invariant pseudo-differential A on P. So we have the following.

Proposition 4.6. *Let A, B be two $U(1)$-invariant pseudo-differential operators on P. Then,*

(1) $[A_\lambda, B_\lambda] = [A,B]_\lambda$,

(2) $\sigma([A_\lambda,B_\lambda]) = \frac{1}{i}\{\sigma(A_\lambda),\sigma(B_\lambda)\}_\lambda$, *where $\{\cdot,\cdot\}_\lambda$ is the Poisson bracket in $(P_\lambda,\Omega_\lambda)$.*

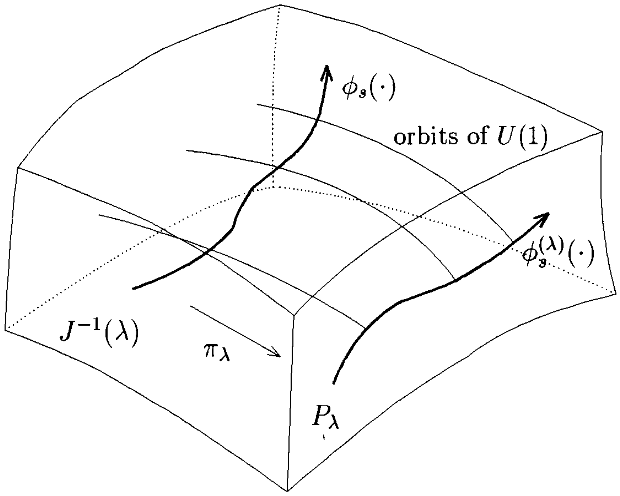

Figure 3

5. Helton's theorem – Spectrum of the Bochner-Laplacian and motion in the magnetic field

In the principal $U(1)$-bundle $\pi : P \to M$ we suppose P is endowed with a connection $\widetilde{\nabla}$ and M is endowed with a Riemannian metric m as in §2. Let us take the invariant metric on the group $U(1)$, and we introduce a Riemannian metric $\widetilde{m}$ on P (called a Kaluza-Klein metric) which satisfies (1) the map $\mathfrak{u}(1) \ni v \mapsto v_P \in V_p(\subset T_pP)$ is an isometry for every $p \in P$, (2) the horizontal space H_p and the vertical space V_p are orthogonal to each other for every $p \in P$, and (3) H_p and $T_{\pi(p)}M$ are isometric with each other by the map $\pi_*|_{H_p}$.

From the metric $\widetilde{m}$ on P we have the Hamiltonian dynamical system $(T^*P, \Omega, \widetilde{H})$. Since $\widetilde{H}$ is $U(1)$-invariant, we get the Hamiltonian function $\widetilde{H}_\mu$ on $P_\mu(\mu \in \mathfrak{u}(1)^*)$. Thus, we have the reduced Hamiltonian system $(P_\mu, \Omega_\mu, \widetilde{H}_\mu)$. Let $(P_\mu, \Omega_\mu, H_\mu)$ be the Hamiltonian system introduced in §2. Then, we have $\widetilde{H}_\mu = H_\mu + |\mu|^2$ (see Figure 1). Therefore, we have the following.

Lemma 5.1. *Two Hamiltonian systems $(P_\mu, \Omega_\mu, H_\mu)$ and $(P_\mu, \Omega_\mu, \widetilde{H}_\mu)$ are isomorphic as the dynamical systems, i.e., the flows of two systems are the same.*

Let $\widetilde{\Delta}$ be the Laplace-Beltrami operator on P defined from the Riemannian metric $\widetilde{m}$. Since $\widetilde{\Delta}$ is $U(1)$-invariant, we have the differential operator $\widetilde{\Delta}_\lambda$ on E_λ $(\lambda \in \Lambda^*)$

Lemma 5.2. *Let $L^{(\lambda)}$ be the Bochner-Laplacian on E_λ. Then,*

$$\widetilde{\Delta}_\lambda = L^{(\lambda)} + |\lambda|^2$$

holds good.

Proof. See [10, p.127], which treats the case G to be a torus. □

Now, we consider the spectrum of the Bochner-Laplacian $L^{(\lambda)}$, which consists of non-negative eigenvalues

$$\nu_1^2 \leq \nu_2^2 \leq \cdots \leq \nu_p^2 \leq \cdots \uparrow \infty \quad (\nu_p \geq 0).$$

We state the main result, which is a generalization of Helton's theorem to the Bochner-Laplacian on the line bundle.

Theorem. *Let Σ be the set of cluster points of the set $\{\nu_p - \nu_q\}$. Then, the every orbit of the flow of $(P_\lambda, \Omega_\lambda, H_\lambda)$ is closed if $\Sigma \neq \mathbf{R}$.*

The proof of the theorem is carried out similarly as [4] on the basis of Egorov's theorem. Let $Q = \sqrt{\tilde{\Delta}}$ (the positive square root of $\tilde{\Delta}$), which is a (formally) self-adjoint, elliptic, $U(1)$-invariant pseudo-differential operator of order one with the principal symbol $\sqrt{\tilde{H}}$. Then, we have

$$Q_\lambda = \sqrt{L^{(\lambda)} + |\lambda|^2}, \qquad \sigma(Q_\lambda) = \sqrt{\tilde{H}_\lambda}.$$

Let $S(s) = \exp isQ$ $(s \in \mathbf{R})$, and let R be a $U(1)$-invariant pseudo-differential operator. Put $R(s) = S(s)^{-1}RS(s)$. By virtue of Lemma 4.5 we get the following from Egorov's theorem for $\sqrt{\tilde{\Delta}}$ (cf. [14, p.147]).

Lemma 5.3(Egorov's theorem on line bundles). *The operator $R(s)$ is a $U(1)$-invariant pseudo-differential operator on P, and*

$$\sigma(R(s)_\lambda)(x,\xi) = \sigma(R_\lambda)(\phi_s^{(\lambda)}(x,\xi))$$

holds, where $\phi_s^{(\lambda)}(\cdot)$ is the flow on $(P_\lambda, \Omega_\lambda)$ which is generated by the Hamiltonian vector field X_λ associated to $\sigma(Q_\lambda) = \sqrt{\tilde{H}_\lambda}$.

Proof of Theorem(cf. Helton [6], Guillemin [4]). Let f be an element of $C_0^\infty(\mathbf{R})$. Let R be an arbitrary $U(1)$-invariant pseudo-differential operator of order zero on P. Consider

$$R_f = \int_{-\infty}^{\infty} \hat{f}(s) S(s)_\lambda^{-1} R_\lambda S(s)_\lambda ds = \int_{-\infty}^{\infty} \hat{f}(s)(S(s)^{-1}RS(s))_\lambda ds.$$

Let $\mu_p = \sqrt{\nu_p^2 + |\lambda|^2}$ $(p = 1, 2, \ldots)$ be the eigenvalues of Q_λ. Then, it follows from the spectral theorem that

$$R_f = \sum_{p,q} f(\mu_p - \mu_q)\Pi_q R_\lambda \Pi_p,$$

where Π_p's are finite rank projection operator satisfying $Q_\lambda = \sum \mu_p \Pi_p$. Suppose $\Sigma \neq \mathbf{R}$. Then, we have an interval I containing only finitely many $(\mu_p - \mu_q)$'s because a cluster point of $(\nu_p - \nu_q)$'s is also a cluster point of $(\mu_p - \mu_q)$'s and vice versa. If the support of f is contained in I, then R_f is of finite rank, and is a smoothing operator. Let r_λ be the principal symbol of R_λ. By virtue of Lemma 5.3 the principal symbol of $\hat{f}(s)S(s)_\lambda^{-1}R_\lambda S(s)_\lambda$ is given by $\hat{f}(s)r_\lambda(\phi_s^{(\lambda)}(\cdot))$, and we can derive

$$\int_{-\infty}^{\infty} \hat{f}(s) r_\lambda(\phi_s^{(\lambda)}(x,\xi))ds = 0 \tag{5.1}$$

for any $(x,\xi) \in P_\lambda$ from the fact that R_f is a smoothing operator. Noticing this equation holds for any r_λ, we can conclude that every orbit of $\phi_s^{(\lambda)}$ is closed as follows. Suppose there exists a orbit $\gamma(s) = \phi_s^{(\lambda)}((x,\xi))$ which is not closed. Given any compactly supported function g and any $K > 0$. Then, there exists $r_\lambda \in C^\infty(P)$ as the principal symbol of a pseudo-differential operator R_λ such that

$$r_\lambda(\gamma(s)) = g(s) + h(s),$$

where $h \in C^\infty(\mathbf{R})$ with $||h||_\infty \leq ||g||_\infty$ and $\mathrm{supp} h \cap [-K, K] = \phi$. Here we notice that $\sqrt{\widetilde{H}_\lambda}$ is constant along the orbit $\gamma(s)$. The equation (5.1) turns out to

$$\int \hat{f}(s)g(s)ds + \int \hat{f}(s)h(s)ds = 0.$$

By letting K go to infinity, we get

$$\int_{-\infty}^{\infty} \hat{f}(s)g(s)ds = 0.$$

This implies $\hat{f} \equiv 0$ and also $f \equiv 0$. This, however, contradicts the fact that (5.1) holds for $f \not\equiv 0$. Thus we have shown that every trajectory of the Hamiltonian vector field X_λ is closed. To complete the proof it remains to prove the following lemma.

Lemma 5.4. *Every orbit of the flow of* $(P_\lambda, \Omega_\lambda, H_\lambda)$ *is closed if and only if every orbit of the flow of* $(P_\lambda, \Omega_\lambda, \sqrt{\widetilde{H}_\lambda})$ *is closed.*

Proof. Note that the flows by $\sqrt{\widetilde{H}_\lambda}$ and H_λ are both contained in each level manifold $E_c : \sqrt{\widetilde{H}_\lambda} = c$ (a positive constant). Let X_λ and Y_λ be Hamiltonian vector fields associated to $\sqrt{\widetilde{H}_\lambda}$ and H_λ, respectively. Since $d\sqrt{\widetilde{H}_\lambda} = (1/2\sqrt{\widetilde{H}_\lambda})dH_\lambda$, we have $Y_\lambda = 2cX_\lambda$ on E_c. Hence, the traces of the associated integral curves are the same. □

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