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On the Construction of a Special Divisor of Some Special Curve

Dedicate to Professor Yoshihiro Ichijyô on his 65th birthday

By

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Abstract

Let $\pi : X \to C$ be a triple covering of curves where C is Brill-Noether general. We prove the existence of a base point free pencil of degree $d = g - [\frac{3h+1}{2}] - 2$ on a curve X which is not composed with π.

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0 Introduction

Let C be a non-singular curve of genus h and let $\pi : X \to C$ be a triple covering i.e. $deg(\pi) = 3$ and X is a non-singular curve. Let g be the genus of X. In this paper, we investigate the problem of the existence of base point free pencils of relatively low degree on X. By a simple application of the Castelnuovo-Severi bound, we have the following:

Lemma A (Castelnuobo-Severi bound) *For any integer* $n \geq \frac{g-3h}{2}$, *a base point free pencil* g^1_n *is a composed with* π *i.e. the morphism* $\pi : X \to \mathbb{P}^1$ *induced by* π *is always factors through* π.

On the other hand, not many things have been known about this problem. In [3], we prove the following:

Theorem A *If* $h \geq 1$ *and* $g \geq (2[\frac{3h+1}{2}]+1)([\frac{3h+1}{2}]+1)$ *and* C *is general (in the sense of Bril-Noether), then there exists a base point free pencil of degree* $d \geq g - [\frac{3h+1}{2}] - 1$ *which is not composed with* π.

Our aim is to give an extension of Theorem A. The main result of this paper is to prove the following result.

Theorem B *Under the same assumption of Theorem A, there exists a base point free pencil of degree* $d = g - [\frac{3h+1}{2}] - 2$ *which is not composed with* π.

NOTATIONS
$\mathcal{O}_A$: The structure sheaf of a variety A
f^*: The pull back defined by a morphism f
f_*: The direct image defined by a morphism f
$\deg(f)$: The degree of a finite morphism f
$|\mathcal{L}|$: The complete linear system defined by an invertible sheaf $\mathcal{L}$
ϕ_V: The rational map defined by a linear system V
$\mathcal{O}_A(D)$: The invertible sheaf associated with a divisor D
$\Gamma(A, \mathcal{F})$: The global sections of a sheaf $\mathcal{F}$
K_A: A canonical divisor on a non-singular variety A
ω_A: The canonical invertible sheaf on a non-singular variety A
$\mathbb{P}(\delta)$: The projective bundle $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n \delta)$ defined by a locally free sheaf δ

1 The proof of Theorem B

First we prove the following:

Proposition 1 *Under the same assumption of Theorem A, we assume that* $h = 2e$. *Then there is a component* Z *of* $W^1_{g-3e-2}(X)_{red}$ *such that*

$$Z \not\subset \pi^* W^1_l(C) + W_{g-3e-2-3l}(X)$$

for any $l \geq 1$.

To prove the above result, we need the following lemma.

Lemma 1 *Let* $\pi_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus \mathcal{E}$ *and let* $\delta = -(C_0^2)$ *where* C_0 *is a minimal section of* $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. *Then there are two line bundles* $\mathcal{L}$, $\mathcal{M}$ *on* C *such that*

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \to 0$$

is exact and

$$deg(\mathcal{M}) = -(\frac{g-3h}{2} + \frac{2-\delta}{2})$$
$$deg(\mathcal{L}) = -(\frac{g-3h}{2} + \frac{2+\delta}{2})$$
$$-h \leq \delta \leq \frac{g-3h+2}{3}.$$

Proof. See [3].

Q.E.D.

Proof of Proposition 1 . Let D be a divisor of degree $5e + 2$ on C. Then

$$dim\Gamma(X, \mathcal{O}(\pi^*D)) = dim\Gamma(C, \mathcal{O}(D)) = 3e + 3$$

$$dim\Gamma(X, \mathcal{O}(\pi^*(D + P + Q))) = dim\Gamma(C, \mathcal{O}(D + P + Q)) = 3e + 5$$

for any $P, Q \in C$ by Lemma 1. Hence we have

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D)) = g - 12e - 4$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*(D) - \pi^*(P + Q))) = g - 12e - 8$$

for any $P, Q \in C$. Now we must prove that there are $P_1, \cdots, P_{g-12e-6} \in X$ such that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-6})) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-6} - \pi^*(P + Q))) = 0$$

for any $P, Q \in C$. We put

$$X_{g-12e-6} \supset U_{P+Q} = \{F | dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - \pi^*(P + Q) - F) = 0\}$$

and put

$$Z_{P+Q} = X_{g-12e-6} \setminus U_{P+Q}.$$

Now we consider $\Phi_{P+Q} : X_{g-12e-6} \longrightarrow W_{g-12e-6}(X)$ and $\phi : X_3 \times X_{g-12e-9} \longrightarrow X_{g-12e-6}$ by

$$\Phi_{P+Q}(F) = \mathcal{O}(K_X - \pi^*D - \pi^*(P + Q) = F, \phi(A, B) = A + B.$$

Let

$$S = S_{P+Q} = \Phi^{-1}(W^1_{g-12e-6}(X))$$
$$W = W_{P+Q} = X_{g-12e-6} \setminus S_{P+Q}$$

and let $p_2 : X_3 \times X_{g-12e-9} \longrightarrow X_{g-12e-9}$ be the second projection. Then

$$p_2(\phi^{-1}(Z_{P+Q})) \supset W.$$

Because $K_X - \pi^*D - \pi^*(P + Q) = F$ is linearly equivalent to an effective divisor E and $deg(E) = g - 3e - 5 \geq 3$. Therefore we can take an effective divisor $A_1 + A_2 + A_3 \subset E$. Hence

$$(A_1 + A_2 + A_3, F) \in \phi^{-1}(Z_{P+Q}).$$

So we have

$$p_2(\phi^{-1}(Z_{P+Q})) = X_{g-12e-9}.$$

Therefore

$$dim\phi^{-1}(Z_{P+Q}) = dimX_{g-12e-9} = g - 12e - 9.$$

We now put

$$Z^1_{P+Q} = \{F \in Z_{P+Q} | dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - \pi^*(P+Q) - F)) \geq 2\}.$$

By the same argument of [1] p.163 (1.7)Lemma, we can easily prove that no component of Z_{P+Q} is entirely contained in Z^1_{P+Q}. So $F \in Z_{P+Q} \setminus Z^1_{P+Q}$ implies $K_X - \pi^*(D) - \pi^*(P+Q) - F$ is linearly equivalent to some effective divisor E_0 such that $dim\Gamma(X, \mathcal{O}(E_0)) = 1$. Therefore we can take a divisor $x_1 + x_2 + x_3 \subset F$ such that

$$dim\Gamma(X, \mathcal{O}(x_1 + x_2 + x_3 + E_0)) = 1.$$

We put $F_0 = F - x_1 - x_2 - x_3$. Then

$$\phi(x_1 + x_2 + x_3, F_0) = F$$

and

$$F_0 \in W.$$

Hence

$$\phi^{-1}(Z_{P+Q} \setminus Z^1_{P+Q}) \cap p_2^{-1}(W) \to Z_{P+Q}$$

is dominating. So we have a component $T \subset \phi^{-1}(Z_{P+Q})$ such that

$$\phi(T) = Z_{P+Q}, \;\; p_2(T) = X_{g-12e-9}.$$

Hence $dimZ_{P+Q} = g - 12e - 9$. Now we consider the locus

$$\bigcup_{P+Q \in X_2} Z_{P+Q} \subset X_{g-12e-6}.$$

As $dimZ_{P+Q} = g - 12e - 9$, therefore

$$dim \overline{\bigcup_{P+Q \in X_2} Z_{P+Q}} \leq g - 12e - 7.$$

Hence

$$\overline{\bigcup_{P+Q \in X_2} Z_{P+Q}} \subsetneqq X_{g-12e-6}.$$

So we can take an $F = P_1 + \cdots P_{g-12e-6} \in X \setminus \overline{\bigcup_{P+Q \in X_2} Z_{P+Q}}$. By the definition of Z_{P+Q}, F satisfies that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - F) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - F - \pi^*(P+Q)) = 0$$

for any $P, Q \in C$. This means $\mathcal{O}(K_X - \pi^*D - F) \in W^1_{g-3e-2}(X)$ and $\mathcal{O}(K_X - \pi^*D - F) \notin \pi^*(W^1_l(C)) + W_{g-3e-2}(X)$ for any $l \geq 1$ and $P, Q \in C$. Therefore we have Proposition 1.

Q.E.D.

Next we prove the following:

Proposition 2 *Under the same assumption of Theorem A, we assume that $h = 2e+1$. Then there is a component Z of $W^1_{g-3e-4}(X)_{red}$ such that*

$$Z \not\subset \pi^* W^1_l(C) + W_{g-3e-4-3l}(X)$$

for any $l \geq 1$.

Proof of Proposition 2. Let D be a divisor of degree $5e+5$ on C. Then

$$dim\Gamma(X, \mathcal{O}(\pi^*D)) = dim\Gamma(C, \mathcal{O}(D)) = 3e+5$$

$$dim\Gamma(X, \mathcal{O}(\pi^*(D+P+Q))) = dim\Gamma(C, \mathcal{O}(D+P+Q)) = 3e+7$$

for any $P, Q \in C$ by Lemma 1. Hence we have

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D)) = g - 12e - 11$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*(D) - \pi^*(P+Q))) = g - 12e - 15$$

for any $P, Q \in C$. Now we must prove that there are $P_1, \cdots, P_{g-12e-13} \in X$ such that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-13})) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-13} - \pi^*(P+Q))) = 0$$

for any $P, Q \in C$. But the proof is completely the same method of the proof of Proposition 1.

Q.E.D.

We now prove the following result.

Proposition 3 *Under the same assumption of Theorem A, we assume that $h = 2e$. Then every component Z of $W^1_{g-3e-2}(X)_{red}$ whose general element has base point is of the form $\pi^* W^1_l(C) + W_{g-3e-2-3l}(X)$ for some $l \geq 1$.*

To prove the above result, we need the following results.

Theorem 1 *If Y is a general curve of genus g, then*

$$dim W^r_d(Y) \geq g - (r+1)(g-d+r).$$

Proof. See [1] p.206 (1.1)Theorem.

Q.E.D.

Theorem 2 *Let Y be a curve of genus g, $\mathcal{L} \in W_n^r(Y) \setminus W_n^{r+1}(Y)$ and let $\mu : \Gamma(Y, \mathcal{L}) \otimes \Gamma(Y, \omega_Y \otimes \mathcal{L}^{\otimes -1}) \to \Gamma(Y, \omega_Y)$ be the cup product map. Then*

$$T_{\mathcal{L}}(W_n(Y)) \cong (im(\mu))^{\perp}$$

where $T_{\mathcal{L}}(W_n(Y)) \subset H^1(Y, \mathcal{O})$ is a tangent space of $W_n(Y)$ and $(im(\mu))^{\perp}$ is a dual space of $im\mu$ by the Serre duality pairing.

Proof. See [1] p.189 (4.2)Proposition.

Q.E.D.

Theorem 3 *Let Y be a non-hyperelliptic curve of genus $g \geq 3$, let d be an integer such that $2 \leq d \leq g-1$ and let r be an integer such that $0 < 2r \leq d$. Then*

$$dim W_d^r(Y) \leq d - 2r - 1.$$

Proof. See [1] p.191 (5.1)Theorem.

Q.E.D.

Proof of Proposition 3. Let $\Sigma \subset W^1_{g-3e-2}(X)$ be a component whose general element has a base point. Thus

$$\Sigma = \Sigma^1_\beta + W_{g-3e-2-\beta}(X)$$

for some $\beta \geq 1$ where Σ^1_β is a subvariety of $W^1_\beta(X)$ whose general element is a base point free. As

$$dim\ W^1_\beta(X) \geq g - 6e - 6$$

by Theorem 1, we have that

$$dim\ \Sigma^1_\beta(X) \geq \beta - 3e - 4.$$

By the assumption of Theorem A and Lemma A, $\beta - 3e - 4 \geq 0$. Let $\mathcal{L} \in \Sigma^1_\beta$ be a general element and let $mu : \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \omega \otimes \mathcal{L}^{\otimes -1}) \to \Gamma(X, \omega_X)$ be the cup product map. Then

$$dim\ (im\mu)^{\perp} = dim\ T_{\mathcal{L}}(W^1_\beta(X)) \geq \beta - 3e - 4$$

by Theorem 2. By the base point free pencil trick,

$$dim\ \Gamma(X, \mathcal{L}^{\otimes 2}) - 3 \geq dim\ (im\mu)^{\perp}.$$

Therefore we have

$$dim\ \Sigma^1_\beta \geq \beta - 3e - 4$$

$$dim\ \Gamma(X, \mathcal{L}^{\otimes 2}) \geq \beta - 3e - 1 \geq 3.$$

We consider a finite map

$$\phi : \Sigma^1_\beta \to W^{\beta-3e-2}_{2\beta}(X)$$

by $\phi(\mathcal{L}) = \mathcal{L}^{\otimes 2}$. So we have

$$dim W^{\beta-3e-2}_{2\beta}(X) \geq \beta - 3e - 4.$$

Hence

$$dim\ W^1_{\beta+3e+3}(X) = dim\ W^1_{2\beta-(\beta-3e-3)}(X) \geq 2(\beta - 3e) - 7.$$

As $\beta \leq g - 3e - 3$, $\beta + 3e + 3 \leq g$. We assume that $\beta + 3e + 3 = g$. We first consider

$$\rho : W_{\beta+3e+3}(X) \rightarrow W_{g-2}(X)$$

by

$$\rho(\mathcal{L}) = \omega_X \otimes \mathcal{L}^{\otimes -1}.$$

This is a finite map. Therefore we have

$$dim\ W^1_{\beta+3e+3}(X) = dim\ W^1_{g-2}(X) = g - 2.$$

Hence

$$g - 2 \geq 2(\beta - 3e) - 7.$$

By the assumption of Theorem A, this is a contradiction. Now we assume that $\beta+3e+3 \leq g - 1$. Then

$$dim\ W^1_{\beta+3e+3}(X) \leq \beta + 3e + 3 - 2 - 1$$

by Theorem 3. Hence

$$\beta \leq 9e + 7$$

By the assumption of Theorem A and Lemma A, this is a contradiction.

Q.E.D.

The following result is also proved by the same method of the proof of proposition 3

Proposition 4 *Under the same assumption of Theorem A, we assume that* $h = 2e + 1$. *Then every component* Z *of* $W^1_{g-3e-4}(X)_{red}$ *whose general element has base point is of the form* $\pi^* W^1_l(C) + W_{g-3e-4-3l}(X)$ *for some* $l \geq 1$.

Proof of Theorem B. By Proposition 2, Proposition 1, Proposition 3 and Proposition 4, we have Theorem B.

Q.E.D.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of Algebraic Curves I, Springer Verlag, 1985

[2] R.Hartshorne, Algebraic Geometry, Springer Verlag, 1974

[3] T. Kato, C. Keem, A. Ohbuchi, On triple coverings of irrational curves, preprint