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Unit Groups of Some Quartic Fields

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Abstract

Using the methods developed in [5] and J.H.E.Cohn's results on some quartic diophantine equation, we shall show the structure of the unit groups of the quartic fields $\mathbf{Q}(\sqrt{F_{6n}^2+4}, \sqrt{L_{6n}^2+16})$, where F_m and L_m are the mth Fibonacci and Lucas numbers. At the same time, we shall show the explicit class number formulae for these quartic fields.

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1. Introduction and preliminary lemmas

In our previous paper [5], we have shown the structure of the unit groups of $\mathbf{Q}(\sqrt{F_{2n}^2+1}, \sqrt{L_{2n}^2+1})$ and $\mathbf{Q}(\sqrt{A_{2n}^2+1}, \sqrt{B_{2n}^2+1})$, where F_m, L_m, A_m and B_m are the mth Fibonacci, Lucas, Pell and companion Pell numbers. Using the methods developed in [5] and J.H.E.Cohn's results on some quartic diophantine equation, we shall show the structure of the unit groups of the bicyclic biquadratic fields $\mathbf{Q}(\sqrt{F_{6n}^2+4}, \sqrt{L_{6n}^2+16})$.

Let d be a positive integer. We assume d is not perfect square and the minimal positive integer solution $x=t$ and $y=u$ of the Pellian equation $x^2-dy^2=-4$ satisfies $t \equiv u \equiv 1 \pmod 2$. We denote $(t+u\sqrt{d})/2$ by ε. Then it is known that:

Lemma 1. (cf. J.H.E.Cohn [2]) *With the above notation, the diophantine equation* $x^2-dy^4=-1$ *has at most one positive integer solution and the only possible solution is* *given by* $x+y^2\sqrt{d}=\varepsilon^3$.

As a corollary of this lemma, we have the following.

Corollary 1. *The diophantine equation* $x^2-5y^4=-1$ *has only one integer solution* $(x,y)=(2,1)$.

Corollary 2. *The diophantine equation* $x^2-125y^4=-1$ *has no integer solutions.*

2. Unit groups of $\mathbf{Q}(\sqrt{\mathbf{F}_{6n}^2+4},\sqrt{\mathbf{L}_{6n}^2+16})$

Let d_1 and d_2 be nonzero rational numbers. We denote $d_1\sim d_2$ when $d_1(\mathbf{Q}^\times)^2=d_2(\mathbf{Q}^\times)^2$ in $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2$. Then it is clear that $d_1\sim d_2\Longleftrightarrow \mathbf{Q}(\sqrt{d_1})=\mathbf{Q}(\sqrt{d_2})$. Put $\ell_n+f_n\sqrt{5}$ $=(2+\sqrt{5})^n$, then $F_{3n}=2f_n$ and $L_{3n}=2\ell_n$. Let K be the bicyclic biquadratic field $\mathbf{Q}(\sqrt{F_{6n}^2+4},\sqrt{L_{6n}^2+16})=\mathbf{Q}(\sqrt{f_{2n}^2+1},\sqrt{\ell_{2n}^2+4})$ $(n\geq 1)$. From the fact f_{2n}^2+1 $=f_{2n-1}\cdot f_{2n+1}$ and Corollaries 1 and 2, we see $f_{2n}^2+1\not\sim 1$ and $f_{2n}^2+1\not\sim 5$ for $n\geq 1$. Hence K contains exactly 3 quadratic subfields $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(\sqrt{f_{2n}^2+1})$ and $\mathbf{Q}(\sqrt{\ell_{2n}^2+4})$. We denote the discriminants of $\mathbf{Q}(\sqrt{f_{2n}^2+1})$ and $\mathbf{Q}(\sqrt{\ell_{2n}^2+4})$ by D_f and D_ℓ. We note $D_f\sim f_{2n}^2+1=f_{2n-1}\cdot f_{2n+1}$ and $D_\ell\sim \ell_{2n}^2+4=5f_{2n-1}\cdot f_{2n+1}$. Let ε_1 be the fundamental unit $(1+\sqrt{5})/2$ of $\mathbf{Q}(\sqrt{5})$. Let ε_f be the unit $f_{2n}+\sqrt{f_{2n}^2+1}$ and ε_ℓ be the unit $(\ell_{2n}+\sqrt{\ell_{2n}^2+4})/2$. We denote the fundamental units of $\mathbf{Q}(\sqrt{f_{2n}^2+1})$ and $\mathbf{Q}(\sqrt{\ell_{2n}^2+4})$ by $\varepsilon_2>1$ and $\varepsilon_3>1$, respectively. Let N be the norm maps from the quadratic fields to $\mathbf{Q}$. Since $N(\varepsilon_f)=N(\varepsilon_\ell)=-1$, one knows $N(\varepsilon_2)=N(\varepsilon_3)=-1$ and $\varepsilon_f=\varepsilon_2^{2i+1}$, $\varepsilon_\ell=\varepsilon_3^{2j+1}$ for some integers i and j.
E_K denotes the unit group of K. Then E_K contains the subgroup $E=<\pm 1,\varepsilon_1,\varepsilon_2,\varepsilon_3>$. As usual, we call the number $Q_K=[E_K:E]$, the unit index of K. For any $\varepsilon\in E_K$, it is known (cf. [6] or [11]) that $\varepsilon^2\in E$ and $E_K/E\cong E_K^2/E^2\subset E/E^2$. Therefore, to find a system of generators of the unit group E_K, we must list up the element of E which are perfect squares in K from among 7 numbers $\varepsilon_1^\alpha\varepsilon_2^\beta\varepsilon_3^\gamma$ $(\neq 1)$, where $\alpha,\beta,\gamma=0$ or 1.
Since $N(\varepsilon_1)=N(\varepsilon_2)=N(\varepsilon_3)=-1$, the number $\varepsilon_1^\alpha\varepsilon_2^\beta\varepsilon_3^\gamma$ does not become totally positive for the values of (α,β,γ) other than $(1,1,1)$. Hence one knows the unit index Q_K of K is 1 or 2 and

$$Q_K=2\Longleftrightarrow\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\in K.$$

Since $\varepsilon_f = \varepsilon^{2i+1}$ and $\varepsilon_\ell = \varepsilon^{2j+1}$ for some $i, j \in \mathbf{Z}$, we see

$$\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} \in K \iff \sqrt{\varepsilon_o\varepsilon_f\varepsilon_\ell} \in K, \text{where } \varepsilon_o = \varepsilon_1^3 = 2 + \sqrt{5}.$$

It is known (cf. [6] or [11])that:

Lemma 2. *Put* $C = Tr_{K/\mathbf{Q}}(\varepsilon_o\varepsilon_f\varepsilon_\ell - \varepsilon_o - \varepsilon_f - \varepsilon_\ell)$, *then* $\sqrt{\varepsilon_o\varepsilon_f\varepsilon_\ell} \in K$ *if and only if* C *is contained in one of* $(\mathbf{Q}^\times)^2$, $5(\mathbf{Q}^\times)^2$, $D_f(\mathbf{Q}^\times)^2$ *and* $D_\ell(\mathbf{Q}^\times)^2$.

Since $D_f \sim f_{2n-1} \cdot f_{2n+1}$ and $D_\ell \sim 5f_{2n-1} \cdot f_{2n+1}$, one sees

$$\sqrt{\varepsilon_o\varepsilon_f\varepsilon_\ell} \in K \iff C \sim 1 \text{ or } C \sim 5 \text{ or } C \sim f_{2n-1} \cdot f_{2n+1} \text{ or } C \sim 5f_{2n-1} \cdot f_{2n+1}.$$

From the definition of C, we have $C = 4(f_{2n}\ell_{2n} + (\ell_{2n}^2 + 4)/2 - 2 - f_{2n} - \ell_{2n}/2)$ $= 2(2f_{2n} + \ell_{2n})(\ell_{2n} - 1)$. Since $2f_{2n} + \ell_{2n} = f_{2n+1}$ and $\ell_{2n} - 1 \sim 2$ when n is odd and ~ 10 when n is even, we have $C \sim f_{2n+1}$ when n is odd and $C \sim 5f_{2n+1}$ when n is even. Hence we have shown

$$\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} \in K \iff \sqrt{\varepsilon_o\varepsilon_f\varepsilon_\ell} \in K \iff f_{2n\pm1} \sim 1 \text{ or } 5 \; (n \geq 1).$$

From Corollaries 1 and 2, we see $f_{2n\pm1} \sim 1$ or 5 if and only if $n = 1$. Hence, we have shown:

Theorem. *With the above notation,*
$E_K =< \pm1, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3 >$ *for* $n = 1$,
$E_K =< \pm1, \varepsilon_1, \varepsilon_2, \varepsilon_3 >$ *for* $n \geq 2$.

Let h_f and h_ℓ be the class numbers of the quadratic fields $\mathbf{Q}(\sqrt{f_{2n}^2 + 1})$ and $\mathbf{Q}(\sqrt{\ell_{2n}^2 + 4})$. Let h_K be the class number of K. Since the class number of $\mathbf{Q}(\sqrt{5})$ is one, we have the following.

$$h_K = \frac{Q_K h_f h_\ell}{4}.$$

From the above theorem, one sees the unit index $Q_K = 2$ for $n = 1$ and $Q_K = 1$ for $n \geq 2$. Hence we have:

Corollary 3. *With the above notation,*

$$h_K = \frac{h_f h_\ell}{2} \quad for \; n = 1,$$

$$h_K = \frac{h_f h_\ell}{4} \quad for \; n \geq 2.$$

Remark. In the above discussion, it is not so obvious that there are infinitely many bicyclic biquadratic fields of the form $\mathbf{Q}(\sqrt{F_{6n}^2+4}, \sqrt{L_{6n}^2+16}) = \mathbf{Q}(\sqrt{f_{2n}^2+1}, \sqrt{5})$. The infinity of these quartic fields is equivalent to the infinity of the real quadratic fields expressed in the form $\mathbf{Q}(\sqrt{f_{2n}^2+1})$ $(n = 1, 2, \ldots)$. Suppose, on the contrary, the number of real quadratic fields of this type is finite. Then there exists a constant r such that, for any $n > r$, $\mathbf{Q}(\sqrt{f_{2n}^2+1}) = \mathbf{Q}(\sqrt{f_{2k}^2+1})$ for some k $(1 \leq k \leq r)$. By the Dirichlet's theorem on primes in arithmetical progressions, there exists a prime p such that $p \equiv 1 (mod\ (2r+1)!\ )$. Consider the real quadratic field $\mathbf{Q}(\sqrt{f_{p+1}^2+1})$. Then, from the assumption, there exists k $(1 \leq k \leq r)$, which satifies $f_{p+1}^2 + 1 = f_p f_{p+2} \sim f_{2k}^2 + 1 = f_{2k-1} f_{2k+1}$. Hence, there exist nonzero integers x and y such that $x^2 f_p f_{p+2} = y^2 f_{2k-1} f_{2k+1}$. Since $(p, p+2) = (p, 2k-1)$ $= (p, 2k+1) = 1$, one sees $(f_p, f_{p+2}) = (f_p, f_{2k-1}) = (f_p, f_{2k+1}) = 1$. Hence one sees f_p is always a perfect square, which contradicts Corollary 1.

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