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Kaehlerian Finsler Manifolds

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Abstract

In the present paper, continued the preceding paper [10], we are mainly concerned with a Kaehlerian Finsler manifold (M, f, g). First, in the Kaehlerian Finsler manifold, we define a generalized Finsler metric $\tilde{g}$ by $\tilde{g} = (g + {}^t fgf)/2$. We investigate the relation between the Finsler metric g, the generalized Finsler metric $\tilde{g}$, the complex structure f and several Finsler connections derived from g and $\tilde{g}$. In consequence of it, we obtain that the Kaehlerian Finsler manifold is a Landsberg space and the generalized Finsler metric $\tilde{g}$ can be regarded as a real representation of a complex Finsler metric in a sense. Finally we find a necessary and sufficient condition for an Hermitian structure on the tangent bundle over a Kaehlerian Finsler manifold to be a Kaehler structure.

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§1 Preliminaries

In the preceding paper [10], we have obtained the following:

Let M be a $2n$-dimensional manifold admitting an almost complex structure $f^i_j(x)$ and a Finsler metric $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x, y)$.

If the fundamental function $L(x, y)$ satisfies the so-called Rizza condition, that is,

$$L(x, \phi_\theta y) = L(x, y) \tag{1.1}$$

for any $\theta \in \boldsymbol{R}$, where

$$\phi^i_{\theta j} = \cos\theta \cdot \delta^i_j + \sin\theta \cdot f^i_j, \tag{1.2}$$

then M is called an almost Hermitian Finsler manifold or simply a *Rizza manifold.* In connection with the Rizza manifold, we can show

Theorem 1.1. *The tangent space at any point of a Rizza manifold is a*

In the present paper, the Latin indices $a, b, \ldots, i, j, k \ldots$ run over the range $1, 2, \ldots, 2n$; and the Greek indices $\alpha, \beta, \ldots, \lambda, \mu, \ldots$ run over the range $1, 2, \ldots, n$; and the indices $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \ldots$ stand for $\alpha + n, \beta + n, \gamma + n \ldots$ respectively.

complex Banach space.

Theorem 1.2. *Let M be a manifold admitting an almost complex structure $f_j^i(x)$ and a Finsler metric $g_{ij}(x, y)$. The condition for the couple $(f_j^i(x), g_{ij}(x, y))$ to construct a Rizza structure is given by $L(x, \phi_\theta y) = L(x, y)$. This condition is equivalent to any one of the following:*

$$g_{pq}(x, \phi_\theta y)\phi_{\theta i}^p \phi_{\theta j}^q = g_{ij}(x, y), \tag{1}$$

$$g_{ij}(x, y) f_m^i(x) y^m y^j = 0, \tag{2}$$

$$(g_{im}(x, y) - g_{pq}(x, y) f_i^p(x) f_m^q(x)) y^m = 0, \tag{3}$$

$$g_{im}(x, y) f_j^m(x) + g_{jm}(x, y) f_i^m(x) + 2C_{ijm}(x, y) f_r^m(x) y^r = 0. \tag{4}$$

Theorem 1.3. *If a Finsler metric $g_{ij}(x, y)$ and an almost complex structure $f_j^i(x)$ satisfy the condition*

$$g_{pq}(x, y) f_i^p(x) f_j^q(x) = g_{ij}(x, y),$$

then g_{ij} is a Riemann metric, that is, (f, g) is an almost Hermitian structure.

Theorem 1.4. *A Rizza manifold is a complex manifold if $\overset{*}{\nabla}_k f_j^i = 0$ holds good where $\overset{*}{\nabla}$ means h-covariant derivative with respect to the Cartan's Finsler connection $(\overset{*}{\Gamma}{}_{jk}^i, G_j^i)$.*

Now a Rizza manifold satisfying $\overset{*}{\nabla}_k f_j^i = 0$ is said to be a *Kaehlerian Finsler manifold*, and a Rizza manifold whose complex structure is integrable is said to be an *Hermitian Finsler manifold.*

Let M be a Rizza manifold. If we put

$$\tilde{g}_{ij}(x, y) = \frac{1}{2}(g_{ij}(x, y) + g_{pq}(x, y) f_i^p(x) f_j^q(x)), \tag{1.3}$$

then $\tilde{g}_{ij}$ is a homogeneous symmetric generalized metric, which is called a *generalized Finsler metric*, and $\tilde{g}_{ij}$ satisfies

$$\tilde{g}_{pq}(x, y) f_i^p(x) f_j^q(x) = \tilde{g}_{ij}(x, y), \tag{1.4}$$

Concerning these two metrics $g_{ij}(x, y)$ and $\tilde{g}_{ij}(x, y)$, because of the Rizza condition (3), we have

Theorem 1.5. *In a Rizza manifold, the relation*

$$g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j \left(\frac{1}{2} \tilde{g}_{pq}(x, y) y^p y^q \right) \tag{1.5}$$

holds true.

Moreover we have shown

Theorem 1.6. *Let M be a manifold admitting an almost complex structure* $f_j^i(x)$. *In order that M admits a Finsler metric which constructs a Rizza structure together with* $f_j^i(x)$, *it is necessary and sufficient that M admits a generalized Finsler metric* $\tilde{g}_{ij}(x, y)$ *satisfying the conditions*

(1) $\tilde{g}_{jk}(x, y) = \tilde{g}_{pq}(x, y) f_j^p(x) f_k^q(x)$, (2) $\dot{\partial}_k \tilde{g}_{pq}(x, y) y^p y^q = 0$,

(3) $(\tilde{g}_{jk}(x, y) + \dot{\partial}_k \tilde{g}_{jm}(x, y) y^m) \xi^j \xi^k$ is positive definite.

§2 Finsler connections in a Kaehlerian Finsler manifold

Let M be a $2n$-dimensional manifold endowed with a Kaehlerian Finsler structure (f, g). It is directly seen that $\overset{*}{\nabla}_k \tilde{g}_{ij} = 0$ where $\tilde{g}$ is the induced generalized Finsler metric given by (1.3). That is to say, we have

$$\overset{*}{\nabla}_k f_j^i = 0, \quad \overset{*}{\nabla}_k \tilde{g}_{ij} = 0. \tag{2.1}$$

Now the condition $\overset{*}{\nabla}_k f_j^i = 0$ leads us to

$$y^m \partial_m f_j^i + G_m^i f_j^m - f_m^i G_j^m = 0.$$

Differentiating this partially with respect to y^k, we have

$$\overset{B}{\nabla}_k f_j^i = 0, \tag{2.2}$$

where $\overset{B}{\nabla}$ means the h-covariant derivative with respect to the Berwald connection (G_{jk}^i, G_j^i).

Now we put

$$f_{ij}(x, y) = g_{im}(x, y) f_j^m(x), \tag{2.3}$$

$$\tilde{f}_{ij}(x, y) = \tilde{g}_{im}(x, y) f_j^m(x). \tag{2.4}$$

By virtue of (1.4) we have

$$\tilde{f}_{ij} = -\tilde{f}_{ji}, \quad \tilde{f}_{im} f_j^m = -\tilde{g}_{ij}, \quad \tilde{f}_{ij} = \frac{1}{2}(f_{ij} - f_{ji}). \tag{2.5}$$

On the other hand, the relations $\overset{B}{\nabla}_k g_{ij} = -2 y^m \overset{*}{\nabla}_m C_{ijk}$ and $\overset{*}{\Gamma}{}_{jk}^i = G_{jk}^i - y^m \overset{*}{\nabla}_m C_{jk}^i$ are well-known [13]. So, we see that the condition $\overset{*}{\nabla}_k f_j^i = 0$ and (2.2) lead us to $y^m\overset{*}{\nabla}_m C^k_{ri} f^r_j = f^k_r y^m \overset{*}{\nabla}_m C^r_{ij}$. Using this and (2.5), we see

$$\begin{aligned}\overset{B}{\nabla}_k \tilde{f}_{ij} &= \frac{1}{2}(\overset{B}{\nabla}_k g_{ir} f^r_j - \overset{B}{\nabla}_k g_{jr} f^r_i)\\ &= -y^m\overset{*}{\nabla}_m C_{ikr} f^r_j + y^m\overset{*}{\nabla}_m C_{jkr} f^r_i\\ &= -f_{kr} y^m\overset{*}{\nabla}_m C^r_{ij} + f_{kr} y^m\overset{*}{\nabla}_m C^r_{ij}\\ &= 0.\end{aligned}$$

Then (2.2), (2.5) and the above result lead us directly to $\overset{B}{\nabla}_k \tilde{g}_{ij} = 0$. Hence, in a Kaehlerian Finsler manifold, the relations

$$\overset{B}{\nabla}_k f^i_j = 0, \quad \overset{B}{\nabla}_k \tilde{g}_{ij} = 0, \quad \overset{B}{\nabla}_k \tilde{f}_{ij} = 0 \tag{2.6}$$

hold true.

Now let us put

$$\overset{M}{\Gamma}{}^i_{jk} = \frac{1}{2}\tilde{g}^{im}(X_k \tilde{g}_{jm} + X_j \tilde{g}_{km} - X_m \tilde{g}_{jk}), \tag{2.7}$$

where we put

$$X_k = \partial_k - G^m_k \dot{\partial}_m. \tag{2.8}$$

Then $\overset{M}{\Gamma}{}^i_{jk}$ is symmetric with j and k and satisfies the transformation rule of a linear connection. In the present paper, from now on, we say this $\overset{M}{\Gamma}{}^i_{jk}$ is the modified Cartan connection of $\tilde{g}$, and we denote by $\overset{M}{\nabla}$ the h-covariant derivative with respect to $(\overset{M}{\Gamma}{}^i_{jk}, G^i_j)$. Then direct calculation leads us to

$$\overset{M}{\nabla}_k \tilde{g}_{ij} = 0. \tag{2.9}$$

Thus we have $\overset{*}{\nabla}_k \tilde{g}_{ij} = \overset{B}{\nabla}_k \tilde{g}_{ij} = \overset{M}{\nabla}_k \tilde{g}_{ij} = 0$. Since the coefficients $\overset{*}{\Gamma}{}^i_{jk}$, G^i_{jk}, $\overset{M}{\Gamma}{}^i_{jk}$ are all symmetric with j and k, and the used non-linear connections are common to these three Finsler connections. So we obtain

$$\overset{*}{\Gamma}{}^i_{jk} = G^i_{jk} = \overset{M}{\Gamma}{}^i_{jk}. \tag{2.10}$$

Hence the Finsler manifold is a Landsberg space [13]. Consequently we obtain

Theorem 2.1. *Let M be a Kaehlerian Finsler manifold. Then M is a Landsberg space. Moreover the modified Cartan connection of $\tilde{g}$ also coincides with the Cartan's Finsler connection of g, which satisfies* $\overset{*}{\nabla}_k f_j^i = 0$ and $\overset{*}{\nabla}_k g_{ij} = \overset{*}{\nabla}_k \tilde{g}_{ij} = 0$.

§3 The tangent bundle over a Kaehlerian Finsler manifold

First, for the present, we assume that M is a Rizza manifold. Let $T(M)$ be the tangent bundle over M. Here we adopt (X_i, Y_i) as a local frame of $T(M)$ where we put

$$X_i = \partial_i - G_i^m \dot{\partial}_m, \quad Y_i = \dot{\partial}_i. \tag{3.1}$$

Then we can define globally on $T(M)$ a (1, 1)-tensor field F such that

$$F(X_i) = f_i^m X_m, \quad F(Y_i) = f_i^m Y_m. \tag{3.2}$$

It is apparent that F is an almost complex structure on $T(M)$[7]. Moreover we can define an inner product $\langle\ ,\ \rangle$ such that

$$\langle X_i, X_j \rangle = \tilde{g}_{ij}, \ \langle X_i, Y_j \rangle = 0, \ \langle Y_i, Y_j \rangle = \tilde{g}_{ij}. \tag{3.3}$$

Then the inner product gives $T(M)$ a globally defined Riemann metric $\tilde{G}$ [7]. The components of F and $\tilde{G}$ in terms of the frame (X_i, Y_i) are written as

$$F = \begin{pmatrix} f_j^i(x), & 0 \\ 0, & f_j^i(x) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{g}_{ij}(x, y), & 0 \\ 0, & \tilde{g}_{ij}(x, y) \end{pmatrix}. \tag{3.4}$$

In addition we have ${}^tF\tilde{G}F = \tilde{G}$. Therefore we obtain

Theorem 3.1. *If a manifold M is a Rizza manifold, then its tangent bundle $T(M)$ admits an almost Hermitian structure $(F, \tilde{G})$.*

In what follows, we show this structure $(F, \tilde{G})$ is an Hermitian structure if M is a Kaehlerian Finsler manifold.

For the Kaehlerian Finsler manifold M, as is shown in Theorem 1.4, the given almost complex structure $f_j^i(x)$ is integrable, that is, M is a complex manifold. So M is covered by a system of local complex coordinate neighbourhoods $\{(U, z^\alpha)\}$. If we express $z^\alpha = x^\alpha + \sqrt{-1}x^{\bar{\alpha}}$, then $(x^i) = (x^\alpha, x^{\bar{\alpha}})$ is an admissible real local coordinate of U. With respect to this (x^i), the complex structure f has the components $\begin{pmatrix} 0, & -\delta_\beta^\alpha \\ \delta_\beta^\alpha, & 0 \end{pmatrix}$, which we denote by $J = (J_j^i)$ hereafter. We call this local coordinate (x^i) as the canonical coordinate of a Kaehlerian Finsler manifold. With respect to this canonical coordinate, the condition $\overset{*}{\nabla}_k f^i_j = 0$ can be written as

$$\overset{*}{\Gamma}{}^i_{mk} J^m_j = J^i_m \overset{*}{\Gamma}{}^m_{jk}, \tag{3.5}$$

which leads us to $\overset{*}{\Gamma}{}^\alpha_{\beta j} = \overset{*}{\Gamma}{}^{\bar\alpha}_{\bar\beta j}$ and $\overset{*}{\Gamma}{}^\alpha_{\bar\beta j} = -\overset{*}{\Gamma}{}^{\bar\alpha}_{\beta j}$.
Hence we have

$$G^\alpha_\beta = G^{\bar\alpha}_{\bar\beta}, \quad G^\alpha_{\bar\beta} = -G^{\bar\alpha}_\beta. \tag{3.6}$$

And, at the same time, (3.5) leads us to

$$J^k_m G^m_j = G^k_m J^m_j. \tag{3.7}$$

As to the almost complex structure F defined by (3.2), we see

$$\begin{aligned} F(\partial_i) &= F(X_i + G^m_i Y_m) = J^m_i X_m + G^m_i J^r_m Y_r \\ &= J^m_i X_m + J^m_i G^r_m Y_r = J^m_i \partial_m, \\ F(\dot\partial_i) &= J^m_i \dot\partial_m. \end{aligned}$$

Hence F is a complex structure of $T(M)$. Thus $T(M)$ is covered by a system of local complex coordinate neighbourhoods $\{(\pi^{-1}(U), (z^\alpha, \xi^\alpha))\}$. If we express $z^\alpha = x^\alpha + \sqrt{-1}\,x^{\bar\alpha}$, $\xi^\alpha = y^\alpha + \sqrt{-1}\,y^{\bar\alpha}$, then $(x^i; y^i) = (x^\alpha, x^{\bar\alpha}; y^\alpha, y^{\bar\alpha})$ is an admissible local real coordinate of $\pi^{-1}(U)$. With respect to this $(x^\alpha, x^{\bar\alpha}; y^\alpha, y^{\bar\alpha})$, the complex structure F has the components $\begin{pmatrix} J, & 0 \\ 0, & J \end{pmatrix}$. Hence we obtain

Theorem 3.2. *On the tangent bundle over a Kaehlerian Finsler manifold, the almost Hermitian structure* $(F, \tilde G)$, *which is defined by* (3.2) *and* (3.3), *is an Hermitian structure.*

Now the Finsler metric function $L(x^i, y^i)$ can be written as

$$L(x^i, y^i) = L\left(\frac{z^\alpha + \bar z^\alpha}{2}, \frac{z^\alpha - \bar z^\alpha}{2\sqrt{-1}}, \frac{\xi^\alpha + \bar\xi^\alpha}{2}, \frac{\xi^\alpha - \bar\xi^\alpha}{2\sqrt{-1}}\right) := H(z^\alpha, \bar z^\alpha; \xi^\alpha, \bar\xi^\alpha).$$

Here we consider the Rizza condition with respect to this complex coordinate. Since $(\phi_\theta y)^i = \cos\theta \cdot y^i + \sin\theta \cdot J^i_m y^m$, it is easy to see

$$(\phi_\theta y)^\alpha = \frac{1}{2}(e^{i\theta}\xi^\alpha + \overline{e^{i\theta}\xi^\alpha}), \quad (\phi_\theta y)^{\bar\alpha} = \frac{1}{2\sqrt{-1}}(e^{i\theta}\xi^\alpha - \overline{e^{i\theta}\xi^\alpha}).$$

That is, the Rizza condition (1.1) can be rewritten as

$$H(z^\alpha, \bar z^\alpha; \xi^\alpha, \bar\xi^\alpha) = H(z^\alpha, \bar z^\alpha; e^{i\theta}\xi^\alpha, \overline{e^{i\theta}\xi^\alpha}). \tag{3.8}$$

We represent by g^*_{ij} and $\tilde{g}^*_{ij}$ the components of g_{ij} and $\tilde{g}_{ij}$ respectively with respect to the complex coordinate $(z^i; \xi^i) = (z^\alpha, \bar{z}^\alpha; \xi^\alpha, \bar{\xi}^\alpha)$. Due to the relation

$$\left(\frac{\partial x^i}{\partial z^j}\right) = \left(\frac{\partial y^i}{\partial \xi^j}\right) = \begin{pmatrix} \frac{1}{2}\delta^\alpha_\beta, & \frac{1}{2}\delta^\alpha_\beta \\ \frac{-\sqrt{-1}}{2}\delta^\alpha_\beta, & \frac{\sqrt{-1}}{2}\delta^\alpha_\beta \end{pmatrix} \quad \text{and}$$

$$g^*_{ij} = \frac{1}{2}\frac{\partial^2 H^2}{\partial \xi^i \partial \xi^j} = g_{pq}\frac{\partial x^p}{\partial z^i}\frac{\partial x^q}{\partial z^j},$$

we see

$$(3.9) \quad \begin{cases} g^*_{\alpha\beta} = \frac{1}{4}\{(g_{\alpha\beta} - g_{\bar{\alpha}\bar{\beta}}) - \sqrt{-1}(g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}})\}, \\ g^*_{\alpha\bar{\beta}} = \frac{1}{4}\{(g_{\alpha\beta} + g_{\bar{\alpha}\bar{\beta}}) - \sqrt{-1}(g_{\bar{\alpha}\beta} - g_{\alpha\bar{\beta}})\}, \\ g^*_{\bar{\alpha}\beta} = \frac{1}{4}\{(g_{\alpha\beta} + g_{\bar{\alpha}\bar{\beta}}) + \sqrt{-1}(g_{\bar{\alpha}\beta} - g_{\alpha\bar{\beta}})\}, \\ g^*_{\bar{\alpha}\bar{\beta}} = \frac{1}{4}\{(g_{\alpha\beta} - g_{\bar{\alpha}\bar{\beta}}) + \sqrt{-1}(g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}})\}. \end{cases}$$

Since $\tilde{g}_{\alpha\beta} = \tilde{g}_{\bar{\alpha}\bar{\beta}} = \frac{1}{2}(g_{\alpha\beta} + g_{\bar{\alpha}\bar{\beta}})$, $\tilde{g}_{\alpha\bar{\beta}} = -\tilde{g}_{\bar{\alpha}\beta} = \frac{1}{2}(g_{\alpha\bar{\beta}} - g_{\bar{\alpha}\beta})$, so we get

$$(3.10) \quad \begin{cases} \tilde{g}^*_{\alpha\beta} = 0, \\ \tilde{g}^*_{\alpha\bar{\beta}} = \frac{1}{4}\{(g_{\alpha\beta} + g_{\bar{\alpha}\bar{\beta}}) - \sqrt{-1}(g_{\bar{\alpha}\beta} - g_{\alpha\bar{\beta}})\}, \\ \tilde{g}^*_{\bar{\alpha}\beta} = \frac{1}{4}\{(g_{\alpha\beta} + g_{\bar{\alpha}\bar{\beta}}) + \sqrt{-1}(g_{\bar{\alpha}\beta} - g_{\alpha\bar{\beta}})\}, \\ \tilde{g}^*_{\bar{\alpha}\bar{\beta}} = 0. \end{cases}$$

That is, we have $\tilde{g}^*_{\alpha\bar{\beta}} = g^*_{\alpha\bar{\beta}}$. Hence we get

$$(3.11) \quad \tilde{g}^*_{\alpha\beta} = \tilde{g}^*_{\bar{\alpha}\bar{\beta}} = 0, \quad \tilde{g}^*_{\alpha\bar{\beta}} = \overline{\tilde{g}^*_{\bar{\alpha}\beta}} = \frac{1}{2}\frac{\partial^2 H^2}{\partial \xi^\alpha \partial \bar{\xi}^\beta}.$$

This fact tells us that the generalized Finsler metric $\tilde{g}_{ij}$ coincides with the real representation of a complex Finsler metric in the sense of Aikou [1] etc..

Moreover, in a Kaehlerian Finsler manifold, if we examine, with respect to the canonical coordinate, the components of the torsion tensor

$$R^i_{jk} = \partial_k G^i_j - \dot{\partial}_m G^i_j G^m_k - \partial_j G^i_k + \dot{\partial}_m G^i_k G^m_j,$$

we can show, by using (3.6),

$$R^i_{jk} + f^i_r R^r_{mk} f^m_j + f^i_r R^r_{jm} f^m_k - R^i_{mr} f^m_j f^r_k = 0. \tag{3.12}$$

§4 The Kaehlerian form on the tangent bundle over a Kaehlerian Finsler manifold

Let M be a Kaehlerian Finsler manifold and $T(M)$ be its tangent bundle. As is shown in Theorem 3.2, $T(M)$ admits an Hermitian structure $(F, \tilde{G})$, which is defined by (3.2) and (3.3). Here we consider the condition that the Hermitian structure is Kaehlerian.

For this purpose, we consider the so-called Kaehler 2-form

$$\Omega = \tilde{f}_{ij} dx^i \wedge dx^j + \tilde{f}_{ij} \delta y^i \wedge \delta y^j \tag{4.1}$$

where $(dx^i, \delta y^i)$ is the local dual coframe of (X^i, Y^i), that is,

$$\delta y^i = dy^i + G^i_m dx^m. \tag{4.2}$$

It is directly seen that

$$d\Omega = X_k \tilde{f}_{ij} dx^k \wedge dx^i \wedge dx^j + (\dot{\partial}_k \tilde{f}_{ij} + \tilde{f}_{km} R^m_{ij}) \delta y^k \wedge dx^i \wedge dx^j$$
$$+ \overset{B}{\nabla}_k \tilde{f}_{ij} dx^k \wedge \delta y^i \wedge \delta y^j + \dot{\partial}_k \tilde{f}_{ij} \delta y^k \wedge \delta y^i \wedge \delta y^j.$$

By virtue of (2.5) and (2.6), we have that $(F, \tilde{G})$ is a Kaehler structure if and only if

(1) $X_k \tilde{f}_{ij} + X_i \tilde{f}_{jk} + X_j \tilde{f}_{ki} = 0,$

(2) $\dot{\partial}_k \tilde{f}_{ij} + \tilde{f}_{km} R^m_{ij} = 0,$

(3) $\dot{\partial}_k \tilde{f}_{ij} + \dot{\partial}_i \tilde{f}_{jk} + \dot{\partial}_j \tilde{f}_{ki} = 0.$

However, since $\overset{B}{\nabla}_k \tilde{f}_{ij} = 0$, so $X_k \tilde{f}_{ij} = G^m_{ik} \tilde{f}_{mj} - G^m_{jk} \tilde{f}_{mi}$. Hence it is easy to see that (1) holds identically. Similarly, since $f_{ij} = g_{im} f^m_j$, so $\dot{\partial}_k f_{ij} = 2C_{kim} f^m_j$. Hence $\dot{\partial}_k f_{ij} = \dot{\partial}_i f_{kj}$ holds true. Therefore (2.5) tells us that (3) holds identically. Thus we obtain

Theorem 4.1. *Let M be a Kaehlerian Finsler manifold and $T(M)$ be its tangent bundle. The Hermitian structure $(F, \tilde{G})$ shown in Theorem 3.2 is a Kaehler structure on $T(M)$ if and only if*

$$\dot{\partial}_k \tilde{f}_{ij} + \tilde{f}_{km} R^m_{ij} = 0 \tag{4.3}$$

holds true.

If the given Finsler metric g_{ij} is a Riemann metric, we see directly that

$$\tilde{g}_{ij} = g_{ij},\ \tilde{f}_{ij} = f_{ij},\ R^k_{ij} = R^k_{hij} y^h.$$

Then the condition (4.3) is rewritten as $R^k_{hij} = 0$. Hence we obtain also

Theorem 4.2. *The Hermitian structure* $(F, \tilde{G})$ *on the tangent bundle over a Kaehler manifold is a Kaehler structure if and only if the base manifold is a flat Kaehler manifold.*

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