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Some Integral Formulas in Fubini-Study Spaces

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§1. Introduction

In the previous paper [1], we got some integral formulas in a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. In this paper, firstly we will extend all formulas in [1] to those in Fubini-study spaces of constant holomorphic sectional curvature 4λ ($\lambda>0$). Next we will give the complex version of the formula (14.70) in [2].

Let $C^{n+1}=\{z=(z^0,\dots, z^n)\}$ be the complex Euclidean $(n+1)$-space with natural inner product $(z, w)=\sum_{k=0}^{n} z^k\bar{w}^k$, for $z, w\in C^{n+1}$. The Euclidean metric g on C^{n+1} is given by $g(z, w)=Re(z, w)$. Put $S^{2n+1}(\lambda^{-1/2})=\{z\in C^{n+1}, g(z, z)=1/\lambda\}$. Then it is a principal fibre bundle over the complex projective n-space $P^n(C)$ with structure group S^1 and projection π. We may regard $z=(z^0,\dots, z^n)$ as the homogeneous coordinate system of the point $[z]\in P^n(C)$, where $[z]=\pi(z)$. For $z\in S^{2n+1}(\lambda^{-1/2})$, we may put $T_zS^{2n+1}=\{w\in C^{2n+1}; g(z, w)=0\}$. The space given by $T'_z=\{w\in C^{n+1}; g(z, w=0, g(iz, w)=0\}$ is a subspace of T_zS^{n+1} whose orthogonal complement is $\{iz\}$. The projection π induces a linear isomorphism π_* of T'_z onto $T_{[z]}P^m(C)$. The standard Riemannian metric on $S^{2n+1}(\lambda^{-1/2})$ is given by $(1/\lambda)g(W, Z)$, for $W, Z\in T_zS^{2n+1}$. We define the Fubini-Study metric g of constant holomorphic sectional curvature 4λ by

$$g(X, Y)=\frac{1}{\lambda}g(X', Y'),$$

where $X, Y\in T_{[z]}P^n(C)$ and X', Y' are their respective horizontal lifts at z.

§2. Total volumes of complex Grassmann manifolds

Let $a_0,\dots, a_n$ be a unitary frame field on C^{n+1}. Put $da_k=\sum_{j=0}^{n}\omega_{jk}a_j$. Let L_r^0 be a fixed projective r-space of $P^n(C)$. Assume that L_r^0 is defined by the points $a_0,\dots, a_r$. Then we have $\omega_{jk}=0$ for $0\leqq k\leqq r$ and $r+1\leqq j\leqq n$. Thus a density for projective r-spaces which is invariant under the unitary group $U(n+1)$ may be given by

(2.1) $$dL_r=\left(\frac{\sqrt{-1}}{2\lambda}\right)^{(n-r)(r+1)}\bigwedge_{j,k}(\omega_{jk}\wedge\bar{\omega}_{jk}),\ (0\leqq k\leqq r,\ r+1\leqq j\leqq n).$$

For $r=0$, we get the density for points, that is, the volume element of $P^n(C)$ deduced from the Fubini-Study metric given in §1.

The point $\lambda^{-1/2}a_0$ moves on the sphere $S^{2n+1}(\lambda^{-1/2})$ centerd at the origin. Since $d(\lambda^{-1/2}a_0)=\sum_{j=0}^{n}(\lambda^{-1/2})\omega_{j0}a_j$. The volume element ds^{2n+1} for $S^{2n+1}(\lambda^{-1/2})$ is given by

(2.2) $$ds^{2n+1}=(\sqrt{-1}/2\lambda)^n(-\sqrt{-1/\lambda}\,\omega_{00})\bigwedge_i(\omega_{i0}\wedge\omega_{i0}),\quad(1\leqq i\leqq n).$$

The restriction of the form $\sqrt{-1}/\lambda\omega_{00}$ to each fibre of the fibre bundle π: $S^{2n+1}(\lambda^{-1/2})\to P^n(C)$ is regarded as the standard volume element of $S^1(\lambda^{-1/2})$. Hence the total volume $m(P^n(C))$ is given

(2.3) $$m(P^n(C))=\frac{1}{2\pi\sqrt{\lambda}}m(S^{2n+1}(\lambda^{-1/2}))=\frac{\pi^n}{\lambda^n n!}.$$

Let L_{n-1} be the $(n-1)$-plane in $P^n(C)$ perpendicular to a_0. Put $L_{r-1}^{n-1}=L_r\cap L_{n-1}$. Then we have the density for $(r-1)$-planes L_{r-1}^{n-1} in L_{n-1} as follows.

(2.4) $$dL_{r-1}^{n-1}=\left(\frac{\sqrt{-1}}{2\lambda}\right)^{(n-r)r}\wedge(\omega_{hi}\wedge\bar{\omega}_{hi}),\ (1\leqq i\leqq r,\ r+1\leqq h\leqq n).$$

If ds^{2r+1} denotes the volume element of $S^{2r+1}(\lambda^{-1/2})$ in L_r, we have

(2.5) $$dL_r\wedge ds^{2r+1}=dL_{r-1}^{n-1}\wedge ds^{2n+1}.$$

Successive exterior multiplication by ds^{2r-1}, $ds^{2r-3},\dots,ds^3$, ds^1 gives

(2.6) $$dLr\wedge ds^{2r+1}\wedge ds^{2r-1}\wedge\cdots\wedge ds^3\wedge ds^1=ds^{2(n-r)+1}\wedge ds^{2(n-r)+3}\wedge\cdots\wedge ds^{2n+1}.$$

Integrating over the spheres $S^{2n+1}(\lambda^{-1/2})$, $S^{2n-1}(\lambda^{-1/2}),\dots,S^3(\lambda^{-1/2})$, $S^1(\lambda^{-1/2})$, we get (see [1], [2])

Proposition 1. *The total volume of the r-planes in the Fubini-Study space $P^n(C)$ of constant holomorphic sectional curvature 4λ, that is, the total volume of the complex Grassmann manifold $G_{r+1,n-r}$ of $(r+1)$-planes in C^{n+1}, is given by*

$$m(G_{r+1,n-r})=\frac{1!2!\cdots\cdots\cdots r!}{n!(n-1)!\cdots(n-r)!}\left(\frac{\pi}{\lambda}\right)^{(n-r)(r+1)}.$$

§3. Densities for linear subspaces

Let L_r^0 be a fixed projective r-space of $P^n(C)$. Assume that L_r^0 is defined by the points $a_0,\dots,a_r$. Let L_q^0 be a fixed projective q-subspace contained in L_r^0.

Suppose that $a_0,\dots, a_q$ span L_q^0. Then the density for projective r-spaces containing L_q^0 is

$$(3.1)\qquad dL_{r[q]}=\left(\frac{\sqrt{-1}}{2\lambda}\right)^{(n-r)(r-q)}\bigwedge_{i,j}(\omega_{hi}\wedge\bar{\omega}_{hi}),\ (q+1\leqq i\leqq r,\ r+1\leqq h\leqq n).$$

Let L_{n-q-1} be the projective $(n-q-1)$-subspace perpendicular to L_q^0. Ecah $L_{r[q]}$ can be defined by the intersection $L_{r[q]}\cap L_{n-q-1}$, which is a projective $(r-q-1)$-subspace, and consequently the density of all $L_{r[q]}$ is equal to the density of all L_{r-q-1} in L_{n-q-1}, that is,

$$(3.2)\qquad dL_{p[q]}=dL_{n-q-1}^{r-q-1}.$$

Let L_r and L_q be a moving projective r-subspace and a fixed projective q-subspace respectively in $P^n(C)$. Assume that $r+q>n$. Denote by L_{r+q-n} the intersection $L_q\cap L_r$. Take a unitary frame field $\{a_0,\dots, a_n\}$ such that $a_0,\dots, a_{r+q-n}$ span L_{r+q-n} and $a_{r+q-n+1},\dots, a_r$ lie on L_r. Moreover take points $b_{r+q-n+1},\dots, b_n$ such that $a_0,\dots, a_{r+q-n}$, $b_{r+q-n+1},\dots, b_n$ form a unitary frame field and $a_0,\dots, a_{r+q-n}$, $b_{r+q-n+1},\dots, b_r$ span L_q. As similarly as (3.5) in [1], we get

$$(3.3)\qquad dL_r=|\Delta|^{2(r+q-n+1)}dL_{r[r+q-n]}\wedge dL_{r+q-n}^q,$$

where $\Delta=det(a_h, b_k)$, $(r+1\leqq k, h\leqq n)$. By putting $N=2n-r-q-1$, $v=r+q-n+1$, $\rho=n-q-1$, we get (see Proposition 2 in [1])

$$(3.4)\qquad \int_{G_{N-\rho,\rho+1}}|\Delta|^{2v}dL^N=\frac{m(G_{N-\rho,v+\rho+1})}{m(G_{n-\rho,v})}.$$

Let $F(L_r)$ be an integrable function that depends only on $L_{r+q-n}^q=L\cap L_q$. From (3.3), it follows that

$$\int f(L_r)\,dL_r=\int|\Delta|^{2(r+q-n+1)}\,dL_{r[r+q-n]}\int F(L_{r+q-n}^q)dL_{r+q-n}^q.$$

Applying (3.2) and (3.4), we obtain

Proposition 2. *Let $F(L_r)$ be an integrable function that depends only on $L_{r+q-n}^q=L_r\cap L_q$. Then*

$$\int F(L_r)dL_r=\frac{m(G_{r+1,n-r})}{m(G_{r+q-n+1,n-r})}\int F(L_{r+q-n}^q)dL_{r+q-n}^q.$$

§4. Intersections of projective subspaces and submanifolds

Let L_r be a moving projective r-space and M^q be a Kaehlerian submanifold in $P^n(C)$. Let $\{a_0,\dots, a_n\}$ be a unitary frame field such that $a_0,\dots, a_r$ span L_r. For any submanifold X of $P^n(C)$, denote by PT_xX the projective tangent space of X at

$x \in X$, that is, $\pi((d\pi)^{-1}(T_xX))$, where π: $C^{n+1} \to P^n(C)$. We may assume that $x = a_r \in L_r \cap M^q$ and $a_0, \dots, a_{r+q-n}$ span $PT_x(L_r \cap M^q)$. Let $b_{r+1}, \dots, b_n$ be a set of unitary vectors such that PT_xM is spanned by $a_1, \dots, a_{r+q-1}, b_{r+1}, \dots, b_n$. We can put, up to a constant factor,

$$(4.1)\quad dL_r = \bigwedge_h (\omega_{hr} \wedge \bar{\omega}_{hr}) \bigwedge_{j,k} (\omega_{kj} \wedge \bar{\omega}_{kj}), \quad (r+1 \leqq h \leqq n,\ 1 \leqq j \leqq r-1,\ r+1 \leqq k \leqq n).$$

Let L'_{n-r} be the linear $(n-r)$-space in C^{n+1} orthogonal to L_r and $L_{r-1[x]}$ be the projective $(r-1)$-space in L_r orthogonal to $x = a_r$. The volume element of L'_{n-r} is given by, up to a constant factor,

$$(4.2)\quad d\sigma_{n-r} = \bigwedge_h (\omega_{hr} \wedge \bar{\omega}_{hr}), \quad (r+1 \leqq h \leqq n).$$

Hence we get, up to a constant factor,

$$(4.3)\quad dL_r = d\sigma_{n-r} \wedge dL_{r-1[x]}.$$

As $x \in M^q$, it holds $dx = \sum_{i=0}^{q+r-n} \lambda_i a_i + \sum_{k=r+1}^{n} \beta_k b_k$, where λ_i and β_k are differential 1-forms. We have

$$\omega_{hr} = -(dx, a_h) = -\sum_{k=r+1}^{n} \beta_k (b_k, a_h).$$

If follows from (4.2), up to a constant factor,

$$(4.4)\quad d\sigma_{n-r} = |\Delta|^2 \bigwedge_k (\beta_k \wedge \bar{\beta}_k), \quad (r+1 \leqq k \leqq n),$$

where $\Delta = det(b_k, a_h)$. If we denote by $\sigma_{r+q-n}(x)$ and $\sigma q(x)$ the volume elements of $L_r \cap M^q$ and M^q respectively. We have, up to a constant factor,

$$(4.5)\quad d\sigma_q(x) = \bigwedge_k (\beta_k \wedge \bar{\beta}_k), \quad (r+1 \leqq k \leqq n).$$

From (4.3), (4.4) and (4.5), it follows, up to a constant factor,

$$(4.6)\quad d\sigma_{r+q-n}(x) \wedge dL_r = |\Delta|^2 d\sigma^q(x) \wedge dL_{r-1[x]}.$$

Since Δ is independent of x, integrating the both sides of (4.6) over all projective r-space intersecting M^q, we obtain

$$(4.7)\quad \int_{L_r \cap M^q \neq \emptyset} m(M^q \cap L_r) dL_r = Cm(M^q),$$

where C is a constant. We need to determine it. For this purpose, we calculate the both side of (4.7) for $M^q = L_q = P^q(C)$. Using (2.3) and Proposition 1, we get

$$C = \frac{m(G_{r+1,n-r}) \times m(P^{q+r-n}(C))}{m(P^q(C))} = \frac{1! \cdots\cdots r! q!}{n! \cdots (n-r)!(q+r-n)!} \left(\frac{\pi}{\lambda}\right)^{(n-r)r}.$$

Thus we obtain

Proposition 3. *Let M^q be a q-dimensional compact Kaehler submanifold of a Fubini-Study space $P^n(C)$ of constant holomorphic sectional curvature 4λ. Let L_r a projective r-space in $P^n(C)$. Denote by $m(M^q)$ and $m(M^q \cap L_r)$ the volumes of M^q and $M^q \cap L_r$ respectively. Then we have*

$$\int_{M^q \cap L_r \neq \emptyset} m(M^q \cap L_r)\, dL_r = \frac{1!2!\cdots\cdots\cdots r!q!}{n!(n-1)!\cdots(n-r)!(q+r-n)!}\left(\frac{\pi}{\lambda}\right)^{(n-r)r} m(M^q),$$

where dL_r is the density for projective r-spaces in $P^n(C)$.

When $q+r=n$, the intersection of a q-dimensional submanifold M^q and a projective r-space L_r is, in general, a finite set of points in $P^n(C)$. In this case, the above formula becomes

Corollary. *Let denote by $\sharp(M^{n-r} \cap L_r)$ be the number of the points in $M^{n-r} \cap L_r$. Then it holds*

$$\int_{M^{n-r} \cap L_r \neq \emptyset} \sharp(M^{n-r} \cap L_r)\, dL_r = \frac{1!2!\cdots\cdots r!}{n!(n-1)!\cdots(n+r+1)!}\left(\frac{\pi}{\lambda}\right)^{(n-r)r} m(M^{n-r}).$$

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