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The Conductor of Some Special Points in P^2

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Abstract

We describe a way of calculating the conductor of some "special" points in $\mathbf{P}^2$, which are constructed by complete intersection finite sets of points. As examples, we calculate the conductor of pure configurations in $\mathbf{P}^2$. Furthermore we give a necessary and sufficient condition for a pure configuration to have the Cayley-Bacharach property.

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Introduction

Let A be the homogeneous coordinate ring of a set of s points $X = \{P_1, \cdots, P_s\}$ in $\mathbf{P}^n = \mathbf{P}_k^n$, where k is an algebraically closed field, and let $\overline{A}$ be the integral closure of A in its total quotient ring $Q = Q(A)$, i.e., $\overline{A} \cong \prod_{i=1}^s k[t_i]$, where $k[t_i]$ is isomorphic to the homogeneous coordinate ring of P_i. We denote by C_X the *conductor* of A in $\overline{A}$, namely

$$C_X = \{ \ a \in \overline{A} \ \mid \ a\overline{A} \subset A \ \}.$$

F. Orecchia [7, Theorem 4.3] showed that

$$C_X = \prod_{i=1}^{s} t_i^{e_i} k[t_i],$$

where e_i is the least degree of any hypersurface which passes through all of X except for P_i. Accordingly we call e_i the *degree of conductor* of P_i in X and write $d_X(P_i) = e_i$. Also, we refer to C_X as the conductor of X.

In this note, we describe a way of calculating the conductor of some "special" points in $\mathbf{P}^2$, which are constructed by complete intersection finite sets of points (see Theorem 3.1). Theorem 3.1 is viewed as an extension of Cayley-Bacharach Theorem in $\mathbf{P}^2$ (see Remark 2.2 (2)). As examples, we calculate the conductor of pure configurations in $\mathbf{P}^2$ (see Theorem 4.1). Furthermore we give a necessary and sufficient condition for a pure configuration to have the Cayley-Bacharach property (see Corollary 4.7).

1. Preliminaries

Throughout this note, let k be an algebraically closed field. Let $R = k[x_0, x_1, \cdots, x_n]$ be a homogeneous coordinate ring of $\mathbf{P}^n = \mathbf{P}^n_k$ and let I be a homogeneous ideal of R. The ring $A = R/I = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra of finite type. Hence the dimension of A_i as a k-vector space is finite. The *Hilbert function* of A is defined by $H(A, i) = \dim_k A_i$ for all $i = 0, 1, \cdots$, and the *Hilbert series* of A is defined by $F(A, \lambda) = \sum_{i \geq 0} H(A, i)\lambda^i \in \mathbf{Z}[[\lambda]]$. We put $d = \dim A$. Then it is well-known that we can write $F(A, \lambda)$ in the form

$$F(A, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d}$$

for certain integers $h_0, h_1, \cdots, h_s$ satisfying $\sum h_i \neq 0$ and $h_s \neq 0$. We put $s(A) = s$ and $e(A) = \sum_{i=0}^{s(A)} h_i$.

Assume that A is Cohen-Macaulay, and let

$$0 \longrightarrow \bigoplus_{i=1}^{t_g} R(-l_{g,i}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{t_1} R(-l_{1,i}) \longrightarrow R \longrightarrow A \longrightarrow 0$$

be a minimal free resolution of A, where $g = n + 1 - d$. The *socle type* of A is defined by

$$S(A, \lambda) = \sum_{i \geq 0} (\dim_k [Tor_g^R(A, k)(g)]_i)\lambda^i \quad \text{, i.e.,} \quad S(A, \lambda) = \sum_{i=1}^{t_g} \lambda^{l_{g,i}-g} .$$

It is well-known that $l_{g,i} - g \leq s(A)$. We say that A is *level* if $l_{g,i} - g = s(A)$ for all $i = 1, \cdots, t_g$. The *Cohen-Macaulay type* of A is defined by

$$r(A) = \dim_k Tor_g^R(A, k) \quad \text{, i.e.,} \quad r(A) = S(A, 1).$$

Next, let A be the homogeneous coordinate ring of a finite set X of points in $\mathbf{P}^n$, i.e., $A = R/I(X)$, where $I(X)$ is the homogeneous ideal of X generated by $\{f \in R \mid f$ is homogeneous and $f(P) = 0$ for all $P \in X\}$. We note that A is an 1-dimensional reduced ring. The Hilbert function, the Hilbert series, the socle type and the Cohen-Macaulay type of X are defined by $H(X, i) = H(A, i)$ for all $i \geq 0$, $F(X, \lambda) = F(A, \lambda)$, $S(X, \lambda) = S(A, \lambda)$ and $r(X) = r(A)$, respectively. We denote by $\mid X \mid$ the number of points in X and $\mu(X)$ the minimal number of generators of $I(X)$. Furthermore we put $e(X) = e(A)$ and $s(X) = s(A)$.

Remark 1.1. Let A be the homogeneous coordinate ring of a finite set X of points in $\mathbf{P}^n$, and let $y \in A_1$ be a non zero-divisor. Put $B = A/yA$ and $Soc(B) = \{f \in B \mid fg = 0$ for all $g \in \bigoplus_{i\geq 1} B_i\} = \bigoplus_{i\geq 0} Soc(B)_i$. It is well-known that $Soc(B) \cong Tor_n^R(A,k)(n)$ as graded k-vector spaces. Furthermore, we note that $Soc(B) \supset B_{s(A)}$ and $Soc(B)_i = (0)$ for all $i > s(A)$, and it is easy to check that X is level (i.e., A is level) if and only if $Soc(B) = B_{s(A)}$.

Finally, we shall recall some basic facts about Hilbert functions of points in $\mathbf{P}^n$.

Proposition 1.2 (cf. [3]). *Let X be a finite set of points in $\mathbf{P}^n$. Then*
(1) $e(X) = \mid X \mid$.
(2) $H(X,i) \leq H(X,i+1)$ *for all* $i \geq 0$.
(3) $H(X,i) = H(X,i+1) \Rightarrow H(X,i+2) = H(X,i)$.
(4) $H(X,i) = \mid X \mid$ *for all* $i >> 0$.
(5) $s(X) = \min\{i \mid H(X,i) = \mid X \mid\}$.
(6) *If* $Y \subset X$ *then* $s(Y) \leq s(X)$.

2. The Cayley-Bacharach property

A. V. Geramita, P. Maroscia and L. Roberts gave a simple combinatorial characterization of those sequences $S = \{b_i\}_{i\geq 0}$ which are the Hilbert function of some set of points in $\mathbf{P}^{b_1}$, namely, $S = \{b_i\}_{i\geq 0}$ is the Hilbert function of some set of points in $\mathbf{P}^{b_1}$ if and only if S is a zero-dimensional differentiable O-sequence (cf. [3, Theorem 4.1] for the details).

Definition (cf. [3]). Let S be a zero-dimensional differentiable O-sequence. We say that ζ is a *permissible value* for S if the sequence $S' = \{b'_i\}_{i\geq 0}$, where

$$b'_i = \begin{cases} b_i & 0 \leq i < \zeta, \\ b_i - 1 & \zeta \leq i, \end{cases}$$

is a zero-dimensional differentiable O-sequence.

Remark 2.1. Let X be a finite set of points in $\mathbf{P}^n$ and let $P \in X$. We can check that the degree of conductor of P in X, $d_X(P)$ is necessarily a permissible value for $\{H(X,i)\}_{i\geq 0}$, i.e.,

$$H(X\backslash\{P\},i)=\begin{cases} H(X,i) & 0\le i<d_X(P),\\ H(X,i)-1 & d_X(P)\le i.\end{cases}$$

Also we have $d_X(P)\le s(X)$ for all $P\in X$ (cf. [3, Lemma 3.3]). Furthermore, there exists a point $P\in X$ such that $d_X(P)=s(X)$ (cf. [3, Theorem 3.4]).

Definition. Let X be a finite set of points in $\mathbf{P}^n$. We say that X has the *Cayley-Bacharach property* (CBP for short) if $d_X(P)=s(X)$ for all $P\in X$.

Remark 2.2. (1) We can easily calculate the Hilbert function of a finite set X of points in P^1, that is

$$H(X,i)=\begin{cases} i+1 & 0\le i<\mid X\mid,\\ \mid X\mid & \mid X\mid\le i.\end{cases}$$

Hence $\{H(X,i)\}_{i\ge 0}$ has the unique permissible value $\zeta=\mid X\mid -1$. Thus by Remark 2.1, we obtain $d_X(P)=\mid X\mid -1$ for all $P\in X$. Therefore, all finite sets of points in $\mathbf{P}^1$ has CBP.

(2) In general, if X is a finite set of points in $\mathbf{P}^n$ such that the coordinate ring of X is Gorenstein, then X has CBP (cf. [1, Theorem 5]). Hence, all complete intersection finite sets of points in $\mathbf{P}^n$ has CBP (Cayley-Bacharach Theorem). Therefore if $X\subset\mathbf{P}^n$ is a complete intersection of type $(a_1,\cdots,a_n)$, i.e., X is a set of $a_1\cdots a_n$ points which is the intersection of hypersurfaces of degree a_i $(1\le i\le n)$, then $d_X(P)=s(X)=a_1+\cdots+a_n-n$ for all $P\in X$.

The following result tells us the relation betwen the degree of conductor and the socle type of finite set of points in $\mathbf{P}^n$.

Proposition 2.3. *Let X be a finite set of points in $\mathbf{P}^n$, and let $S(X,\lambda)=\sum_{i=0}^{s(X)}a_i\lambda^i$ be the socle type of X. Then we have*

$$d_X(P)\in\{i\mid a_i\ne 0\}$$

for all $P\in X$.

PROOF. We may assume that x_0 is not a zero-divisor on $A=R/I(X)$. Put $B=R/(I(X),x_0)=\oplus_{i=0}^{s(X)}B_i$ and $Y=X\backslash\{P\}$. By Remark 2.1, we obtain

$$H(Y,i)=\begin{cases} H(X,i) & 0\le i<d_X(P),\\ H(X,i)-1 & d_X(P)\le i.\end{cases}$$

Hence we ha

$$\Delta H(Y,i) = \begin{cases} \Delta H(X,i) - 1 & i = d_X(P), \\ \Delta H(X,i) & \text{otherwise,} \end{cases}$$

where $\Delta H(X,i)$ is the difference function of X which is defined by

$$\Delta H(X,i) = H(X,i) - H(X,i-1) \quad (\text{here } H(X,-1) = 0)$$

and is equal to $H(B,i)$. Therefore it holds

$$\dim_k J_i = \begin{cases} 1 & i = d_X(P), \\ 0 & \text{otherwise,} \end{cases}$$

where $J = \bigoplus J_i$ is the image of $I(Y)$ in B. Thus, there is an element $\xi \in J_{d_X(P)}$ such that $\xi \neq 0$, and we have $B_1\xi = (0)$. Hence $\xi \in Soc(B)$. This implies our assertion.

The following is clear from Proposition 2.3, so we omit the proof.

Corollary 2.4. If X is level, then X has CBP.

Remark 2.5. In general, the converse of Corollary 2.4 is not true. For example, we consider the following set X of 7-points in $\mathbf{P}^2$

```
o   o
o o
o o o
```

It is easy to check that X has CBP and $S(X,\lambda) = \lambda^2 + \lambda^3$.

3. An extension of Cayley-Bacharach Theorem in $\mathbf{P}^2$

Let $g, g' \in R = k[x_0, x_1, x_2]$. We write $g \mid g'$ if $g' \in gR$ and $\deg g < \deg g'$.

The main theorem of this note is the following.

Theorem 3.1. *Let $Y_1, \cdots, Y_t$ be finite sets of points in $\mathbf{P}^2$ which are complete intersection, i.e., there exist forms $g_i, h_i \in R = k[x_0, x_1, x_2]$ such that $I(Y_i) = (g_i, h_i)(1 \leq i \leq t)$. Put $X = \cup_{i=1}^t Y_i$ and $s(Y_i) = \deg g_i + \deg h_i - 2$ for all $i = 1, \cdots, t$. Assume that $g_{i+1} \mid g_i$ and $h_i \mid h_{i+1}$ for all $i = 1, \cdots, t-1$, and g.c.d.$\{g_1, h_t\} = 1$. Then we have*

$$d_X(P) = \max_{1 \leq i \leq t} \{s(Y_i) \mid P \in Y_i\}$$

for all $P \in X$.

We need some lemmas to prove Theorem 3.1.

Lemma 3.2. *Let* X *be a finite set of points in* $\mathbf{P}^n$, *let* $g \in R = k[x_0, x_1, \cdots, x_n]$ *be a homogeneous polynomial and put* $Z = \{P \in X \mid g(P) \neq 0\}$ *and* $X \backslash Z = \{P \in X \mid P \notin Z\}$. *Then we have the following.*

(1) $d_X(P) \leq d_Z(P) + \deg g$ *for all* $P \in Z$.

(2) *If* $I(X \backslash Z) = (I(X), g)$, *then* $d_X(P) = d_Z(P) + \deg g$ *for all* $P \in Z$.

PROOF. Let $P \in Z$ and let f be a homogeneous polynomial such that $\deg f = d_Z(P)$, $f(Q) = 0$ for all points $Q \in Z \backslash \{P\}$ and $f(P) \neq 0$. Since $g(Q) = 0$ for all $Q \in X \backslash Z$, $fg(Q) = 0$ for all Q of X except for P. Furthermore since $f(P) \neq 0$ and $g(P) \neq 0$, we have $fg(P) \neq 0$. Hence $\deg fg \geq d_X(P)$. This implies the assertion of (1).

Next, let $P \in Z$ and let f be a homogeneous polynomial such that $\deg f = d_X(P)$, $f(Q) = 0$ for all points Q of X except for P and $f(P) \neq 0$. Since $f(Q) = 0$ for all $Q \in X \backslash Z$, we have $f \in I(X \backslash Z) = (I(X), g)$. Hence, $f = qg + r$ for some $q \in R$ and $r \in I(X)$. If $q(Q) = 0$ for all points Q of Z except for P and $q(P) \neq 0$, then $\deg q \geq d_Z(P)$. By noting $d_X(P) = \deg f = \deg q + \deg g$, we have $d_X(P) \geq d_Z(P) + \deg g$. So, it is enough to show that $q(Q) = 0$ for all points Q of Z except for P and $q(P) \neq 0$. Let $Q \in Z$ and $Q \neq P$. Since $f(Q) = 0$ and $r(Q) = 0$, we have $qg(Q) = 0$. Furthermore since $Q \in Z$, $g(Q) \neq 0$. Hence $q(Q) = 0$. Also, since $f(P) \neq 0$ and $r(P) = 0$, $qg(P) \neq 0$. Furthermore since $P \in Z$, $g(P) \neq 0$. Hence $q(P) \neq 0$. This completes the proof of (2).

Lemma 3.3. *With the same notations as in Theorem 3.1, we have the following.*

(1) *The set* $\{g_1, g_2h_1, g_3h_2, \cdots, g_th_{t-1}, h_t\}$ *is a minimal generating set for the ideal* $I(X)$ *of* X.

(2) $Y_i \not\subset \bigcup_{j \neq i} Y_j$ *for all* $i = 1, \cdots, t$.

PROOF. (1) Obviously, $I(X) = \bigcap_{i=1}^{t}(g_i, h_i)$. Put

$$I = (g_1, g_2h_1, g_3h_2, \cdots, g_th_{t-1}, h_t).$$

Let $P \in Y_j$, i.e., $g_j(P) = 0$ and $h_j(P) = 0$. Then we have $g_ih_{i-1}(P) = 0$ for all $i \leq j$ in view of $g_{i+1} \mid g_i$. On the other hand, we have, for all $i > j$, $g_ih_{i-1}(P) = 0$ by noting $h_i \mid h_{i+1}$. This implies that $I(X) \supset I$. Next, we show that $I(X) \subset I$. We prove this by induction on t. Our assertion is true for $t = 1$. Let $t > 1$. By the assumption of induction, we have

$$\bigcap_{i=1}^{t-1} I(Y_i) = (g_1, g_2h_1, \cdots, g_{t-1}h_{t-2}, h_{t-1}).$$

For any element $f \in I(X)$, we can write f in the forms

$$f = g_1a_1 + g_2h_1a_2 + \cdots + g_{t-1}h_{t-2}a_{t-1} + h_{t-1}a_t$$

or

$$f = g_tb + h_tc$$

for certain elements $a_i, b, c \in R$. Hence we have

$$g_t(g_1'a_1 + g_2'h_1a_2 + \cdots + g_{t-1}'h_{t-2}a_{t-1} - b) = h_{t-1}(h_t'c - a_t),$$

where $g_i = g_i'g_t$ and $h_t = h_t'h_{t-1}$. Thus, since $g.c.d.\{g_t, h_{t-1}\} = 1$, we have $h_t'c - a_t \in g_tR$, i.e., $a_t = g_tu + h_t'c$ $(u \in R)$. Hence

$$f = g_1a_1 + g_2h_1a_2 + \cdots + g_{t-1}h_{t-2}a_{t-1} + g_th_{t-1}u + h_tc \in I.$$

Next we show that $\mu(X) = t + 1$. If

$$g_1 \in (g_2h_1, \cdots, g_th_{t-1}, h_t)$$

or

$$h_t \in (g_1, g_2h_1, \cdots, g_th_{t-1}),$$

then we have $h_1 \mid g_1$ or $g_t \mid h_t$, a contradiction. Assume that

$$g_ih_{i-1} \in (g_1, g_2h_1, \cdots, g_{i-1}h_{i-2}, g_{i+1}h_i, \cdots, g_th_{t-1}, h_t)$$

for some i $(1 < i \leq t)$. Then there exist $a_i \in R$ such that

$$g_ih_{i-1} = g_1a_1 + g_2h_1a_2 + \cdots + g_{i-1}h_{i-2}a_{i-1} + g_{i+1}h_ia_i + \cdots + g_th_{t-1}a_{t-1} + h_ta_t.$$

Hence we have

$$g_ih_{i-1} = g_{i-1}u + h_iv,$$

where $g_j = g_j'g_{i-1}$ for all $j \leq i-2$, $h_j = h_j'h_i$ for all $j \geq i+1$, $u = g_1'a_1 + g_2'h_1a_2 + \cdots + h_{i-2}a_{i-1}$ and $v = g_{i+1}a_i + g_{i+2}h_{i+1}'a_{i+1} + \cdots + h_t'a_t$. Thus $g_i(h_{i-1} - g_{i-1}'u) = h_iv$, where $g_{i-1} = g_{i-1}'g_i$. Since $g.c.d.\{g_i, h_i\} = 1$, we have $h_{i-1} - g_{i-1}'u \in h_iR$, i.e., $h_{i-1} - g_{i-1}'u = h_ir$ for some $r \in R$. Hence $h_{i-1}(1 - h_i'r) = g_{i-1}'u$, where $h_i = h_{i-1}h_i'$. Thus since $g.c.d.\{h_{i-1}, g_{i-1}'\} = 1$, we have $1 - h_i'r \in g_{i-1}'R$. Therefore $1 \in (h_i', g_{i-1}')R$. But, since g_{i-1}' and h_i' are homogeneous polynomials with positive degree, $1 \notin (g_{i-1}', h_i')$, a contradiction.

(2) Put $X' = \bigcup_{j \neq i} Y_j$. We show that $I(X') \neq I(X)$. From (1), we have $g_{i+1}h_{i-1} \in I(X')$. Assume that $I(X') = I(X)$. Hence $g_{i+1}h_{i-1} \in I(X)$. By noting $I(X) \subset I(Y_i)$, we can write $g_{i+1}h_{i-1} = g_iu + h_iv$, where $u, v \in R$. Hence $g_{i+1}(h_{i-1} - g_i'u) = h_iv$, where $g_i = g_i'g_{i+1}$.

Thus, since $g.c.d.\{g_{i+1}, h_i\} = 1$, we have $h_{i-1} - g_i'u \in h_iR$, i.e., $h_{i-1} - g_i'u = h_ir$ ($r \in R$). Hence, $h_{i-1}(1 - h_i'r) = g_i'u$, where $h_i = h_i'h_{i-1}$. Thus $1 - h_i'r \in g_i'R$. Therefore $1 \in (g_i', h_i')$. But, since g_i' and h_i' are homogeneous polynomials with positive degree, $1 \notin (g_i', h_i')$, a contradiction.

Lemma 3.4. *With the same notations as in Theorem 3.1, we put* $Z = \{P \in X \mid g_i(P) \neq 0\}$ *and* $W = \{P \in X \mid h_i(P) \neq 0\}$ *for some* i. *Then we have* $I(X \backslash Z) = (I(X), g_i)$ *and* $I(X \backslash W) = (I(X), h_i)$.

PROOF. Obviously, $I(X\backslash Z) \supset (I(X), g_i)$. By noting Lemma 3.3 and $g_{j+1} \mid g_j$, we have

$$I(Y_i \cup \cdots \cup Y_t) = (g_i, g_{i+1}h_i, \cdots, g_t h_{t-1}, h_t) = (I(X), g_i).$$

Since $X\backslash Z \supset Y_i \cup \cdots \cup Y_t$, we have $I(X\backslash Z) \subset I(Y_i \cup \cdots \cup Y_t)$. Hence $I(X\backslash Z) \subset (I(X), g_i)$. Thus $I(X\backslash Z) = (I(X), g_i)$. The proof of the equality $I(X\backslash W) = (I(X), h_i)$ is the same as above. We note that $X\backslash Z = Y_i \cup \cdots \cup Y_t$ and $X\backslash W = Y_1 \cup \cdots \cup Y_i$.

Lemma 3.5. *With the same notations as in Lemma 3.4, let* Y_j' $(1 \leq j < i)$ *be subsets of* Y_j *defined by* g_j', h_j, *where* $g_j = g_j'g_i$. *Furthermore let* Y_j' $(i < j \leq t)$ *be subsets of* Y_j *defined by* g_j, h_j', *where* $h_j = h_j'h_i$. *Then* $Z = \bigcup_{j=1}^{i-1} Y_j'$ *and* $W = \bigcup_{j=i+1}^{t} Y_j'$.

PROOF. Let $P \in Z$, i.e., $g_i(P) \neq 0$. Hence, since $g_j(P) \neq 0$ for all $j \geq i$, we have $P \notin Y_j$ for all $j \geq i$. Therefore $P \in Y_j$ for some j, where $1 \leq j < i$. Since $g_j(P) = 0$ and $g_i(P) \neq 0$, we have $g_j'(P) = 0$. Thus $P \in Y_j'$, i.e., $Z \subset \bigcup_{j=1}^{i-1} Y_j'$. Let $P \in \bigcup_{j=1}^{i-1} Y_j'$, i.e., $P \in Y_j'$ for some j $(1 \leq j < i)$. Hence $g_j'(P) = 0$. We note that Y_j is the set of $(\deg g_j)(\deg h_j)$ distinct points. Hence, by Bezout's Theorem, the intersection number of C_1 and C_2 at P is one, where C_1 and C_2 are the curves defined by g_j and h_j, respectively. Thus, since $g_j'(P) = 0$ and $g_j = g_j'g_i$, the intersection number of C_1' and C_2 at P is zero, where C_1' is the curve defined by g_i. Therefore $g_i(P) \neq 0$, i.e., $Z \supset \bigcup_{j=1}^{i-1} Y_j'$. Hence $Z = \bigcup_{j=1}^{i-1} Y_j'$.

The proof of $W = \bigcup_{j=i+1}^{t} Y_j'$ is the same as above.

Lemma 3.6. *With the same notations as in Theorem 3.1, we have*

$$s(X) = \max\{s(Y_i) \mid 1 \leq i \leq t\}.$$

PROOF. We use induction on t. By Remark 2.2 (2), our assertion is true for $t = 1$. Let

$t > 1$. By Lemma 3.3, we have

$$I(Y_1 \cup \cdots \cup Y_{t-1}) = (g_1, g_2h_1, \cdots, g_{t-1}h_{t-2}, h_{t-1}).$$

Hence, by $g_t \mid g_i$ for all $1 \leq i \leq t-1$ and $h_{t-1} \mid h_t$, we have

$$I(Y_1 \cup \cdots \cup Y_{t-1}) + I(Y_t) = (g_t, h_{t-1}).$$

Thus we obtain the following exact sequence

$$0 \longrightarrow R/I(X) \longrightarrow R/I(Y_1 \cup \cdots \cup Y_{t-1}) \oplus R/I(Y_t) \longrightarrow R/(g_t, h_{t-1}) \longrightarrow 0.$$

Hence, in view of Proposition 1.2 (5) and Remark 2.2 (2), we obtain

$$s(X) \leq \max\{s(Y_1 \cup \cdots \cup Y_{t-1}), s(Y_t), s(R/(g_t, h_{t-1}))\}.$$

From the assumption of induction, we have

$$s(Y_1 \cup \cdots \cup Y_{t-1}) = \max\{s(Y_i) \mid 1 \leq i \leq t-1\}.$$

Furthermore we can check that

$$s(Y_{t-1}) \geq \deg g_t + \deg h_{t-1} - 2 = s(R/(g_t, h_{t-1})).$$

Thus, we have

$$s(X) \leq \max\{s(Y_i) \mid 1 \leq i \leq t\}.$$

On the other hand, from Proposition 1.2 (6), we have $s(Y_i) \leq s(X)$ for all $i = 1, \cdots, t$. Hence we obtain

$$s(X) = \max\{s(Y_i) \mid 1 \leq i \leq t\}.$$

We now start to prove Theorem 3.1.

Proof of Theorem 3.1. We use induction on t. By Remark 2.2 (2), our assertion is true for $t = 1$. Let $t > 1$. By Lemma 3.6, there exists an integer j such that $s(X) = s(Y_j)$. Since $d_X(P) \geq d_{Y_j}(P)$ for all $P \in Y_j$, we have $d_X(P) = s(Y_j)$ for all $P \in Y_j$ by Remark 2.1. Put $Y = \{P \in X \mid P \notin Y_j\}$, $Z = \{P \in X \mid g_j(P) \neq 0\}$ and $W = \{P \in X \mid h_j(P) \neq 0\}$. Obviously $Y = Z \cup W$. By Lemma 3.4, we have $I(X \backslash Z) = (I(X), g_j)$ and $I(X \backslash W) = (I(X), h_j)$. Hence, by Lemma 3.2, we have, for all $P \in X$,

$$d_X(P) = \begin{cases} d_Z(P) + \deg g_j & \text{if } P \in Z, \\ d_W(P) + \deg h_j & \text{if } P \in W. \end{cases}$$

Thus by Lemma 3.5 and by the assumption of induction, we have, for all $P \in X$,

$$d_X(P) = \begin{cases} \max_{1\leq i<j}\{s(Y_i') \mid P \in Y_i'\} + \deg g_j & \text{if } P \in Z, \\ \max_{j<i\leq t}\{s(Y_i') \mid P \in Y_i'\} + \deg h_j & \text{if } P \in W. \end{cases}$$

Therefore it follows that for all $P \in X$,

$$d_X(P) = \max_{1\leq i\leq t}\{s(Y_i) \mid P \in Y_i\}.$$

This completes the proof.

The following is clear from Theorem 3.1, so we omit the proof.

Corollary 3.7. *With the same notations of Theorem 3.1, if $s(Y_i) = s(Y_{i+1})$ for all $i = 1, \cdots, t-1$, then X has CBP.*

4. Examples

Let $R = k[x_0, x_1, x_2]$ be a homogeneous coordinate ring of $\mathbf{P}^2$. In this section, we calculate the conductor of pure configurations in $\mathbf{P}^2$:

Definition (cf. [5]). A *pure configuration* in $\mathbf{P}^2$ is a finite set X of points in $\mathbf{P}^2$ which satisfies the following conditions:

There exist distinct elements $c_1, \cdots, c_u \in k$ such that

(i) X is the disjoint union of $X \cap L_1, \cdots, X \cap L_u$, where L_i is the line defined by $x_2 - c_i x_0 = 0$.

(ii) $\varphi(X \cap L_i) \supset \varphi(X \cap L_{i+1})$ for all $i = 1, \cdots, u-1$, where $\varphi : \mathbf{P}^2\backslash\{(0,0,1)\} \longrightarrow \mathbf{P}^1$ is the map defined by sending the point (a_0, a_1, a_2) to the point (a_0, a_1).

We put $d_i = \mid X \cap L_i \mid$ for all $i = 1, \cdots, u$. By the condition (ii), we have $d_1 \geq \cdots \geq d_u$. The *type* of X is defined by $type(X) = (d_1, \cdots, d_u)$ and we write $X = X(d_1, \cdots, d_u)$.

If $d_1 > d_u$, then we put $r_1 = \min\{j \mid d_j > d_{j+1}\}$, and, inductively, if $d_{r_i+1} > d_u$, then $r_{i+1} = \min\{j > r_i \mid d_j > d_{j+1}\}$. If $d_1 = d_u$, then we put $r_1 = u$. Furthermore we denote by $t(X)$ the number of distinct natural numbers in $\{d_1, \cdots, d_u\}$. The *r-type* of X is defined by $(r_1, \cdots, r_{t(X)})$, where $r_{t(X)} = u$.

Let $\{b_1, \cdots, b_{d_1}\}$ be the x_1-coordinates of points in $X \cap L_1$, and let L_j' be the line defined by $x_1 - b_j x_0 = 0$ for all $j = 1, \cdots, d_1$. Note that X is the disjoint union of $X \cap L_1', \cdots, X \cap L_{d_1}'$. We may assume that $\mid X \cap L_j' \mid \geq \mid X \cap L_{j+1}' \mid$ for all $j = 1, \cdots, d_1 - 1$.

Put

$$g_i = \prod_{j=1}^{d_{r_i}} (x_1 - b_j x_0) \qquad \text{and} \qquad h_i = \prod_{j=1}^{r_i} (x_2 - c_j x_0)$$

for all $i = 1, \cdots, t$ $(= t(X))$. Furthermore we put

$$Y_i = \{P \in X \mid g_i(P) = 0 \text{ and } h_i(P) = 0\}$$

for all $i = 1, \cdots, t$ $(= t(X))$, and we call $Y_1, \cdots, Y_t$ the *CI-subsets* of X.

The following theorem can be proved by using Theorem 3.1. So we omit the proof.

Theorem 4.1. *Let $X = X(d_1, \cdots, d_u)$ be a pure configuration in $\mathbf{P}^2$, $(r_1, \cdots, r_t)$ be the r-type of X where $t = t(X)$ and let $Y_1, \cdots, Y_t$ be the CI-subsets of X. Then we have*

$$d_X(P) = \max \{d_{r_i} + r_i - 2 \mid P \in Y_i\}$$

for all $P \in X$.

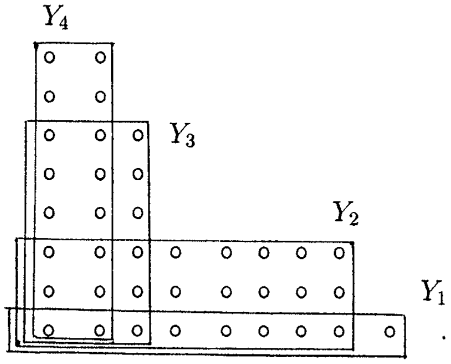
Example 4.2. Let X be the following set of 38 points in $\mathbf{P}^2$

```
o o
o o
o o o
o o o
o o o
o o o o o o o o
o o o o o o o o
o o o o o o o o o .
```

Then we have $X = X(9, 8, 8, 3, 3, 3, 2, 2)$, and the r-type of X is $(1, 3, 6, 8)$. Furthermore we obtain

i :	1	2	3	4
r_i :	1	3	6	8
d_{r_i} :	9	8	3	2
$d_{r_i} + r_i - 2$:	8	9	7	8 .

Also, the CI-subsets Y_1, Y_2, Y_3, Y_4 of X are as follows.



Accordingly we can calculate the degree of conductor of all points in X as follows.

$$d_X(P) = \begin{cases} 7 & \text{for 3 points such that } P \in Y_3, P \notin Y_2 \text{ and } P \notin Y_4, \\ 8 & \text{for 11 points such that } P \in Y_1 \cup Y_4 \text{ and } P \notin Y_2, \\ 9 & \text{for 24 points } P \in Y_2. \end{cases}$$

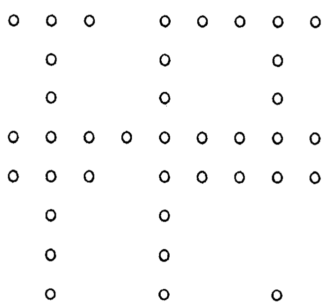
Consequently we have

$$C_X = \prod_{i=1}^{3} t_i^7 k[t_i] \prod_{i=4}^{14} t_i^8 k[t_i] \prod_{i=15}^{38} t_i^9 k[t_i] \, .$$

The following is clear from Theorem 4.1, so we omit the proof.

Corollary 4.3. *Let X and X' be pure configurations in $\mathbf{P}^2$. If $type(X) = type(X')$ then $C_X = C_{X'}$.*

Example 4.4. Let X' be the following set of 38 points in $\mathbf{P}^2$



Then we have $X' = X'(9, 8, 8, 3, 3, 3, 2, 2)$. Thus by Corollary 4.3, we obtain

$$C_{X'} = \prod_{i=1}^{3} t_i^7 k[t_i] \prod_{i=4}^{14} t_i^8 k[t_i] \prod_{i=15}^{38} t_i^9 k[t_i] \, .$$

Next, we recall some results about pure configurations in $\mathbf{P}^2$ from [5].

Definition. Let $\tau_1 = \{\tau_{1,i}\}, \cdots, \tau_m = \{\tau_{m,i}\}$ be sequences of non-negative integers. We denote by $H^{(\tau_1,\cdots,\tau_m)}$ the sequence obtained as follows.

Write down the sequences $\tau_1, \cdots, \tau_m$, successively shifted to the right and add:

$$\begin{array}{rccccccc}
\tau_m : & & & & & \tau_{m,0}, & & \cdots \\
\tau_{m-1} : & & & & \tau_{m-1,0}, & \tau_{m-1,1}, & & \cdots \\
\cdots & & & & \cdots & & & \cdots \\
\tau_2 : & & \tau_{2,0}, & \tau_{2,1}, & \cdots & & & \cdots \\
\tau_1 : & \tau_{1,0}, & \tau_{1,1}, & \tau_{1,2}, & \cdots & & & \cdots \\
\hline
H^{(\tau_1,\cdots,\tau_m)} : & & & & \cdots & & & \cdots
\end{array}$$

Hence

$$H^{(\tau_1,\cdots,\tau_m)}(i) = \sum_{j=1}^{m} \tau_{j,i+1-j}, \quad \text{where} \quad \tau_{j,l} = 0 \quad \text{for} \quad l < 0.$$

Theorem 4.5 (cf. [5, Theorem 3.1]). *Let $X = X(d_1, \cdots, d_u)$ be a pure configuration in $\mathbf{P}^2$, and $(r_1, \cdots, r_t)$ be the r-type of X where $t = t(X)$. For all $i = 1, \cdots, u$, let τ_i be the sequence $1, 2, \cdots, d_i, \longrightarrow$ (continuing with this constant value d_i). Then*

(1) $H(X) = H^{(\tau_1,\cdots,\tau_u)}$.

(2) $\mu(X) = t + 1$.

(3) $S(X, \lambda) = \sum_{i=1}^{t} \lambda^{d_{r_i}+r_i-2}$.

(4) $r(X) = t$.

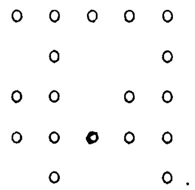
(5) *X is level if and only if $d_{r_i} + r_i = d_{r_{i+1}} + r_{i+1}$ for all $i = 1, \cdots, t-1$.*

Remark 4.6. Let X be a finite set of points in $\mathbf{P}^2$ and $S(X, \lambda) = \sum_{i=0}^{s(X)} a_i \lambda^i$ the socle type of X. In general, for an integer i such that $a_i \neq 0$, it does not necessarily exist a point $P \in X$ such that $d_X(P) = i$. For example, see Remark 2.5. But it follows from Lemma 3.3 (2) and Theorem 4.1 that if X is a pure configuration, then for each i $(1 \leq i \leq t)$, there exists a point $P \in X$ such that $d_X(P) = d_{r_i} + r_i - 2$.

Corollary 4.7. *Let $X = X(d_1, \cdots, d_u)$ be a pure configuration in $\mathbf{P}^2$ and $(r_1, \cdots, r_t)$ be the r-type of X. Then X has CBP if and only if $d_{r_i} + r_i = d_{r_{i+1}} + r_{i+1}$ for all $i = 1, \cdots, t-1$.*

PROOF. The assertion follows from Theorem 4.1 and Remark 4.6.

Example 4.8. Let X be the following set of points in $\mathbf{P}^2$



Then we have $X = X(5,5,4,2,2)$, and the r-type of X is (2,3,5). Therefore, we obtain

i :	1	2	3
r_i :	2	3	5
d_{r_i} :	5	4	2
$d_{r_i} + r_i$:	7	7	7 .

Thus by Corollary 4.7, X has CBP in this case.

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