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Surfaces of Revolution in the Lorentzian 3-Space

By

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§1. Introduction

Let $L^3=(R^3, dx^2+dy^2-dz^2)$ be the Lorentzian 3-space. Surfaces of revolution are obtained by rotating about their axes the generating curves. There are three types of surfaces of revolution in L^3, that is, surfaces rotating about a space-like axis, a time-like axis and a null axis. By using a method similar to Kenmotus's in [7], for a given continuous function $H(s)$, we can solve the differential equations about the generating curves of surfaces of revolution whose mean curvature is $H(s)$.

Using these solutions, we can give a Delaunay's characterizlation of the surfaces of revolution in L^3 with constant mean curvature, which may be stated roughly as the following; Generating curves of surfaces of revolution in L^3 with constant mean curvature are roulettes of conics. This pboblem was already studied by Hano and Nomizu in [5]. But they use the method of Hsiang and Yu [6] and treated only space-like surfaces of revolution. On the other hand, in the present paper, we depend on the Kenmotsu's method and deal with space-like and time-like surfaces together. This gives better geometric interpretation of generating curves of surfces of revolutions.

In the Lorentz 2-space $L^2=(R^2; dy^2-dz^2)$, there are two kinds of conics, horizontal conics and vertical conics. Moreover, for a given conic we have its roulettes rolling along a space-like line and along a time-like line. The roulette of a vertical (resp., horizontal) conic generates a space-like (resp., time-like) surface of revolution with constant mean curvature, and the roulette of a conic rolling along a space-like (resp., time-like) line is a generating curve of a surface of revolution in L^3 with constant mean cuurvature, which rotates about a space-like (resp., time-like) axis (see, for details, Theorem 4).

The Gauss map of a surface in L^3 with constant mean curvature is also a harmonic mapping. Hence the surfaces of revolution with constant curvature constructed in the paper give the harmonic Gauss maps of the surfaces to the sphere S^2 or the hyperbolic space H^2.

§2. The outline

A surface in L^3 is called a surface of revolution with axis l if it is invariant under the action of the group of motions in L^3 which fix each point of the line l. A surface of revolution with space-like axis is given by

$$S(s, t)=(z(s)\sinh t,\ y(s),\ z(s)\cosh t),\quad (S\text{-}axis), \tag{2.1}$$

where $(y(s), z(s))$ is a curve in $L^2=\{R^2, dy^2-dz^2\}$ which is parametrized by the arc length and defined on some open interval I. Hence it holds

$$y'^2-z'^2=\varepsilon, \tag{2.2}$$

where ε is 1 or -1 according to the space-like curve or the time-like curve. The curve is called the generating curve of the surface. A surface of revoluution with time-like axis is given by

$$T(s, t)=(y(s)\cos t,\ y(s)\sin t,\ z(t)),\quad (T-\text{axis}), \tag{2.3}$$

where $(y(s), z(s))$, $s\in I$, is a curve in L^2 satisfying (2.2). A surface of revolution with null axis is given by

$$N(s, t)=(y(s)+z(s)-t^2z(s),\quad -2tz(s),\quad y(s)-z(s)-t^2z(s)),\quad (N\text{-axis}), \tag{2.4}$$

where $(y(s)+z(s), y(s)-z(s))$ is a curve in L^2 with

$$4y'z'=\varepsilon. \tag{2.5}$$

Though the true generating curve is $(y(s)+z(s), y(s)-z(s))$ in this case, we call simply the curve $(y(s), z(s))$ the generating curve of the surface. In all cases, surfaces of revolution are space-like or time-like according to $\varepsilon=1$ or $\varepsilon=-1$.

We can solve the differential equations about the generating curves of surfaces of revolution with given mean curvature function $H(s)$, $s\in I$. In particular, concerning surfaces of revolution with constant mean curvature, we have explicit solutions. For a non-zero constant H and non-negative constant d, we put

$$\begin{cases} f(s)=1+d^2-2d\cosh(2Hs),\ g(s)=d\cosh(2Hs)-1. \\ h(s)=1-d^2-2d\sinh(2Hs),\ k(s)=d\sinh(2Hs)-1, \\ u_1(s)=d\sin^{-1}(s/d),\ u_{-1}(s)=d\log(s+\sqrt{d^2-s^2}), \\ v_1(t)=(t+\log((t-1)/(t+1)))/(2|H|), \\ v_{-1}(t)=(t-2\tan^{-1}t)/(2|H|). \end{cases} \tag{2.6}$$

Theorem 1. *The generating curves $(y(s), z(s))$ of surfaces of revolution with constant mean curvature H are the following, correspondisng to rotating about S-axis, T-axis and N-axis*:

S-axis, $H=0$, $S_0(s)=(u_\varepsilon, \sqrt{d^2-s^2})$, $(0\leqq s\leqq d)$,

$$H\neq 0,\ S_1(H,\,d;\ s)=(\delta\int g/\sqrt{\varepsilon f}\,ds,\ \sqrt{\varepsilon f}/(2|H|)),\ \text{if } \varepsilon=-1,\ d\geqq 0,$$

$$S_2(H,\,d;\ s)=(\delta\int^s k/\sqrt{\varepsilon h}\,ds,\ \sqrt{\varepsilon h}/(2|H|)),\ \text{if } \varepsilon=-1,\ d\neq 0,$$

$$S_3(H;\ t)=(\pm v_\varepsilon,\ t/(2|H|)),$$

T-axis, $H=0,\ T_0(s)=(\sqrt{d^2-s^2},\ u_{-\varepsilon}),\ (0\leqq s\leqq d),$

$$H\neq 0,\ T_1(H,\,d;\ s)=(\sqrt{-\varepsilon f}/(2|H|),\ \delta\int^s g/\sqrt{-\varepsilon f}\,ds),\ \text{if } \varepsilon=1,\ d\geqq 0,$$

$$T_2(H,\,d;\ s)=(\sqrt{-\varepsilon h}/(2|H|),\ \delta\int^s k/\sqrt{-\varepsilon h}\ ds),\ \text{if } \varepsilon=1,\ d\neq 0,$$

$$T_3(H;\ t)=(t/(2|H|),\ \pm v_{-\varepsilon}),$$

N-axis, $H=0,\ N_0(s)=(\varepsilon a^{3/2}/(3d),\ d\,s^{1/2}),\ s>0,\ d>0,$

$$H\neq 0,\ N_1(H,\,d;\ t)=\left(\frac{\varepsilon}{16H^2d}\left(\frac{2t}{1-t^2}-\log((1-t)/(1+t)\right),\ dt\right),\ d>0,$$

$$N_2(H,\,d;\ t)=\left(\frac{\varepsilon}{8H^2d}\left(\tan^{-1}\mathrm{t}-\frac{\mathrm{y}^2}{1+\mathrm{y}^2}\right),\ \mathrm{dt}\right)\ d>0,$$

$$N_3(N,\ t)=\left(-\varepsilon\frac{e^{Ht}}{4H^2},\ e^{-Ht}\right),$$

where δ is the sign of H. The arc-length parameter s and the proper parameter t are taken on open intervals so that the functions in consideration have meaning. If $\varepsilon=1$ (resp., -1), the above curves and the correspondisng surfaces of revolution are space-like (resp., time-like).

A space-like surface with vanishing mean curvature is said to be a maximal surface. The surfaces corresponding to S_0, T_0 and N_0 are the same as those constructed by O. Kobayashi in [8].

The curve $S_1(H,\,d;\ s)=(y(s),\ z(s))$ satisfies

$$\pm y'=\frac{d\cosh\,(2Hs)-1}{2|H|z},\ \varepsilon(2|H|z)^2=1+d^2-2d\cosh\,(2Hs).$$

From these, we get

$$z^2\pm\frac{z}{H}\frac{dy}{ds}+\varepsilon\frac{d^2+1}{4H^2}=0.$$

For other cases, similarly we can obtain the corresponding differential equations.

i	S	T	N
0	$z=d(\varepsilon(1-\left(\frac{dz}{dy}\right)^2))^{1/2}$	$y=d(\varepsilon(\left(\frac{dy}{dz}\right)^2-1))^{1/2}$	$\frac{dz}{dx}=\varepsilon\frac{d^2}{y^2}$
1	$z^2\pm\frac{z}{H}y'+\varepsilon\frac{1-d^2}{4H^2}=0$	$y^2\pm\frac{y}{H}z'-\varepsilon\frac{1-d^2}{4H^2}=0$	$\frac{dy}{dz}=\frac{\varepsilon \quad y^2}{4H^2(d^2-y^2)^2}$
2	$z^2\pm\frac{z}{H}y'+\varepsilon\frac{1+d^2}{4H^2}=0$	$y^2\pm\frac{y}{H}z'-\varepsilon\frac{1+d^2}{4H^2}=0$	$\frac{dy}{dz}=\frac{\varepsilon \quad y^2}{4H^2(d^2-y^2)^2}$
3	$z^2\pm\frac{z}{H}y'+\frac{\varepsilon}{4H^2}=0$	$y^2\pm\frac{y}{H}z'-\frac{\varepsilon}{4H^2}=0$	$\frac{dy}{dz}=\frac{\varepsilon}{4H^2y^2}$

Lemma 2. *The generating curves S_i, T_i, N_i, S_i', T_i', N_i' ($i=1, 2, 3, 4$) satisfy the following differential equations respectively.*

We will consider the conics in the Lorentz 2-space $L^2=(R^2; dy^2-dz^2)$. Let F be a fixed point and D a fixed line in L^2. Put $E=e$, $e>0$ or $E=ie$, $e\neq 0$. The conic C of focus F, directrix D and eccentricity E is, by definition, the locus of a point such that $\|PF\|_L/\|PD\|_L=E$, where $\|PF\|_L$ is thhe Lorentzian distance between P and F, and $\|PD\|_L$ is the Lorentzian distance from P to D. Put $A=\|PD\|/2$ for $E=1$ and $A=E\|FD\|_L/|E^2-1|$ for other E. C is said to be horizontal (resp., vertical) If A is a real (resp., pure imaginary) number. According to $E=1$, $0<E<1$ or $1<E$, the conic is a *parabola*, an *ellipse* or a *hyperbola*. We derive equations describing the roulette of a conic C, that is, the trace of a focus F of a conic C as C rolls along a line. Let s be the arc-length of the roulette. We put

$$(2.7)\qquad S=\begin{cases} s & \text{if the roulette is a space-like curve,} \\ is & \text{if the roulette is a time-like curve.} \end{cases}$$

Lemma 3. *Let C be a conic with E and A. Let Γ_y (resp., Γ_z) be the roulette of the conic C as C rolls along the y-axis* (resp., z-axis). The roulette Γ_y and Γ_z are space-like curves if and only if C is vertical (resp., horizontal). If C is a parabola, that is, $E=1$, the roulette Γ_y (resp., Γ_z) satisfies

$$(2.8)\qquad A=iz\frac{dy}{dS}\left(\text{resp., } A=iy\frac{dz}{dS}\right).$$

If $E=e$ ($e>1$ or $1>e>0$)) or $ie(e>0)$, the roulette Γ_y (resp., Γ_z) satisfies

$$(2.9)\qquad z^2\pm 2iAz\frac{dy}{dS}+(E^2-1)A^2=0 \text{ (resp., } y^2\pm 2iAy\frac{dz}{dS}-(E^2-1)A^2=0).$$

From Theorem 1, Lemma 2 and Lemma 3, we obtain

Theorem 4. *The generating curves of surfaces of revolution with constant mean curvature, rotating about S-axis or T-axis are characterized as follows.*

(*I*) *Space-like surfaces*
Rotating about S-axis (resp., T-axis)

(1) *The roulette of the vertical parabola rolling algong y-axis (resp.,, z-axis), which is exactly* $S_0(s)$ *(resp.,* $T_0(s)$*).*

(2) *The undulary, the roulette of an vertical ellipse with* $E=d$ $(0<d<1)$ *and* $A=i\frac{1}{2H}$, *rolling along y-axis (resp.,, x-axis), which is exactly* $S_1(d, H; s)$ *(resp.,* $T_1(d, H; s)$*) for* $0<|d|<1$.

(3) *The nodary, the roulette of a vertical hyperbola with* $E=d$ $(d>1)$ *and* $A=i\frac{1}{2H}$ *rolling along y-axis (resp., x-axis), which is exactly* $S_1(d, H; s)$ *(resp.,* $T_1(d, H; s)$*) for* $1<|d|$.

(4) *The roulette of the vertical conic with* $E=id$ $(d>0)$ *and* $A=i\frac{1}{2H}$ *rolling along y-axis (x-axis), which is exactly* $S_2(d, H; s)$ *(resp.,* $T_2(d, H; s)$*).*

(5) *The curve* $S_3(d, H; s)$ *(resp.,* $T_3(d, H; s)$*).*
Rotatisng about S-axis,

(6) *The circle* $y^2-z^2=\left(\frac{1}{H}\right)^2$ *with radius H, which is exactly* $S_1(1, H; s)$.

(7) *The line* $z=\frac{1}{2H}$, *which is exactly* $S_1(0, H; s)$.

(*II*) *Time-like surfaces*
Rotating about S-axis (resp., T-axis)
If we replace vertical conics by horizontal conics in (1), (2), (3) and (4) of (I) respectively, we get corresponding generating curves of time-like surfaces of revolution. Moreover, we have

(8) *The curve* $S_3'(d, H; s)$ *(resp.,* $T_3'(d, H; s)$*).*

Rotating about T-axis,

(9) *The circle* $y^2-z^2=-\left(\frac{1}{H}\right)^2$ *with radius iH, which is exactly* $T_1'(1, H; s)$.

(10) *The line* $y=\frac{1}{2H}$ *which is exactly* $T_1'(0, H; s)$.

§3. Surfaces of revolution

The first and second fundamental forms of the surfaces given by (2.1) are

$$\varepsilon ds^2+z^2(s)dt^2 \text{ and } (y''(s)z'(s)-y'(s)z''(s))ds^2-y'(s)z(s)dt^2,$$

respectively. Hence the mean curvature $H(s)$ satisfies

$$2H(s)z(s)+y'(s)+\varepsilon z(s)\ (y'(s)z''(s)-y''(s)z'(s))=0,\ (S\text{-}axis). \tag{3.1}$$

Similarly, as the first and second fundamental forms of the surface given by (2.3) are $\varepsilon ds^2+y^2(s)dt^2$ and $(y'(s)z''(s)-y''(s)z'(s))ds^2+y(s)z'(s)dt^2$, respectively, we have

$$2H(s)y(s)-z'(s)-\varepsilon y(s)\ (y'(s)z''(s)-y''(s)z'(s))=0,\ (T\text{-}axis). \tag{3.2}$$

For the surface given by (2.4), those forms are $\varepsilon ds^2+4z^2(s)dt^2$ and $2(y''(s)z'(s)-y'(s)z''(s))ds^2-4z(s)z'(s)dt^2$. Hence we get

$$2H(s)z(s)+z'(s)+\varepsilon 2z(s)\ (y'(s)z''(s)-y''(s)z'(s))=0,\ (S\text{-}axis). \tag{3.3}$$

From the facts described above, it is evident that surfaces of revolution are space-like (resp., time-like) if and only if their generating curves are space-like (resp., time-like). Multiplying (2.1) by $y'(s)$ and using (1.2), we get

$$2H(s)z(s)y'(s)+(z(s)z(s))'+\varepsilon=0.$$

If we multiply (2.1) by $z'(s)$ and use (1.2), we have

$$2H(s)z(s)z'(s)+(z(s)y'(s))'=0.$$

From these equation, it follows

$$u'(s)+2H(s)u(s)-\varepsilon=0,\ v'(s)-2Hv'(s)-\varepsilon=0\ (S\text{-}axis,\ T\text{-}axis), \tag{3.4}$$

where

$$u(s)=-z(s)\ (y'(s)+z'(s)),\ v(s)=z(s)\ (y'(s)-z'(s))\ (S\text{-}axis). \tag{3.5}$$

Similarly, from (3.2) and (2.2) it follows that the generating curves of surfaces of revolution with T-axis satisfy the same differential equations (3.4). But in this case, we put

$$u(s)=y(s)\ (y'(s)-z'(s)),\ v(s)=y(s)\ (y'(s)+z'(s))\ (T\text{-}axis). \tag{3.6}$$

Using (2.5), we get from (3.3) the following equations.

$$u'(s)-2H(s)u(s)-\varepsilon=0,\ v'(s)+2H(s)v(s)=0\ (N\text{-}axis), \tag{3.7}$$

where

$$u(s)=2z(s)y'(s),\ v(s)=2z(s)z'(s). \tag{3.8}$$

To solve the above differential equations, we introduce the following functions.

$$(3.9)\qquad F(s)=\int_0^s \sinh\left(2\int_0^u H(t)dt\right)du,\ G(s)=\int_0^s \cosh\left(2\int_0^u H(t)dt\right)du.$$

The general solutions of (3.4) are given by

$$(3.10)\qquad u=(G'-F')\ (b_1+\varepsilon(G+F)),\ v=(G'+F')\ (b_2+\varepsilon(G-F)),$$

where b_1 and b_2 are integral constants. Similarly, as the general solutions of (3.7), we have

$$(3.11)\qquad u=(G'+F')\ (b_1+\varepsilon(G-F)),\ v=b_2(G'-F').$$

As we get $z^2=-\varepsilon uv$ and $y'=(v-u)/(2z)$ from (2.2) and (3.5), for the generating curve of a surface with *S-axis*, by putting $b_1+b_2=2c_1$, $b_1-b_2=2c_2$ and

$$(3.12)\qquad H(c_1, c_2)=((F+c_1)^2-(G+c_2)^2),\ I(c_1, c_2)=(F'(G+c_2)-G'(F+c_1)).$$

we have

$$(3.13)\qquad y=\int^s I(c_1, c_2)/\sqrt{\varepsilon H(c_1, c_2)}ds,\ z=\sqrt{\varepsilon H(c_1, c_2)},\ (S\text{-}axis).$$

Similarly, the generating curve of a surface with T-axis is given by

$$(3.14)\qquad y=\sqrt{-\varepsilon H(c_1, c_2)},\ z=\int^s I(c_1, c_2)/\sqrt{-\varepsilon H(c_1, c_2)}ds,\ (T\text{-}axis).$$

The generating curve of a surface with *N-axis has the following expression.*

$$(3.15)\qquad y=\varepsilon\int^s (G'+F')\ (c_1+G-F)/(2\sqrt{K})ds,\ z=\sqrt{K},\ (N\text{-}axis),$$

where we put $K=c_2\ (c_1+G-F)$ and c_1, c_2 are integral constants. Set

$$(3.16)\qquad \begin{cases} S(H, \varepsilon)=\{(c_1, c_2)\in R^2,\ \varepsilon H(c_1, c_2)>0 \quad \text{for all} \quad s\in I\}, \\ T(H, \varepsilon)=\{(c_1, c_2)\in R^2,\ -\varepsilon H(c_1, c_2)>0 \quad \text{for all} \quad s\in I\}, \\ N(H, \varepsilon)=\{(c_1, c_2)\in R^2,\ K(c_1, c_2)>0 \quad \text{for all} \quad s\in I\}. \end{cases}$$

For a given continuous function $H(s)$ on some interval I, the sets defined in (3.16) may be empty. Now we have the following theorem corresponding to the main resuult in [7].

Theorem 5. *If the genreating curve of a surface of revolution is parametrized by the arc lengths, its mean curvature is a function of the s. The arc length parametrized generating curve* $(y(s), z(s))$, $s\in I$, *of a surface of revolution with mean curvature* $H(s)$ *is given by* (3.13), (3.14) *or* (3.15) *for some constants* c_1, c_2, *according as it rotates abouut a* *space-like axis, time-like axis or a nuull axis. A surface of revolution is space-like (resp., time-like) if and only if its generating curve is space-like, that is,* $\varepsilon=1$ *(resp., time-like, that is,* $\varepsilon=-1$*). Conversely, for a given continuous function* $H(s)$, $s\in I$ *with* $S(H,\varepsilon)\neq\phi$, *by taking a point* $(c_1,c_2)\in S(H,\varepsilon)$ *and using* (3.13), *we construct a surface of revolution with mean curvatuure* $H(s)$, *which is space-like or time-like according to* ε. *Similarly, if* $T(H,\varepsilon)\neq\phi$ *or* $N(H,\varepsilon)\neq\phi$, *we construct a surface of revolution by* (3.14) *or* (3.15).

Now, we will show Theorem 1 in §2. At first, we consider surfaces of revolution with vanishing mean curvature, that is, $H=0$. Then the functions $F(s)$, $G(s)$ given by (3.9) are reduced to $F(s)=0$, $G(s)=s$. We get the solutions $S_0(s)$, $T_0(s)$ and $N_0(s)$ in Theorem 1, from (3.13), (3.14) and (3.15) respectively. Next, for a constant $H\neq0$, the functions $F(s)$, $G(s)$ become

$$F(s)=\frac{1}{2H}\cosh 2Hs,\quad G(s)=\frac{1}{2H}\sinh 2Hs. \tag{3.17}$$

Hence, from (3.12), we get, after some parallel translations of the arc length

$$H(c_1,c_2)=\begin{cases}(1+d^2-2d\cos h\ 2Hs)/(4H^2), & \text{if } c_1^2>c_2^2,\\ (1-d^2-2d\sin h\ 2Hs)/(4H^2), & \text{if } c_1^2<c_2^2,\\ (1+4Hce^{-2Hs})/(4H^2), & \text{if } c_1=c_2=c,\\ (1+4Hce^{2Hs})/(4H^2), & \text{if } c_1=-c_2=c,\end{cases} \tag{3.18}$$

$$I(c_1,c_2)=\begin{cases}(d\cos h\ 2Hs-1)/(2H), & \text{if } c_1^2>c_2^2,\\ (d\sin h\ 2Hs-1)/(2H), & \text{if } c_1^2<c_2^2,\\ -(1+2Hce^{-2Hs})/(2Hs),. & \text{if } c_1=c_2=c,\\ -(1+2Hce^{2Hs})/(2Hs), & \text{if } c_1=-c_2=c,\end{cases} \tag{3.18}$$

where $d=-sgn(c_1)2H\sqrt{c_1^2-c_2^2}$ (resp., $-sgn(c_2)2H\sqrt{c_2^2-c_1^2}$) if $c_1^2>c_2^2$ (resp., $c_1^2>c_2^2$). Using these, we obtain Theorem 1.

§4. The Lorentzian plane

Let L^2 be the Lorentzian plane. At first, we describe some facts about angles in Lorenzian geometry. Details are found in [4]. For a vector $p=(y,z)\in L^2$, the *Lorentian norm* $\|p\|_L$ is defined to be $\|p\|$ if p is space-like or null, and $i\|p\|_L$ if p is time-like, where $\|p\|$ is the (absolute) norm of p and $i=\sqrt{-1}$. If p is non-zero and non-null, its *polar coordinates* R, Ω is defined by $R=\|p\|_L$, $\Omega=\theta+\imath\omega$, where $y=R\,cos(\theta+i\omega)$, $iz=R\,sin(\theta+i\omega)$, $\theta=0,\pi/2,\pi,3\pi/2$, $-\infty<\omega<\infty$. We call Ω the *angle of the vector p*. For non-zero and non-null vectors $p,q\in L^2$, let Ω_2 and Ω_q be the angles of p and q respectively. The *oriented angle from p to q* is defined to be $\Omega_q-\Omega_p$ (mod 2π) and denote by $\widehat{pq}$. If p is moved to p' counterclockwise with fixing the beginning point of p so that $\widehat{p'q}=0$, we say that $\widehat{pq}$ is the positively oriented angle. The (unoriented) angle is $\widehat{pq}$ (resp., $\widehat{qp}$) if pq (resp., $\widehat{qp}$) is positiviely oriented. In the Lorentzian plane, we have

$$\cos \widehat{pq}=\frac{<p, q>}{\|p\|_L\|q\|_L}. \tag{4.1}$$

For any points A, $B\in L^2$, Let $\overrightarrow{AB}$ the corresponding vector. Let ΔABC be a triangle in L^2 with non-null $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$. If we denote by $\hat{A}$, $\hat{B}$, $\hat{C}$ the angles of the triangle, then we have $\hat{A}+\hat{B}+\hat{C}=\pi$. If $\hat{C}=\pi/2$, it follows from (4.1) that $\|\overrightarrow{BC}\|_L=\|\overrightarrow{AB}\|_L \cos \hat{B}$ and $\|\overrightarrow{AC}\|_L=\|\overrightarrow{AB}\|_L \sin \hat{B}$.

Next, we investigate rolling curves along a line in L^2 and follow the path of any chosen tracing point on the curve.Let C be a convex curve which can roll along a line l. The tracing point P can be placed inside, on, or outside C. If P is regarded as the origin, then C can be described by polar coordinates R, Ω. Assume the tangent line l is the y-axis. If Φ is the angle between the y-axis and radial line of C, then C traces out the curve Γ given by $y=\sigma-R\cos\Phi$, $iz=R\sin\Phi$, where σ is the arc-length of the curve C. *Let* C be on the moving plane $\{(u, v)\}$. Let Θ be the angle between the y-axis and the u-axis. Then as we have $i\frac{dv}{du}=\tan\Theta$ and $\Theta=\pi-\Omega-\Phi$, we get $\tan\Phi=R\frac{d\Omega}{dR}$. Since $ds^2=dR^2+R^2d\Omega^2=(1+\tan^2\Phi)dR^2$, we may put $\frac{\mathrm{ds}}{\mathrm{dR}}=\frac{1}{\cos\Phi}$. Thus we obtain $i\frac{dz}{dy}=\cot\Phi$. Hence the radial line is normal to Γ. Let s be the arc length of Γ, if Γ is space-like (resp., time-like), we put $dS=ds$ (resp., $dS=ids$). Hence, it follows

$$\frac{dy}{dS}=\sin\Phi, \qquad i\frac{dz}{dS}=\cos\Phi. \tag{4.2}$$

If C rolls along the z-line, we have similarly

$$\frac{dy}{dS}=\cos\Phi, \qquad i\frac{dz}{dS}=\sin\Phi. \tag{4.3}$$

We describe some properties of conics in L^2. (a) C is a parabola. The standard equation of a vertical parabola is $y^2=-4az$. Here $F=(0, a)$, $D: z=-a$ and $A=ia$. Let l be the tangent to C at a point K, which intersects z-axis at P and y-axis at Q. Using (4.1), we have

$$\overrightarrow{FP}\perp l, \qquad \angle FQP=\begin{cases} \angle FKP & \text{for } a\leqq|z| \\ \pi-\angle FKP & \text{for } |z|\leqq a. \end{cases} \tag{4.4}$$

$F=(0, -a)$ and $D': z=a$ are regarded as the focus and the directix of the parabola. The standard equation of a horizontal parabola is given by $z^2=-4ay$, where $F=(a, 0)$ and

D: $y=-a$. This has the same property as the above.
(b) C is a ellipse. The standard equation of a vertical ellipse with focus $F=(0, ea)$ (resp., $F'=(0, -ea)$), directrix D: $z=a/e$ (resp., D': $z=-a/e$), eccentricity $E=e$ and $A=ia$ is $(1-e^2)z^2-y^2=a^2(1-e^2)$. For a point $K=(y, z)$ on C, we have

$$KF \sim KF'=2a \text{ for } |z| \geqq a/e, \quad KF+KF'=2a \text{ for } a \leqq |z| \leqq a/e. \tag{4.5}$$

Take a line l tangent to C at a point $K=(y, z)$. Through F (resp., F') we draw a line perpendicular to l, intersecting l at Q (resp., Q'). Denote by FQ the oriented length of FQ, that is, $FQ=\|FQ\|$ or $-\|FQ\|$ according as FQ is positively oriented or negatively oriented. Now we have

$$FQ \cdot FQ' = \begin{cases} (1-e^2)a^2 & \text{for } a \leqq |z| \leqq a/e, \\ -(1-e^2)a^2 & \text{for } a/e \leqq |z|. \end{cases} \tag{4.6}$$

By making use of (4.1), we get

$$\angle FKQ = \begin{cases} \angle F'KQ' & \text{for } a/e \leqq |z|, \\ \pi - \angle F'KQ' & \text{for } a \leqq |z| \leqq a/e. \end{cases} \tag{4.7}$$

The standard equation of the horizontal ellipse with focus $F=(ea, 0)$ (resp., $(-ea, 0)$), directrix D: $y=a/e$ (resp., D': $y=-a/e$) and $A=a$ is $(1-e^2)y^2-z^2=a^2(1-e^2)$. This curve has the properties similar to the above.
(c) C is a hyperbola. The standard equation of a vertical hyperbola with focus $F=(0, ea)$ (resp., $F'=(0, ea)$), directrix D: $z=a/e$ (resp., D': $z=-a/e$), eccentricity $E=e$ and $A=ia$ is given by the same equation as one of the ellipse. In this case, we obtain for a point $K=(y, z)$ on C, we have

$$KF+KF'=2a \text{ for } |z| \leqq a/e, \quad KF \sim KF'=2a \text{ for } a/e \leqq |z| \leqq a. \tag{4.8}$$

Let l be the tangent line to C at a point $K=(y, z)$. Through F (resp., F') we draw a line perpendicular to l, intersecting l at Q (resp., Q'). Then we have

$$FQ \cdot FQ' = \begin{cases} -(1-e^2)a^2 & \text{for } a/e < |z| < a \\ (1-e^2)a^2 & \text{for } |z| < a/e. \end{cases} \tag{4.9}$$

$$\angle FKQ = \begin{cases} \angle F'KQ' & \text{for } |z| < a/e, \\ \pi - \angle F'KQ' & \text{for } a/e < |z| < a. \end{cases} \tag{4.10}$$

For horizontal hyperbolas, we have the corresponding facts.
(d) C is a conic with imaginary eccentricity. The standard equation of the vertical conic with focus $F=(ea, 0)$ (resp., $F'=(-ea, 0)$), directrix D: $y=-a/e$ (resp., D': $y=a/e$), eccentricity $E=ie$ and $A=ia$ is $z^2-(1+e^2)y^2=(1+e^2)a^2$. In the present case, we obtain for a point $K=(y, z)$ on C

(4.12) $$KF+KF'=2a \text{ for } |y|<a/e,\ KF\sim KF'=2a \text{ for } a/e<|y|.$$

Let l be the tangent line to C at a point $K=(y, z)$. Through F(resp., F') we draw a line perpendicular to l, intersecting l at Q (resp., Q'). Then we have

(4.13) $$FQ\cdot FQ'=\begin{cases}(e^2+1)a^2 & \text{for } |y|<a/e\\ -(e^2+1)a^2 & \text{for } |y|>a/e\end{cases},$$

$$\angle QKF=\begin{cases}\pi-\angle Q'KF & \text{for } |y|<a/e\\ \angle Q'KF & \text{for } |y|>a/e\end{cases}.$$

Horizontal conics with eccentricity $E=ie$ satisfy properties similar to the above.

§5. The differential equations of rolling curves

Let Γ be the trace of a focus $F=(y, z)$ of a conic which rolls along a line l in $L^2=\{(y, z)\}$. We will find a differential equation whose solution gives the curve Γ, that is, we will prove Lemma 3.

(a) C is a parabola. (1) l is the y-axis. At first, Let C be the vertical parabola given in (a) of §4. Assume that C is tangent to y-axis at a point K. Let B be the vertex of C and l' be the line through B and perpendendicular to the axis of C. Assume that y-axis intersects l' at P and the axis of C at Q. Let Φ be the angle between y-axis and the line through the focus $F=(y, z)$ of C and K. We get from (4.4)$\angle BPF=\angle PQF=\Phi$ (or $\pi-\Phi$). As we have $\|\overrightarrow{FB}\|_L=\|\overrightarrow{FP}\|_L \sin\Phi$, we may put $a=z\sin\Phi$. Hence from (4.2), it holds $a=z\, dy/ds$, that is,

$$z=a\sqrt{1-(dz/dy)^2},\ dS=ds.$$

Assume next that C is horizontal. Let points K, B, P and Q be taken as similarly as in the above case. Let Φ also the angle between z-axis and the line through the focus F and K. Now we have $\angle FPB=\pi/2-\angle BFP=\pi/2-(\angle FQP+\angle QPF)=-\Phi$. Hence, from $\|\overrightarrow{BF}\|_L=\|\overrightarrow{FP}\|_L \sin\angle FPB$, we get $a=-iz\sin\Phi$. In this case, it should be $dS=ids$ in the formula (4.2). Thus we obtain $a=z(dy/ds)$, $ds^2=dz^2-dy^2$. In other word, we show (2.8) for Γ_y, $A=ia$ and $dS=ids$.

(2) l is the z-axis. Let C be the vertical parabola as in (1). Then as similarly as in the above, we obtain the equation $ia=y\sin\Phi$, where Φ is the angle between the z-axis and the tangent line. From (4.2), we get $i(dz/ds)=\cos(\pi-\Phi)=\sin\Phi$. thus we obtain (2.8) for Γ_Z, $A=ia$ and $dS=ds$. For the horizontal C, we have $a=y\sin\Phi$. Hence as we must put $dS=ids$ in (4.2), we get (2.8) for Γ_Z, $A=a$ and $dS=ids$.

(b) C is an ellipse.

(1) l is the y-axis. Let C be the vertical ellipse given in (b) of §4. Let $F=(y, z)$ and $F'=(y',$

z') be foci of C. The y-line is tangent to C at a point K. Through F(resp., F'), draw a line perpendicular to y-line and intersecting it at Q (resp., Q'). From (4.6), (4.7) and (4.8), we get $FQ \cdot F'Q' = (1-e^2)a^2$, $KF + KF' = 2a$ and $\angle Q'KF' = \pi - \angle QPF$. Using (4.2), we have $\pm z/KF = sin\angle QKF = dy/dS$ and $\pm z'/KF' = sin\angle Q'KF' = ay/dS$, where the double sign is for $z, z' > 0$ and $z, z' < 0$, respectively. Thus we obtain (2.9) for Γ_y, $E = e$ $(0 < e < 1)$, $A = ia$ and $dS = ds$. Assume that C is horizontal. Take points F, F', Q, Q', and K as above. Then we have $KF \sim KF' = 2a$, $FQ \cdot F'Q' = -(1-e^2)a^2$, $iz/KF = \pm sin\angle QKF = dy/dS$ and $\pm iz'/KF' = sin\angle Q'KF' = dy/dS$. Hence, we must put $dS = ids$ and we get $z + z' = \pm 2a\, dy/ds$ and $zz' = -(1-e^2)a^2$. Thus we have (2.9) for Γ_y, $E = e$ $(0 < e < 1)$, $A = a$ and $dS = ids$.

((2) l is the Z-line. C is vertical. In this case, we have $FQ \cdot F'Q' = -(1-e^2)a^2$, $KF \sim KF' = 2a$, $y = \pm KF(dz/ds)$ and $y' = \pm KF'(dz/ds)$. Thus we have (2.9) for Γ_z, $E = e$ $(0 < e < 1)$, $A = ia$ and $dS = ds$. Similarly, when C is horizontal, we get (2.9) for Γ_Z, $E = e$ $(0 < e < 1)$, $A = a$ and $dS = ids$.

(c) C is a hyperbola.

(1) l is thhe y-axis. Let C be vertical. Assume that C is tangent to y-axis at K. Let $F = (y, z)$ and $F' = (y', z')$ be the foci of C. Take points Q and Q' as in (1) of (b). Then we have $FK \sim F'K = 2a$, $FQ \cdot F'Q' = -(e^2-1)a^2$. $z/FQ = \pm dy/dS$ and $z'/F'Q' = \pm dy/dS$. Hence we obtain (2.9) for Γ_y, $E = e$ $(1 < e)$, $A = ia$ and $dS = ds$. If C is horizontal, it holds that $KF + F'K = 2a$, $FQ \cdot F'Q' = (e^2-1)a^2$, $iz/FK = \pm dy/dS$ and $iz'/F'K = \pm dy/dS$. Hence, we get (2.9) for Γ_y, $A = a$, $E = e$ $(e > 1)$ and $dS = ids$.

(2) l is the Z-axis. If C is vertical, taking points K, Q and Q' as in (1), we have $KF + KF' = 2a$, $FQ \cdot F'Q' = (e^2-1)a^2$, $y/iKF = \pm idz/dS$ and $y'/iKF' = \pm idz/dS$. Hence we get (2.9) for Γ_Z, $E = e$ $(1 < e)$, $A = ia$ and $dS = ids$. When C is horizontal, we get $KF \sim F'K = 2a$, $FQ \cdot F'Q' = -(e^2-1)a^2$, $y/FK = \pm idz/dS$ and $y'/F'K = \pm idz/dS$. Thus we obtain (2.9) for Γ_Z, $E = e$ $(1 < e)$, $A = a$ and $dS = ids$.

(d) C is a conic with imaginary eccentricity.

(1) l is the y-axis. Let C be vertical. Take points K, Q, Q' as above. Then we have $KF + KF' = 2a$, $FK \cdot F'K = (1+e^2)a^2$, $z/KF = \pm dy/dS$ and $z'/KF' = \pm dy/dS$. Hence we have (2.9) for Γ_y, $E = ie$, $A = ia$ and $dS = ids$. Similarly, for a horizontal C, we have (2.9) for Γ_y, $E = ie$, $A = a$ and $dS = ids$.

(2) l is the z-axis. For a vertical C, we have $KF \sim KF' = 2a$, $yy' = -(1+e^2)a^2$, $y/(iKF) = \pm idz/dS$ and $y'/(iKF') = \pm idz/dS$. Thus we have (2.9) for Γ_Z, $E = ie$, $A = ia$ and $dS = ds$. Similarly, for a horizontal C, we get (2.9) for Γ_Z, $E = ie$, $A = a$ and $dS = ids$.

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