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On Periodic Solutions of Autonomous Systems with the First Variation Equation Having a Double Characteristic Multiplier One

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§1. Introduction

In the previous paper [8], we have considered a periodic solution of a periodic differential system whose first variation equation has a characteristic multiplier one and we have proposed a method for computing singular points of nonlinear equations defined by solutions of periodic systems involving parameters. Further, we have also given a method for computing bifurcation points of periodic solutions of periodic systems.

In the present paper, we shall consider a periodic solution of an autonomous system whose first variation equation has a double characteristic multiplier one and we shall discuss singular points and bifurcation points of nonlinear equations defined by solutions of autonomous systems involving parameters.

Roughly speaking, as has been shown in [8], a characteristic multiplier one corresponds to a periodic solution of the first variation equation. In the case of an autonomous system, on the other hand, if the system has a periodic solution, then the first variation equation necessarily has a periodic solution. Therefore we must consider another periodic solution of the first variation equation independent of such a periodic solution.

Although such a difficulty exists in the case of an autonomous system, the theory and method used for overcoming difficulties arising from the singularity of the Jacobian matrix of a nonlinear equation implicit in form are the same as the ones used in the previous paper [8].

In this paper, in §2, we consider turning points, cusp points, etc. of nonlinear equations defined by solutions of autonomous systems involving parameters. And, in §3, we consider bifurcation points of periodic solutions of autonomous systems.

Lastly, in order to illustrate our theory and method, we present some examples of singular points and bifurcation points in §4. These examples show the usefulness of our theory and method.

§ 2. Singular Points of Nonlinear Equations Defined by Solutions of Autonomous Systems Involving Parameters

We consider an ω-periodic solution of a real n-dimensional autonomous system

$$\frac{dx}{d\tau}=X(x, B), \tag{2.1}$$

where $X(x, B)$ is continuously differentiable with respect to (x, B) in the region Δ. Here Δ is some region of the (x, B)-space, and B is a parameter and we assume that the dimension of the parameter B is m (≥ 1). Of course, the period ω is unknown.

At first, let us suppose that the values of $(m-1)$ components of B are given and one and only one component of B is unknown. For the sake of simplicity, we denote such an unknown component of B by B and this is called the parameter B with dimension one.

Now, transforming τ to t by $\tau=\frac{\omega}{2\pi}t$ in (2.1), we have

$$\frac{dx}{dt}=\frac{\omega}{2\pi}X(x, B). \tag{2.2}$$

Therefore the problem of finding an ω-periodic solution of (2.1) is reduced to the one of finding a 2π-periodic solution of (2.2).

As is well-known, when $x(t)$ is a solution of (2.2), $x(t+\alpha)$ is also a solution of (2.2) for an arbitrary constant α. This fact tells us that no 2π-periodic solution of (2.2) is uniquely determined by the periodic boundary condition alone. Then we must give one more condition (cf. [1]). In order to simplify the following argument, we adopt the condition

$$x_2(0)=\beta \tag{2.3}$$

as an additional condition, where $x(0)=(x_1(0), x_2(0),\dots, x_n(0))^T$ and β is a constant number. About how to choose an additional condition, see Section 5. Here $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$.

Setting $u=(x, \omega)^T$ and $V(u, B)=\left(\frac{\omega}{2\pi}X(x, B), 0\right)^T$, we rewrite (2.2) in the following form:

$$\frac{du}{dt}=V(u, B). \tag{2.4}$$

Thus, we consider a 2π-periodic solution $u=u(t)=(x(t), \omega)^T$ of (2.4) satisfying the condition (2.3).

Let $(\varphi(t, u(0), B), \omega)^T$ be a solution of (2.4) at a given B such that $(\varphi(0, u(0), B), \omega)^T=(x(0), \omega)^T$, where $(x(0), B)\in\Delta$. Then we consider the equation

(2.5)
$$F(u(0), B)=\begin{pmatrix} \varphi(0, u(0), B)-\varphi(2\pi, u(0), B) \\ x_2-\beta \end{pmatrix}$$
$$=\begin{pmatrix} x(0)-\varphi(2\pi, u(0), B) \\ x_2-\beta \end{pmatrix}=0,$$

where $u(0)=(x(0), \omega)^T$, $x(0)=(x_1(0), x_2(0),\dots, x_n(0))^T=(x_1, x_2,\dots, x_n)^T$. By the assumption on $X(x, B)$, the function $F(u(0), B)$ defined by the equality (2.5) is continuously differentiable with respect to $(u(0), B)$ and we denote by $F_u(u(0), B)$ the Jacobian matrix of $F(u(0), B)$ with respect to $u(0)=(x(0), \omega)^T$. Then we have

(2.6)
$$F_u(u(0), B)=\begin{pmatrix} E_{n,n+1}-\Psi_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix},$$

where $E_{n,n+1}=(E_n, 0)$ (E_n is the $n\times n$ unit matrix and 0 is the n-dimensional zero vector), and $\Psi_1(t)$ is the $n\times(n+1)$ matrix satisfying $\Psi(t)=\begin{pmatrix}\Psi_1(t)\\ 0\cdots 01\end{pmatrix}$. Here $\Psi(t)$ is the fundamental matrix of the linear homogeneous system

(2.7)
$$\frac{dk}{dt}=V_u(u(t), B)k$$

satisfying the initial condition $\Psi(0)=E_{n+1}$ ($(n+1)\times(n+1)$ unit matrix), where

(2.8)
$$V_u(u, B)=\begin{pmatrix} \frac{\omega}{2\pi}X_x(x, B) & \frac{1}{2\pi}X(x, B) \\ 00\cdots\cdots 0 & 0 \end{pmatrix}$$

and $X_x(x, B)$ above denotes the Jacobian matrix of $X(x, B)$ with respect to x. Further, $\Psi_1(t)$ can be written in the following form:

(2.9)
$$\Psi_1(t)=\left(\Phi_1(t) \quad \frac{t}{2\pi}\Phi_1(t)c\right),$$

where $\Phi_1(t)$ is the fundamental matrix of the linear homogeneous system

(2.10)
$$\frac{dh}{dt}=\frac{\omega}{2\pi}X_x(x(t), B)h$$

satisfying the initial condition $\Phi_1(0)=E_n$, and $c=X(x(0), B)$.

Now we consider the case where the first variation equation (2.10) has a double characteristic multiplier one. That is, we consider the case where the matrix $\Phi_1(2\pi)$ has a double eigenvalue one.

Assume that there exists a point $(\hat{u}(0), \hat{B})$ ($(\hat{x}(0), \hat{B})\in\Delta$) satisfying the equation (2.5) and also satisfying

$$\text{(2.11)} \qquad \operatorname{rank}\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}=n,$$

where $\hat{u}(0)=(\hat{x}(0),\ \hat{\omega})^T$, $\hat{\Psi}_1(t)=(\hat{\Phi}_1(t)\ \frac{t}{2\pi}\hat{\Phi}_1(t)\hat{c})$ $(\hat{c}=X(\hat{x}(0),\ \hat{B}))$. Here $\hat{\Phi}_1(t)$ is the fundamental matrix of (2.10) at $u=\hat{u}(t)=(\hat{x}(t),\ \hat{\omega})^T$ and $B=\hat{B}$ satisfying the initial condition $\hat{\Phi}_1(0)=E_n$, where $\hat{u}(t)=(\hat{x}(t),\ \hat{\omega})^T$ is a 2π-periodic solution of (2.3)–(2.4) at $B=\hat{B}$ through $(\hat{x}(0),\ \hat{\omega})^T$ at $t=0$. The condition (2.11) can be rewritten in the form

$$\text{(2.12)} \qquad \operatorname{rank} F_u(\hat{u}(0),\ \hat{B})=n.$$

This point $(\hat{u}(0),\ \hat{B})$ is called a "*singular point*" of the nonlinear equation (2.5). We discuss the singular point $(\hat{u}(0),\ \hat{B})$ and we propose a method for computing it.

For the sake of simplicity, without loss of generality, we assume that

$$\text{(2.13)} \qquad n=\operatorname{rank}\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) \\ 10\cdots 0 \end{pmatrix},$$

where $\hat{D}_1(2\pi)$ is the $n\times n$ matrix obtained from $E_{n,n+1}-\hat{\Psi}_1(2\pi)$ by deleting the first column vector. As for the permissibility of the assumption (2.13), see Section 5. Then, by the condition (2.13), the equation

$$\text{(2.14)} \qquad \begin{cases} \begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix}k=0, \\ k_1-1=0 \end{cases} \qquad (\text{where } k=(k_1,\dots,\ k_n,\ k_{n+1})^T)$$

has a solution $\hat{k}=(\hat{k}_1,\dots,\ \hat{k}_n,\ \hat{k}_{n+1})^T$ and this solution $\hat{k}$ is really the initial value of a 2π-periodic solution of (2.7) at $u=\hat{u}(t)$ and $B=\hat{B}$ satisfying $k_1(0)=k_1=1$. That is, $\hat{\Psi}(t)\hat{k}$ is a 2π-periodic solution of (2.7) at $u=\hat{u}(t)$ and $B=\hat{B}$. Hence, when $\hat{\Phi}_1(2\pi)$ has a double eigenvalue one and the condition (2.13) is satisfied, in order to obtain the singular point $(\hat{u}(0),\ \hat{B})$, we have only to find a 2π-periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$ of the system

$$\text{(2.15)} \qquad \begin{cases} \dfrac{du}{dt}=V(u,\ B), \\ \dfrac{dk}{dt}=V_u(u,\ B)k \end{cases}$$

satisfying the conditions

$$\text{(2.16)} \qquad \begin{cases} x_2-\beta=0, \\ k_1-1=0, \\ k_2=0, \end{cases}$$

where $u(0)=(x(0),\ \omega)^T$, $x(0)=(x_1(0),\dots,\ x_n(0))^T=(x_1,\dots,\ x_n)^T$ and $k(0)=(k_1(0),\dots,\ k_n(0),\ k_{n+1})^T=(k_1,\dots,\ k_n,\ k_{n+1})^T$.

As is seen from the above argument, the problem (2.15)–(2.16) certainly has a 2π-periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$. Indeed, $\hat{k}(t)=\hat{\Psi}(t)\hat{k}$.

Now we discuss the isolatedness of this periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$.

Let $\left(\begin{pmatrix}\varphi(t,\ \boldsymbol{x})\\ \omega\end{pmatrix},\ \begin{pmatrix}\varphi_1(t,\ \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T$ be a solution of (2.15) such that $\left(\begin{pmatrix}\varphi(0,\ \boldsymbol{x})\\ \omega\end{pmatrix},\ \begin{pmatrix}\varphi_1(0,\ \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T=\left(\begin{pmatrix}x(0)\\ \omega\end{pmatrix},\ \begin{pmatrix}\tilde{k}(0)\\ k_{n+1}\end{pmatrix}\right)^T$, where $\boldsymbol{x}=(u(0),\ k(0),\ B)^T$, $u(0)=(x(0),\ \omega)^T$, $k(0)=(\tilde{k}(0),\ k_{n+1})^T$ and $\tilde{k}(0)=(k_1,\dots,\ k_n)^T$. Then we consider the equation

$$F(\boldsymbol{x})=\begin{pmatrix}\begin{pmatrix}\varphi(0,\ \boldsymbol{x})-\varphi(2\pi,\ \boldsymbol{x})\\ x_2-\beta\end{pmatrix}\\ \begin{pmatrix}\varphi_1(0,\ \boldsymbol{x})-\varphi_1(2\pi,\ \boldsymbol{x})\\ k_2\end{pmatrix}\\ k_1-1\end{pmatrix}=0. \tag{2.17}$$

Of course, the initial value $\hat{\boldsymbol{x}}=(\hat{u}(0),\ \hat{k}(0),\ \hat{B})^T$ of the 2π-periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$ of (2.15)–(2.16) is a solution of (2.17). For this solution $\hat{\boldsymbol{x}}$, we have

Theorem 1.

Assume that $X(x,\ B)$ is twice continuously differentiable with respect to $(x,\ B)$ in the region Δ.

If the conditions

$$\begin{cases} n=\operatorname{rank}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi)\\ 010\cdots\cdots\cdots 0\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots 0\end{pmatrix}\\ <n+1=\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi) & \hat{\xi}_1(2\pi)\\ 10\cdots 0 & 0\end{pmatrix}\end{cases} \tag{2.18}$$

are satisfied, then the matrix $F'(\hat{\boldsymbol{x}})$ is non-singular if and only if

$$\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi) & \hat{l}(2\pi)\\ 10\cdots 0 & 0\end{pmatrix}=n+1, \tag{2.19}$$

where

$F'(\boldsymbol{x})$ denotes the Jacobian matrix of $F(\boldsymbol{x})$ with respect to $\boldsymbol{x}$; $\hat{\xi}_1(2\pi)=\hat{\Phi}_1(2\pi)\int_0^{2\pi}\hat{\Phi}_1^{-1}(s)\frac{\hat{\omega}}{2\pi}X_B(\hat{x}(s),\ \hat{B})\ ds$, where $X_B(x,\ B)$ denotes the partial derivative of $X(x,\ B)$ with respect to B; $\hat{l}(2\pi)=-\hat{\Psi}_2(2\pi)\hat{k}(0)$, where $\hat{\Psi}_1(t)$ and $\hat{\Psi}_2(t)$ are the

$n\times(n+1)$ matrices such that $\hat{\Psi}(t)=\begin{pmatrix}\hat{\Psi}_1(t)\\ 0\cdots 01\end{pmatrix}$ *and* $\hat{\hat{\Psi}}_2(t)=\begin{pmatrix}\hat{\Psi}_2(t)\\ 0\cdots 00\end{pmatrix}$, *respectively. Here* $(\hat{\Psi}(t), \hat{\hat{\Psi}}_2(t))^T$ *is a solution* $(2(n+1)\times(n+1)$ *matrix) of the system*

$$\text{(2.20)}\qquad \begin{cases} \dfrac{dz_1}{dt}=V_u(\hat{u}(t), \hat{B})z_1, \\[2ex] \dfrac{dz_2}{dt}=V_u(\hat{u}(t), \hat{B})z_2+\{V_{uu}(\hat{u}(t), \hat{B})\hat{k}(t)\}z_1 \end{cases}$$

satisfying the initial condition $(\hat{\Psi}(0), \hat{\hat{\Psi}}_2(0))^T=(E_{n+1}, 0)^T$ *(0 denotes the* $(n+1)\times(n+1)$ *zero matrix), where* $V_{uu}(u, B)$ *denotes the second derivative of* $V(u, B)$ *with respect to* u.

PROOF. By the assumptions of the theorem, the function $F(\boldsymbol{x})$ defined by the equality (2.17) is continuously differentiable with respect to $\boldsymbol{x}$. Then, for the solution $\hat{\boldsymbol{x}}$, we have

$$\text{(2.21)}\quad F'(\hat{\boldsymbol{x}})=\begin{pmatrix} \begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix} -\hat{\xi}_1(2\pi) \\ 0 \end{pmatrix} \\[2ex] \begin{pmatrix} -\hat{\Psi}_2(2\pi) \\ 000\cdots\cdots\cdots 0 \end{pmatrix} & \begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix} & \begin{pmatrix} -\hat{\xi}_2(2\pi) \\ 0 \end{pmatrix} \\[2ex] 000\cdots\cdots\cdots 0 & 100\cdots\cdots\cdots 0 & 0 \end{pmatrix},$$

where $(\hat{\hat{\xi}}_1(t), \hat{\hat{\xi}}_2(t))^T$ $(\hat{\hat{\xi}}_i(t)\equiv(\hat{\xi}_i(t), 0)^T$ $(i=1, 2))$ is a solution of the system

$$\text{(2.22)}\qquad \begin{cases} \dfrac{d\zeta_1}{dt}=V_u(\hat{u}(t), \hat{B})\zeta_1+V_B(\hat{u}(t), \hat{B}), \\[2ex] \dfrac{d\zeta_2}{dt}=V_u(\hat{u}(t), \hat{B})\zeta_2+\{V_{uu}(\hat{u}(t), \hat{B})\zeta_1+V_{uB}(\hat{u}(t), \hat{B})\}\hat{k}(t) \end{cases}$$

satisfying the initial condition $(\hat{\hat{\xi}}_1(0), \hat{\hat{\xi}}_2(0))^T=(0, 0)^T$. Here $V_B(u, B)$ and $V_{uB}(u, B)$ denote the partial derivatives of $V(u, B)$ and $V_u(u, B)$ with respect to B, respectively. In fact, $V_B(u, B)$ is of the form

$$V_B(u, B)=\left(\frac{\omega}{2\pi}X_B(x, B), 0\right)^T.$$

From (2.18) and (2.21) it follows that

$$\text{(2.23)}\qquad \det F'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to (2.19)}.$$

This completes the proof. Q. E. D.

The 2π-periodic solution $(\hat{u}(t), \hat{k}(t), \hat{B})^T$ of (2.15)–(2.16) satisfying $\det F'(\hat{\boldsymbol{x}})\neq 0$ is called to be "*isolated*".

When $\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi) & \hat{l}(2\pi)\\ 10\cdots0 & 0\end{pmatrix}=n$, the equation

$$\begin{cases}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi)\\ 010\cdots\cdots\cdots\cdots0\end{pmatrix}p+\begin{pmatrix}\hat{l}(2\pi)\\ 0\end{pmatrix}=0,\\ p_1=0\end{cases}\tag{2.24}$$

has a solution $\hat{p}$, where $p=(p_1,\dots, p_n, p_{n+1})^T$. This solution $\hat{p}$ becomes the initial value of a 2π-periodic solution of

$$\frac{dk_2}{dt}=V_u(\hat{u}(t), \hat{B})k_2+\{V_{uu}(\hat{u}(t), \hat{B})\hat{k}_1(t)\}\hat{k}_1(t),\tag{2.25}$$

where $\hat{k}_1(t)=\hat{k}(t)$. That is, $\hat{k}_2(t)\equiv\hat{\Psi}(t)\hat{p}+\hat{\hat{\Psi}}_2(t)\hat{k}(0)$ is a 2π-periodic solution of (2.25). Hence we have only to find a 2π-periodic solution $(\hat{u}(t), \hat{k}_1(t), \hat{k}_2(t), \hat{B})^T$ of the system

$$\begin{cases}\dfrac{du}{dt}=V(u, B),\\[2mm] \dfrac{dk_1}{dt}=V_u(u, B)k_1,\\[2mm] \dfrac{dk_2}{dt}=V_u(u, B)k_2+\{V_{uu}(u, B)k_1\}k_1\end{cases}\tag{2.26}$$

satisfying the conditions

$$\begin{cases}x_2-\beta=0,\\ k_1^1-1=0,\\ k_1^2=0,\\ k_2^1=0,\\ k_2^2=0,\end{cases}\tag{2.27}$$

where $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, $k_i(0)=(k_i^1,\dots, k_i^n, k_i^{n+1})^T$ $(i=1, 2)$ and the dimension of the parameter B is two. By the dimension of the parameter B being two, is meant the parameter B such that the values of $(m-2)$ components of B are given and its remaining two components either of which is the preceding unknown component are unknown. For the sake of simplicity, we write such two unknown components of B as B_1, B_2 and we denote the parameter B by $B=(B_1, B_2)^T$ and this is called the parameter B with dimension two. Since the equation (2.25) has a 2π-periodic solution, the problem (2.26)–(2.27) certainly has a 2π-periodic solution $(\hat{u}(t), \hat{k}_1(t), \hat{k}_2(t), \hat{B})^T$.

Now let us discuss the isolatedness of the solution $(\hat{u}(t), \hat{k}_1(t), \hat{k}_2(t), \hat{B})^T$.

Let $\left(\begin{pmatrix}\varphi(t, \boldsymbol{x}_2)\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(t, \boldsymbol{x}_2)\\ k_1^{n+1}\end{pmatrix}, \begin{pmatrix}\varphi_2(t, \boldsymbol{x}_2)\\ k_2^{n+1}\end{pmatrix}\right)^T$ be a solution of (2.26) such that $\left(\begin{pmatrix}\varphi(0, \boldsymbol{x}_2)\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(0, \boldsymbol{x}_2)\\ k_1^{n+1}\end{pmatrix}, \begin{pmatrix}\varphi_2(0, \boldsymbol{x}_2)\\ k_2^{n+1}\end{pmatrix}\right)^T = \left(\begin{pmatrix}x(0)\\ \omega\end{pmatrix}, \begin{pmatrix}\tilde{k}_1(0)\\ k_1^{n+1}\end{pmatrix}, \begin{pmatrix}\tilde{k}_2(0)\\ k_2^{n+1}\end{pmatrix}\right)^T$, where $\boldsymbol{x}_2=(u(0), k_1(0), k_2(0), B)^T$, $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, $k_i(0)=(\tilde{k}_i(0), k_i^{n+1})^T$, $\tilde{k}_i(0)=(k_i^1,\dots, k_i^n)^T$ $(i=1, 2)$ and $B=(B_1, B_2)^T$. Then we consider the system

$$F_2(\boldsymbol{x}_2)=\begin{pmatrix}\begin{pmatrix}\varphi(0, \boldsymbol{x}_2)-\varphi(2\pi, \boldsymbol{x}_2)\\ x_2-\beta\end{pmatrix}\\ \begin{pmatrix}\varphi_1(0, \boldsymbol{x}_2)-\varphi_1(2\pi, \boldsymbol{x}_2)\\ k_1^2\end{pmatrix}\\ \begin{pmatrix}\varphi_2(0, \boldsymbol{x}_2)-\varphi_2(2\pi, \boldsymbol{x}_2)\\ k_2^2\end{pmatrix}\\ k_1^1-1\\ k_2^1\end{pmatrix}=0. \tag{2.28}$$

Of course, the initial value $\hat{\boldsymbol{x}}_2=(\hat{u}(0), \hat{k}_1(0), \hat{k}_2(0), \hat{B})^T$ of the solution $(\hat{u}(t), \hat{k}_1(t), \hat{k}_2(t), \hat{B})^T$ of (2.26)–(2.27) is a solution of the system (2.28). For the solution $\hat{\boldsymbol{x}}_2$, we easily get the following theorem.

Theorem 2.

Assume that $X(x, B)$ is three times continuously differentiable with respect to (x, B) in the region Δ.

If the conditions

$$\begin{cases} n=\operatorname{rank}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi)\\ 010\cdots\cdots\cdots\cdots 0\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots 0\end{pmatrix}\\ <n+1=\operatorname{rank}\begin{pmatrix}\hat{D}_1(2\pi) & \hat{\xi}_{11}(2\pi)\\ 10\cdots 0 & 0\end{pmatrix}\end{cases} \tag{2.29}$$

and

$$\operatorname{rank}\begin{pmatrix}\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots 0\end{pmatrix} & \mathbf{0} & \begin{pmatrix}-\hat{\xi}_{11}(2\pi)\\ 0\end{pmatrix} & \begin{pmatrix}-\hat{\xi}_{12}(2\pi)\\ 0\end{pmatrix}\\ \begin{pmatrix}\hat{D}_2(2\pi)\\ 00\cdots 0\end{pmatrix} & \begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots 0\end{pmatrix} & \begin{pmatrix}-\hat{\xi}_{21}(2\pi)\\ 0\end{pmatrix} & \begin{pmatrix}-\hat{\xi}_{22}(2\pi)\\ 0\end{pmatrix}\end{pmatrix}=2(n+1) \tag{2.30}$$

are satisfied, then the matrix $F_2'(\hat{\boldsymbol{x}}_2)$ is non-singular if and only if

(2.31)
$$\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_2(2\pi) \\ 10\cdots0 & 0 \end{pmatrix} = n+1,$$

where

$F_2'(\boldsymbol{x}_2)$ *denotes the Jacobian matrix of* $F_2(\boldsymbol{x}_2)$ *with respect to* $\boldsymbol{x}_2$;

$(\hat{\Psi}(t), \hat{\hat{\Psi}}_2(t), \hat{\hat{\Psi}}_3(t))^T$ *is a solution* $(3(n+1)\times(n+1)$ *matrix) of the system*

(2.32)
$$\begin{cases} \dfrac{dz_1}{dt} = V_u(\hat{u}(t), \hat{B})z_1, \\[2mm] \dfrac{dz_2}{dt} = V_u(\hat{u}(t), \hat{B})z_2 + \{V_{uu}(\hat{u}(t), \hat{B})\hat{k}_1(t)\}z_1, \\[2mm] \dfrac{dz_3}{dt} = V_u(\hat{u}(t), \hat{B})z_3 + 2\{V_{uu}(\hat{u}(t), \hat{B})\hat{k}_1(t)\}z_2 \\ \qquad\qquad + [\{V_{uuu}(\hat{u}(t), \hat{B})\hat{k}_1(t)\}\hat{k}_1(t) + V_{uu}(\hat{u}(t), \hat{B})\hat{k}_2(t)]z_1 \end{cases}$$

satisfying the initial condition $(\hat{\Psi}(0), \hat{\hat{\Psi}}_2(0), \hat{\hat{\Psi}}_3(0))^T = (E_{n+1}, 0, 0)^T$, *where* $V_{uuu}(u, B)$ *denotes the third derivative of* $V(u, B)$ *with respect to* u;

$\hat{D}_1(2\pi)$ and $\hat{D}_2(2\pi)$ *are the* $n\times n$ *matrices obtained from* $E_{n,n+1} - \hat{\Psi}_1(2\pi)$ *and* $-\hat{\Psi}_2(2\pi)$ *by deleting the first column vectors, respectively, where* $\hat{\Psi}_1(2\pi)$ *and* $\hat{\Psi}_2(2\pi)$ *are the* $n\times(n+1)$ *matrices such that* $\hat{\Psi}(2\pi) = \begin{pmatrix}\hat{\Psi}_1(2\pi)\\ 00\cdots01\end{pmatrix}$ *and* $\hat{\hat{\Psi}}_2(2\pi) = \begin{pmatrix}\hat{\Psi}_2(2\pi)\\ 00\cdots00\end{pmatrix}$, *respectively*;

$(\hat{\xi}_{1i}(t), \hat{\xi}_{2i}(t))^T$ $(\hat{\xi}_{ji}(t) \equiv (\xi_{ji}(t), 0)^T$ $(j=1, 2))$ $(i=1, 2)$ *are solutions of*

(2.33)
$$\begin{cases} \dfrac{d\zeta_1}{dt} = V_u(\hat{u}(t), \hat{B})\zeta_1 + V_{B_i}(\hat{u}(t), \hat{B}), \\[2mm] \dfrac{d\zeta_2}{dt} = V_u(\hat{u}(t), \hat{B})\zeta_2 + \{V_{uu}(\hat{u}(t), \hat{B})\zeta_1 + V_{uB_i}(\hat{u}(t), \hat{B})\}\hat{k}_1(t) \end{cases}$$

satisfying the initial condition $(\hat{\xi}_{1i}(0), \hat{\xi}_{2i}(0))^T = (0, 0)^T$ $(i=1, 2)$, *respectively, where* $V_{B_i}(u, B)$ *and* $V_{uB_i}(u, B)$ $(i=1, 2)$ *are the partial derivatives of* $V(u, B)$ *and* $V_u(u, B)$ *with respect to* B_i $(i=1, 2)$, *respectively*;

$\hat{l}_3(2\pi) = -\hat{\Psi}_3(2\pi)\hat{k}_1(0) - 2\hat{\Psi}_2(2\pi)\hat{k}_2(0)$, *where* $\hat{\Psi}_3(2\pi)$ *is the* $n\times(n+1)$ *matrix such that* $\hat{\hat{\Psi}}_3(2\pi) = \begin{pmatrix}\hat{\Psi}_3(2\pi)\\ 00\cdots00\end{pmatrix}$.

More generally, we suppose that d components of the parameter B are unknown $(3\le d\le m)$ and $X(x, B)$ is $(d+1)$ times continuously differentiable with respect to (x, B) in the region Δ. For the sake of simplicity, we write such unknown components of the parameter B as $B_1, B_2, \ldots, B_d$ and we denote the parameter B by $B = (B_1, B_2, \ldots, B_d)^T$ and this is called the parameter B with dimension d.

Put

(2.34) $$V^{(j)}=\sum_{i=0}^{j-1}{}_{j-1}C_i V_u^{(j-1-i)}k_{i+1} \qquad (1\leq j\leq d),$$

where $V^{(0)}=V_u(u, B)$, and $V_u^{(q)}$ $(q=0, 1,\dots, d-1)$ denote the derivatives of $V^{(q)}$ $(q=0, 1,\dots, d-1)$ with respect to u, respectively, and k_r $(r=1,\dots, d)$ are $(n+1)$-dimensional vectors.

We assume that there exists a 2π-periodic solution $(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$ of the system

(2.35) $$\begin{cases} \dfrac{du}{dt}=V(u, B), \\ \dfrac{dk_1}{dt}=V^{(0)}k_1, \\ \dfrac{dk_2}{dt}=V^{(0)}k_2+V^{(1)}k_1, \\ \vdots \\ \dfrac{dk_d}{dt}=\displaystyle\sum_{i=0}^{d-1}{}_{d-1}C_i V^{(i)}k_{d-i} \end{cases}$$

satisfying the conditions

(2.36) $$\begin{cases} x_2-\beta=0, \\ k_1^1-1=0, \\ k_1^2=0, \\ k_2^1=0, \\ k_2^2=0, \\ \vdots \\ k_d^1=0, \\ k_d^2=0, \end{cases}$$

where $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$ and $k_i(0)=(k_i^1,\dots, k_i^n, k_i^{n+1})^T$ $(i=1, 2,\dots, d)$. Moreover we assume that for the solution $(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$, the conditions

(2.37) $$\begin{cases} n=\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) \\ 10\cdots 0 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_1(2\pi) \\ 10\cdots 0 & 0 \end{pmatrix} \\ =\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_2(2\pi) \\ 10\cdots 0 & 0 \end{pmatrix}=\cdots\cdots=\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_{d-1}(2\pi) \\ 10\cdots 0 & 0 \end{pmatrix} \\ <n+1=\operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_d(2\pi) \\ 10\cdots 0 & 0 \end{pmatrix} \end{cases}$$

$$
(2.38)\qquad \operatorname{rank}\left(\begin{array}{ccccccc}
{}_0C_0\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots0\end{pmatrix} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix}-\hat{\xi}_{11}(2\pi)\\ 0\end{pmatrix}\cdots & \begin{pmatrix}-\hat{\xi}_{1d}(2\pi)\\ 0\end{pmatrix}\\
{}_1C_1\begin{pmatrix}\hat{D}_2(2\pi)\\ 00\cdots0\end{pmatrix} & {}_1C_0\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots0\end{pmatrix} & \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix}-\hat{\xi}_{21}(2\pi)\\ 0\end{pmatrix}\cdots & \begin{pmatrix}-\hat{\xi}_{2d}(2\pi)\\ 0\end{pmatrix}\\
{}_2C_2\begin{pmatrix}\hat{D}_3(2\pi)\\ 00\cdots0\end{pmatrix} & {}_2C_1\begin{pmatrix}\hat{D}_2(2\pi)\\ 00\cdots0\end{pmatrix} & {}_2C_0\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots0\end{pmatrix} & \cdots & \mathbf{0} & \begin{pmatrix}-\hat{\xi}_{31}(2\pi)\\ 0\end{pmatrix}\cdots & \begin{pmatrix}-\hat{\xi}_{3d}(2\pi)\\ 0\end{pmatrix}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
{}_{d-1}C_{d-1}\begin{pmatrix}\hat{D}_d(2\pi)\\ 00\cdots0\end{pmatrix} & {}_{d-1}C_{d-2}\begin{pmatrix}\hat{D}_{d-1}(2\pi)\\ 00\cdots\cdots0\end{pmatrix} & {}_{d-1}C_{d-3}\begin{pmatrix}\hat{D}_{d-2}(2\pi)\\ 00\cdots\cdots0\end{pmatrix} & \cdots & {}_{d-1}C_0\begin{pmatrix}\hat{D}_1(2\pi)\\ 10\cdots0\end{pmatrix} & \begin{pmatrix}-\hat{\xi}_{d1}(2\pi)\\ 0\end{pmatrix}\cdots & \begin{pmatrix}-\hat{\xi}_{dd}(2\pi)\\ 0\end{pmatrix}
\end{array}\right) = d(n+1)
$$

are satisfied, where

$(\hat{\Psi}(t), \hat{\hat{\Psi}}_2(t),\dots, \hat{\hat{\Psi}}_{d+1}(t))^T$ $\left(\hat{\Psi}(t)=\begin{pmatrix}\hat{\Psi}_1(t)\\ 0\cdots01\end{pmatrix}\right.$ and $\left.\hat{\hat{\Psi}}_j(t)=\begin{pmatrix}\hat{\Psi}_j(t)\\ 0\cdots00\end{pmatrix}\ (j=2,\dots, d+1)\right)$ is a solution ($(d+1)(n+1)\times(n+1)$ matrix) of the system

$$
(2.39)\qquad \begin{cases}
\dfrac{dz_1}{dt} = \hat{V}^{(0)}z_1\,,\\[2mm]
\dfrac{dz_2}{dt} = \hat{V}^{(0)}z_2+\hat{V}^{(1)}z_1\,,\\
\quad\vdots\\
\dfrac{dz_{d+1}}{dt} = \displaystyle\sum_{i=0}^{d} {}_dC_i\,\hat{V}^{(i)}z_{d+1-i}
\end{cases}
$$

satisfying the initial condition $(\hat{\Psi}(0), \hat{\hat{\Psi}}_2(0),\dots, \hat{\hat{\Psi}}_{d+1}(0))^T=(E_{n+1}, 0,\dots, 0)^T$ (0 is the $(n+1)\times(n+1)$ zero matrix), where $\hat{V}^{(r)}$ $(r=0, 1,\dots, d)$ denote the values of $V^{(r)}$ $(r=0, 1,\dots, d)$ at $(u, k_1,\dots, k_d, B)^T=(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$, respectively;

$$
(2.40)\qquad \hat{l}_j(2\pi)=-\sum_{i=1}^{j} {}_jC_i\hat{\Psi}_{i+1}(2\pi)\hat{k}_{j+1-i}(0)\qquad (1\leq j\leq d)\,;
$$

$\hat{D}_1(2\pi), \hat{D}_2(2\pi),\dots, \hat{D}_d(2\pi)$ are the $n\times n$ matrices obtained from $E_{n,n+1}-\hat{\Psi}_1(2\pi)$, $-\hat{\Psi}_2(2\pi),\dots, -\hat{\Psi}_d(2\pi)$ by deleting the first column vectors, respectively;

$(\tilde{\xi}_{1j}(t),\dots, \xi_{dj}(t))^T$ $(\tilde{\xi}_{qj}(t)\equiv(\hat{\xi}_{qj}(t), 0)^T$ $(q=1, 2,\dots, d))$ $(j=1, 2,\dots, d)$ are solutions of

$$(2.41)\qquad \begin{cases} \dfrac{d\zeta_1}{dt} = \hat{V}^{(0)}\zeta_1 + \hat{V}_{B_j}, \\ \dfrac{d\zeta_2}{dt} = \hat{V}^{(0)}\zeta_2 + \hat{V}^{(1)}\zeta_1 + \hat{V}^{(0)}_{B_j}\hat{k}_1(t), \\ \quad\vdots \\ \dfrac{d\zeta_d}{dt} = \sum_{i=0}^{d-1} {}_{d-1}C_i\, \hat{V}^{(i)}\zeta_{d-i} + \sum_{i=0}^{d-2} {}_{d-2}C_i\, \hat{V}^{(i)}_{B_j}\hat{k}_{d-1-i}(t) \end{cases}$$

satisfying the initial condition $(\tilde{\xi}_{1j}(0),\dots, \tilde{\xi}_{dj}(0))^T=(0,\dots, 0)^T$, respectively, where V_{B_j} and $V^{(q)}_{B_j}$ $(j=1, 2,\dots, d;\ q=0, 1,\dots, d-2)$ are the partial derivatives of $V(u, B)$ and $V^{(q)}$ with respect to B_j, respectively, and $\hat{V}_{B_j}$ and $\hat{V}^{(q)}_{B_j}$ $(j=1, 2,\dots, d;\ q=0, 1,\dots, d-2)$ denote the values of V_{B_j} and $V^{(q)}_{B_j}$ at $(u, k_1,\dots, k_d, B)^T=(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$, respectively.

In order to discuss the isolatedness of the solution $(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$, we consider the equation

$$(2.42)\qquad F_d(\boldsymbol{x}_d)=\begin{pmatrix} \begin{pmatrix} \varphi(0, \boldsymbol{x}_d)-\varphi(2\pi, \boldsymbol{x}_d) \\ x_2-\beta \end{pmatrix} \\ \begin{pmatrix} \varphi_1(0, \boldsymbol{x}_d)-\varphi_1(2\pi, \boldsymbol{x}_d) \\ k_1^2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \varphi_d(0, \boldsymbol{x}_d)-\varphi_d(2\pi, \boldsymbol{x}_d) \\ k_d^2 \end{pmatrix} \\ \psi_d(\boldsymbol{x}_d) \end{pmatrix}=0,$$

where $\boldsymbol{x}_d=(u(0), k_1(0),\dots, k_d(0), B)^T$, $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, $k_i(0)=(\tilde{k}_i(0), k_i^{n+1})^T$, $\tilde{k}_i(0)=(k_i^1, k_i^2,\dots, k_i^n)^T$ $(i=1, 2,\dots, d)$, $B=(B_1,\dots, B_d)^T$, $\psi_d(\boldsymbol{x}_d)=(k_1^1-1, k_2^1,\dots, k_d^1)^T$, and $\left(\begin{pmatrix}\varphi(t, \boldsymbol{x}_d)\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(t, \boldsymbol{x}_d)\\ k_1^{n+1}\end{pmatrix},\dots, \begin{pmatrix}\varphi_d(t, \boldsymbol{x}_d)\\ k_d^{n+1}\end{pmatrix}\right)^T$ is a solution of (2.35) such that $\left(\begin{pmatrix}\varphi(0, \boldsymbol{x}_d)\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(0, \boldsymbol{x}_d)\\ k_1^{n+1}\end{pmatrix},\dots, \begin{pmatrix}\varphi_d(0, \boldsymbol{x}_d)\\ k_d^{n+1}\end{pmatrix}\right)^T = \left(\begin{pmatrix}x(0)\\ \omega\end{pmatrix}, \begin{pmatrix}\tilde{k}_1(0)\\ k_1^{n+1}\end{pmatrix},\dots, \begin{pmatrix}\tilde{k}_d(0)\\ k_d^{n+1}\end{pmatrix}\right)^T$. Then the initial value $\hat{\boldsymbol{x}}_d=(\hat{u}(0), \hat{k}_1(0),\dots, \hat{k}_d(0), \hat{B})^T$ of the solution $(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$ is of course a solution of the system (2.42). For the solution $\hat{\boldsymbol{x}}_d$, we have the following theorem.

Theorem 3.

The matrix $F_d'(\hat{\boldsymbol{x}}_d)$ is non-singular, where $F_d'(\boldsymbol{x}_d)$ denotes the Jacobian matrix of

$F_d(\boldsymbol{x}_d)$ with respect to $\boldsymbol{x}_d$.

PROOF. The proof of the theorem is similar to the one of Theorem 3 of the paper [8]. For the proof, see the paper [8].

The 2π-periodic solution $(\hat{u}(t), \hat{k}_1(t),\dots, \hat{k}_d(t), \hat{B})^T$ of (2.35)–(2.36) satisfying $\det F'_d(\hat{\boldsymbol{x}}_d) \neq 0$ is called to be "*isolated*".

In order to get a highly accurate approximation to the isolated 2π-periodic solution of (2.35)–(2.36), we have applied the Urabe-Galerkin method to (2.35)–(2.36) and we have obtained a desired approximation with high accuracy. For the details of the practical numerical methods, see [6].

Remark 1.

As has been shown in [8], for the solution $\hat{\boldsymbol{x}}_d$ of (2.42), we have

$$\text{(2.43)} \qquad \det F'_d\,(\hat{\boldsymbol{x}}_d) \neq 0 \text{ is equivalent to } \operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{l}_d(2\pi) \\ 10\cdots 0 & 0 \end{pmatrix} = n+1.$$

Remark 2.

Recently, when the dimension of the parameter B is one, R. Seydel [4] has considered a system similar to (2.15). But he did not give any condition for guaranteeing the isolatedness of a solution of the system. Further, he did not describe anything about the case where the dimension of the parameter B is greater than one.

Remark 3.

Since $\det\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}=0$, we may consider the system

$$\text{(2.44)} \qquad G(y)=\begin{pmatrix} \begin{pmatrix} \varphi(0, y)-\varphi(2\pi, y) \\ x_2-\beta \end{pmatrix} \\ g(y) \end{pmatrix}=0$$

instead of the system (2.17), where $y=(u(0), B)^T$, $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, and $(\varphi(t, y), \omega)^T$ is a solution of (2.4) at a given B such that $(\varphi(0, y), \omega)^T=(x(0), \omega)^T$, and $g(y)=\det\begin{pmatrix} E_{n,n+1}-\Psi_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}$. Then the system (2.44) has a solution $\hat{y}=(\hat{u}(0), \hat{B})^T$. Under the same assumptions as in Theorem 1, for the solution $\hat{y}$, we have

$$\text{(2.45)} \qquad \det G'(\hat{y}) \neq 0 \text{ is equivalent to (2.19)},$$

where $G'(y)$ denotes the Jacobian matrix of $G(y)$ with respect to y.

§ 3. Bifurcations of Periodic Solutions

In this section, we consider bifurcations of periodic solutions of the autonomous system (2.1). Throughout this section, we assume that $X(x, B)$ is twice continuously differentiable with respect to (x, B) in the region Δ and the dimension of the parameter B is one.

Now we classify bifurcations into the following two cases.

Case (I). *Concerning the right-hand member $X(x, B)$ of the system (2.1), we assume that for any t*

$$(3.1)\qquad \begin{cases} X(x_0(t+\pi), B) = -X(x_0(t), B), \quad X_B(x_0(t+\pi), B) = -X_B(x_0(t), B) \\ \text{and} \\ X_x(x_0(t+\pi), B) = X_x(x_0(t), B) \end{cases}$$

for an arbitrary 2π-periodic function $x_0(t)$ which satisfies both $x_0(t+\pi) = -x_0(t)$ for any t and $(x_0(t), B) \in \Delta$ for any t, where $X_x(x, B)$ denotes the Jacobian matrix of $X(x, B)$ with respect to x and $X_B(x, B)$ denotes the partial derivative of $X(x, B)$ with respect to B.

Let $B = \hat{B}$ be a bifurcation point and $u = \hat{u}(t) = (\hat{x}(t), \hat{\omega})^T$ be a 2π-periodic solution of (2.3)–(2.4) at $B = \hat{B}$ satisfying $\hat{x}(t+\pi) = -\hat{x}(t)$ for any t. When $\hat{\Psi}(t)$ is the fundamental matrix of (2.7) at $u = \hat{u}(t)$ and $B = \hat{B}$ satisfying the initial condition $\hat{\Psi}(0) = E_{n+1}$, $\hat{\Psi}(t)$ can be written in the form

$$(3.2)\qquad \hat{\Psi}(t) = \begin{pmatrix} \hat{\Psi}_1(t) \\ 0 \cdots 01 \end{pmatrix} = \begin{pmatrix} \hat{\Phi}_1(t) & \frac{t}{2\pi}\hat{\Phi}_1(t)\hat{c} \\ 0 \cdots 0 & 1 \end{pmatrix},$$

where $\hat{\Phi}_1(t)$ is the fundamental matrix of (2.10) at $x = \hat{x}(t)$, $\omega = \hat{\omega}$ and $B = \hat{B}$ satisfying the initial condition $\hat{\Phi}_1(0) = E_n$, and $\hat{c} = X(\hat{x}(0), \hat{B})$. Then we moreover assume that

$$(3.3)\qquad \begin{cases} n-1 = \operatorname{rank}[E_n - \hat{\Phi}_1(\pi)] = \operatorname{rank}[E_n + \hat{\Phi}_1(\pi)], \\ n-2 = \operatorname{rank}[E_n - \hat{\Phi}_1(2\pi)] < n-1 = \operatorname{rank}[E_n - \hat{\Phi}_1(2\pi), \hat{c}] \\ \qquad\qquad = \operatorname{rank}[E_{n,n+1} - \hat{\Psi}_1(2\pi)], \\ n = \operatorname{rank}\begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(2\pi) \\ 010 \cdots\cdots 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(2\pi) & \hat{\xi}_1(2\pi) \\ 010 \cdots\cdots 0 & 0 \end{pmatrix}, \\ n+1 = \operatorname{rank}\begin{pmatrix} E_{n,n+1} + \hat{\Psi}_1(\pi) \\ 010 \cdots\cdots 0 \end{pmatrix}, \end{cases}$$

where $\hat{\xi}_1(t)$ *is defined by*

$$\hat{\xi}_1(t)=\hat{\Phi}_1(t)\int_0^t \hat{\Phi}_1^{-1}(s)\frac{\hat{\omega}}{2\pi}X_B(\hat{x}(s),\hat{B})ds. \tag{3.4}$$

Case (II). *Let* $u=\hat{u}(t)=(\hat{x}(t),\hat{\omega})^T$ *be a* π*-periodic solution of* (2.3)–(2.4) *at* $B=\hat{B}$ *and*

$$\hat{\Psi}(t)=\begin{pmatrix}\hat{\Psi}_1(t)\\ 0\cdots 01\end{pmatrix}=\begin{pmatrix}\hat{\Phi}_1(t) & \frac{t}{2\pi}\hat{\Phi}_1(t)\hat{c}\\ 0\cdots 0 & 1\end{pmatrix} \tag{3.5}$$

be the fundamental matrix of (2.7) *at* $u=\hat{u}(t)$ *and* $B=\hat{B}$ *satisfying the initial condition* $\hat{\Psi}(0)=E_{n+1}$. *We assume that*

$$\begin{cases} n-1=\operatorname{rank}[E_n-\hat{\Phi}_1(\pi)]=\operatorname{rank}[E_n+\hat{\Phi}_1(\pi)],\\ n-2=\operatorname{rank}[E_n-\hat{\Phi}_1(2\pi)]<n-1=\operatorname{rank}[E_n-\hat{\Phi}_1(2\pi),\hat{c}]\\ \qquad\qquad\qquad\qquad\qquad\qquad =\operatorname{rank}[E_{n,n+1}-\hat{\Psi}_1(2\pi)],\\ n=\operatorname{rank}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi)\\ 010\cdots\cdots\cdots 0\end{pmatrix}=\operatorname{rank}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi) & \hat{\xi}_1(2\pi)\\ 010\cdots\cdots\cdots 0 & 0\end{pmatrix},\\ n+1=\operatorname{rank}\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(\pi)\\ 010\cdots\cdots\cdots 0\end{pmatrix}.\end{cases} \tag{3.6}$$

First, we consider Case (I).

It follows from the assumption (3.1) that for $\hat{\Phi}_1(t)$, we have

$$\hat{\Phi}_1(t+\pi)=\hat{\Phi}_1(t)\hat{\Phi}_1(\pi) \tag{3.7}$$

for any t. By (3.7) we see that

$$\hat{\Phi}_1(2\pi)=\hat{\Phi}_1(\pi)^2. \tag{3.8}$$

From the definition of $\hat{\xi}_1(t)$, by (3.1) and (3.8), it is clear that

$$\hat{\xi}_1(2\pi)=-[E_n-\hat{\Phi}_1(\pi)]\hat{\xi}_1(\pi). \tag{3.9}$$

Now we consider the equation

$$\begin{pmatrix}E_{n,n+1}-\hat{\Psi}_1(2\pi)\\ 010\cdots\cdots\cdots 0\end{pmatrix}\tilde{z}=A\tilde{\xi}_1(2\pi) \tag{3.10}$$

for any constant number A, where $\tilde{z}=(z,z_{n+1})^T$, $z=(z_1,\dots,z_n)^T$ and $\tilde{\xi}_1(2\pi)=(\hat{\xi}_1(2\pi),0)^T$.

(i) When $A=0$, the equation (3.10) becomes

$$\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}\tilde{z}=0. \tag{3.11}$$

By the use of (3.2), the equation (3.11) can be rewritten in the following form:

$$\begin{cases} [E_n-\hat{\Phi}_1(2\pi)]z-z_{n+1}\hat{c}=0, & (3.12) \\ z_2=0. & (3.13) \end{cases}$$

Under the assumption (3.3), the necessary and sufficient condition for the existence of a nontrivial solution of the equation (3.12) is

$$z_{n+1}=0. \tag{3.14}$$

Therefore it is sufficient to consider the equation

$$\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix}z=0. \tag{3.15}$$

On the other hand, by the assumption (3.3), the equations

$$\begin{cases} [E_n-\hat{\Phi}_1(\pi)]y=0, & (3.16) \\ [E_n+\hat{\Phi}_1(\pi)]v=0 & (3.17) \end{cases}$$

have nontrivial solutions y and v respectively, where $y=(y_1,\dots, y_n)^T$ and $v=(v_1,\dots, v_n)^T$. These solutions y and v are linearly independant and y and v are also solutions of the equation

$$[E_n-\hat{\Phi}_1(2\pi)]w=0. \tag{3.18}$$

Since rank $[E_n-\hat{\Phi}_1(2\pi)]=n-2$ due to the assumption (3.3), an arbitrary solution of (3.18) is of the form

$$a_1y+a_2v, \tag{3.19}$$

where a_1 and a_2 are arbitrary constants. Since a solution z of (3.15) of course satisfies the equation (3.18), we can write z in the form of (3.19).

However $\hat{c}=X(\hat{x}(0), \hat{B})$ is a solution of the equation (3.17), because $X(\hat{x}(t), \hat{B})$ is certainly a 2π-periodic solution of (2.10) at $x=\hat{x}(t)$, $\omega=\hat{\omega}$ and $B=\hat{B}$ satisfying $X(\hat{x}(t+\pi), \hat{B})=-X(\hat{x}(t), \hat{B})$ for any t. Hence we can take $\hat{c}$ in palce of v in (3.19). Then a solution z of (3.15) can be written in the following form:

$$z=a_1y+a_2\hat{c}. \tag{3.20}$$

The assumption (3.3) tells us that no solution of (3.15) is uniquely determined unless we add some additional condition. So we must give one more condition.

Since y and $\hat{c}$ are linearly independent, there exists a positive integer i ($1 \leq i \leq n$, $i \neq 2$) such that

$$\text{rank}\begin{pmatrix} y_2 & \hat{c}_2 \\ y_i & \hat{c}_i \end{pmatrix} = 2, \tag{3.21}$$

where $\hat{c} = (\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n)^T$.

Now, in order to simplify the following argument, we assume that we can take $i=1$ in (3.21). In other words, we assume that

$$\text{rank}\begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(2\pi) \\ 010 \cdots\cdots\cdots\cdots 0 \end{pmatrix} = \text{rank}\begin{pmatrix} \hat{D}_1(2\pi) \\ 10 \cdots 0 \end{pmatrix} = n, \tag{3.22}$$

where $\hat{D}_1(2\pi)$ is the $n \times n$ matrix obtained from $E_{n,n+1} - \hat{\Psi}_1(2\pi)$ by deleting the first column vector. As for the permissibility of the assumption (3.22), see Section 5. Then we give the condition

$$z_1 = a_1 y_1 + a_2 \hat{c}_1 = 1 \tag{3.23}$$

as an additional condition. Then a solution of (3.15) satisfying the condition (3.23) is uniquely determined and we denote such a solution by $\hat{z}$.

When we put $\hat{\hat{z}} = (\hat{z}, 0)^T$, $\hat{\hat{z}}$ is of course a solution of (3.11). Therefore $\hat{H}(t) \equiv \hat{\Psi}(t)\hat{\hat{z}}$ is a 2π-periodic solution of the system

$$\frac{dk}{dt} = V_u(\hat{u}(t), \hat{B})k \tag{3.24}$$

satisfying the conditions

$$\begin{cases} k_1 = 1, \\ k_2 = 0, \end{cases} \tag{3.25}$$

where $k(0) = (k_1(0), k_2(0), \ldots, k_n(0), k_{n+1})^T = (k_1, k_2, \ldots, k_n, k_{n+1})^T$. Hence, in order to obtain the bifurcation point, we have only to find a 2π-periodic solution $(\hat{u}(t), \hat{k}(t), \hat{B})^T$ of the system

$$\begin{cases} \dfrac{du}{dt} = V(u, B), \\ \dfrac{dk}{dt} = V_u(u, B)k \end{cases} \tag{3.26}$$

satisfying the conditions

$$(3.27)\qquad \begin{cases} x(0)+x(\pi)=0, \\ \tilde{k}(0)-\tilde{k}(2\pi)=0, \\ x_2-\beta=0, \\ k_1-1=0, \\ k_2=0, \end{cases}$$

where $u(t)=(x(t),\ \omega)^T$, $k(t)=(\tilde{k}(t),\ k_{n+1})^T$ and $u(0)=(x(0),\ \omega)^T$, $x(0)=(x_1,\dots,\ x_n)^T$, $k(0)=(\tilde{k}(0),\ k_{n+1})^T$, $\tilde{k}(0)=(k_1,\dots,\ k_n)^T$. As is shown in the above argument, (3.26)–(3.27) certainly has a periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$, and the B-component $\hat{B}$ of this solution is really the desired bifurcation point. In fact, $\hat{x}(t)$ satisfies $\hat{x}(t+\pi)=-\hat{x}(t)$ for any t and $\hat{k}(t)=\hat{H}(t)=\hat{\Psi}(t)\hat{\hat{z}}$.

Next, we discuss the isolatedness of the periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$ of (3.26)–(3.27).

Let $\left(\begin{pmatrix}\varphi(t,\ \boldsymbol{x})\\ \omega\end{pmatrix},\ \begin{pmatrix}\varphi_1(t,\ \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T$ be a solution of (3.26) such that $\left(\begin{pmatrix}\varphi(0,\ \boldsymbol{x})\\ \omega\end{pmatrix},\ \begin{pmatrix}\varphi_1(0,\ \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T=\left(\begin{pmatrix}x(0)\\ \omega\end{pmatrix},\ \begin{pmatrix}\tilde{k}(0)\\ k_{n+1}\end{pmatrix}\right)^T$, where $\boldsymbol{x}=(u(0),\ k(0),\ B)^T$, $u(0)=(x(0),\ \omega)^T$, $x(0)=(x_1,\dots,\ x_n)^T$, $k(0)=(\tilde{k}(0),\ k_{n+1})^T$, $\tilde{k}(0)=(k_1,\dots,\ k_n)^T$. Then we consider the system

$$(3.28)\qquad F(\boldsymbol{x})=\begin{pmatrix} \begin{pmatrix} \varphi(0,\ \boldsymbol{x})+\varphi(\pi,\ \boldsymbol{x}) \\ x_2-\beta \end{pmatrix} \\ \begin{pmatrix} \varphi_1(0,\ \boldsymbol{x})-\varphi_1(2\pi,\ \boldsymbol{x}) \\ k_2 \end{pmatrix} \\ k_1-1 \end{pmatrix}=0.$$

Of course, the initial value $\hat{\boldsymbol{x}}=(\hat{u}(0),\ \hat{k}(0),\ \hat{B})^T$ of the periodic solution $(\hat{u}(t),\ \hat{k}(t),\ \hat{B})^T$ of (3.26)–(3.27) is a solution of the system (3.28). For the solution $\hat{\boldsymbol{x}}$, we have

Theorem 5.

The matrix $F'(\hat{\boldsymbol{x}})$ is non-singular if and only if

$$(3.29)\qquad \operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{\delta} \\ 10\cdots 0 & 0 \end{pmatrix}=n+1,$$

where

$F'(\boldsymbol{x})$ denotes the Jacobian matrix of $F(\boldsymbol{x})$ with respect to $\boldsymbol{x}$;

$\hat{\Psi}_1(t)$ and $\hat{\Psi}_2(t)$ are the $n\times(n+1)$ matrices such that

$$\hat{\Psi}(t)=\begin{pmatrix}\hat{\Psi}_1(t)\\ 0\cdots01\end{pmatrix} \quad\text{and}\quad \hat{\hat{\Psi}}_2(t)=\begin{pmatrix}\hat{\Psi}_2(t)\\ 0\cdots00\end{pmatrix}$$

respectively, where $(\hat{\Psi}(t), \hat{\hat{\Psi}}_2(t))^T$ *is a solution* $(2(n+1)\times(n+1)$ *matrix) of the system*

$$(3.30)\qquad \begin{cases} \dfrac{dz_1}{dt}=V_u(\hat{u}(t), \hat{B})z_1, \\[2ex] \dfrac{dz_2}{dt}=V_u(\hat{u}(t), \hat{B})z_2+\{V_{uu}(\hat{u}(t), \hat{B})\hat{k}(t)\}z_1 \end{cases}$$

satisfying the initial condition $(\hat{\Psi}(0), \hat{\hat{\Psi}}_2(0))^T=(E_{n+1}, 0)^T$;

$\hat{\delta}=-\hat{\Psi}_2(2\pi)\hat{\zeta}-\hat{\xi}_2(2\pi)$, *where* $\hat{\zeta}$ *is a solution of the equation*

$$(3.31)\qquad \begin{pmatrix} E_{n,n+1}+\hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots0 \end{pmatrix}\zeta=-\begin{pmatrix}\hat{\xi}_1(\pi)\\ 0\end{pmatrix}$$

and $(\hat{\hat{\xi}}_1(t), \hat{\xi}_2(t))^T$ $(\hat{\hat{\xi}}_i(t)\equiv(\hat{\xi}_i(t), 0)^T$ $(i=1, 2))$ *is a solution of the system*

$$(3.32)\qquad \begin{cases} \dfrac{d\eta_1}{dt}=V_u(\hat{u}(t), \hat{B})\eta_1+V_B(\hat{u}(t), \hat{B}), \\[2ex] \dfrac{d\eta_2}{dt}=V_u(\hat{u}(t), \hat{B})\eta_2+\{V_{uu}(\hat{u}(t), \hat{B})\eta_1+V_{uB}(\hat{u}(t), \hat{B})\}\hat{k}(t) \end{cases}$$

satisfying the initial condition $(\hat{\hat{\xi}}_1(0), \hat{\xi}_2(0))^T=(0, 0)^T$.

PROOF. For the solution $\hat{\boldsymbol{x}}$, we have

$$(3.33)\qquad F'(\hat{\boldsymbol{x}})=\begin{pmatrix} \begin{pmatrix} E_{n,n+1}+\hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix}\hat{\xi}_1(\pi)\\ 0\end{pmatrix} \\ \begin{pmatrix} -\hat{\Psi}_2(2\pi) \\ 000\cdots\cdots\cdots\cdots0 \end{pmatrix} & \begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots0 \end{pmatrix} & \begin{pmatrix}-\hat{\xi}_2(2\pi)\\ 0\end{pmatrix} \\ 000\cdots\cdots\cdots\cdots0 & 100\cdots\cdots\cdots\cdots0 & 0 \end{pmatrix},$$

from which it follows that

$$(3.34)\qquad \det F'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to (3.29)}.$$

This completes the proof. Q. E. D.

(ii) When $A\neq 0$, since

$$\operatorname{rank}\begin{pmatrix} E_{n,n+1}+\hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots0 \end{pmatrix}=n+1$$

due to the assumption (3.3), the equation

$$\begin{pmatrix} E_{n,n+1}+\hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots0 \end{pmatrix}\tilde{z}=-A\begin{pmatrix} \hat{\xi}_1(\pi) \\ 0 \end{pmatrix} \tag{3.35}$$

has one and only one solution $\hat{\tilde{z}}$, where $\tilde{z}=(z, z_{n+1})^T$, $z=(z_1,\dots, z_n)^T$. The equation (3.35) can be rewritten in the following form:

$$\begin{cases} [E_{n,n+1}+\hat{\Psi}_1(\pi)]\tilde{z}=-A\hat{\xi}_1(\pi), & (3.36) \\ z_2=0. & (3.37) \end{cases}$$

Further, by (3.2), we can rewrite the equation (3.36) in the form

$$[E_n+\hat{\Phi}_1(\pi)]z+\frac{1}{2}z_{n+1}\hat{\Phi}_1(\pi)\hat{c}=-A\hat{\xi}_1(\pi). \tag{3.38}$$

Since $\hat{c}=X(\hat{x}(0), \hat{B})$ satisfies

$$[E_n+\hat{\Phi}_1(\pi)]\hat{c}=0,$$

we have

$$\frac{1}{2}\hat{\Phi}_1(\pi)\hat{c}=-\frac{1}{2}\hat{c} \tag{3.39}$$

and

$$\frac{1}{2}[E_n-\hat{\Phi}_1(\pi)]\hat{c}=\hat{c}. \tag{3.40}$$

The equality (3.39) implies that the equation (3.38) is equivalent to

$$[E_n+\hat{\Phi}_1(\pi)]z-\frac{1}{2}z_{n+1}\hat{c}=-A\hat{\xi}_1(\pi). \tag{3.41}$$

Hence, multiplying the both sides of (3.41) by the matrix $E_n-\hat{\Phi}_1(\pi)$ from the left, by (3.8) and (3.9), we have

$$\begin{cases} [E_{n,n+1}-\hat{\Psi}_1(2\pi)]\tilde{z}=[E_n-\hat{\Phi}_1(2\pi)]z-z_{n+1}\hat{c} \\ =[E_n-\hat{\Phi}_1(\pi)]\,[E_n+\hat{\Phi}_1(\pi)]z-\frac{1}{2}z_{n+1}[E_n-\hat{\Phi}_1(\pi)]\hat{c} \\ =-A[E_n-\hat{\Phi}_1(\pi)]\hat{\xi}_1(\pi)=A\hat{\xi}_1(2\pi). \end{cases} \tag{3.42}$$

This shows that the solution $\hat{\tilde{z}}$ of (3.35) is also a solution of (3.10). Therefore $\hat{\tilde{z}}$ becomes the initial value of a 2π-periodic solution $\hat{K}(t)$ of the system

$$\frac{dK}{dt}=V_u(\hat{u}(t), \hat{B})K+A\cdot V_B(\hat{u}(t), \hat{B}) \tag{3.43}$$

satisfying $h_2=0$, where $K=K(t)=(h(t), h_{n+1})^T$ and $h(0)=(h_1,\dots, h_n)^T$. Indeed, $\hat{K}(0)=\hat{\tilde{z}}$. Moreover, in this case, the h-component $\hat{h}(t)$ of the periodic solution $\hat{K}(t)=(\hat{h}(t), \hat{h}_{n+1})^T$ of (3.43) satisfies $\hat{h}(t+\pi)=-\hat{h}(t)$ for any t, because $\hat{\tilde{z}}$ is originally a solution of (3.35) and, by the assumption (3.1), $X_x(\hat{x}(t), \hat{B})$ is π-periodic in t and $X(\hat{x}(t), \hat{B})$ and $X_B(\hat{x}(t), \hat{B})$ satisfy $X(\hat{x}(t+\pi), \hat{B})=-X(\hat{x}(t), \hat{B})$ and $X_B(\hat{x}(t+\pi), \hat{B})=-X_B(\hat{x}(t), \hat{B})$ for any t, respectively.

Next, we consider Case (II).

In this case, similarly to Case (I), we have

$$\hat{\Phi}_1(t+\pi)=\hat{\Phi}_1(t)\hat{\Phi}_1(\pi) \tag{3.44}$$

for the fundamental matrix $\hat{\Phi}_1(t)$ and we get

$$\hat{\xi}_1(2\pi)=[E_n+\hat{\Phi}_1(\pi)]\hat{\xi}_1(\pi) \tag{3.45}$$

for $\hat{\xi}_1(t)$.

Analogously to Case (I), we consider the equation

$$\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}\tilde{z}=A\hat{\tilde{\xi}}_1(2\pi)=A\begin{pmatrix} \hat{\xi}_1(2\pi) \\ 0 \end{pmatrix} \tag{3.46}$$

for any constant number A, where $\tilde{z}=(z, z_{n+1})^T$, $z=(z_1,\dots, z_n)^T$.

(i) When $A=0$, the equation (3.46) becomes

$$\begin{pmatrix} E_{n,n+1}-\hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix}\tilde{z}=0. \tag{3.47}$$

Since $\hat{\Psi}_1(2\pi)=(\hat{\Phi}_1(2\pi), \hat{c})$, we can rewrite the equation (3.47) in the form

$$\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix}z-z_{n+1}\begin{pmatrix} \hat{c} \\ 0 \end{pmatrix}=0. \tag{3.48}$$

Thus, from the assumption (3.6), it follows that

$$\left\{\begin{array}{l} \text{the equation (3.48) (or (3.47)) has a nontrivial} \\ \text{solution if and only if } z_{n+1}=0. \end{array}\right. \tag{3.49}$$

Hence, it is sufficient to consider the equation

$$\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix}z=0. \tag{3.50}$$

Due to the assumption (3.6), the equations

$$\begin{cases} [E_n - \hat{\Phi}_1(\pi)]y = 0, & (3.51) \\ [E_n + \hat{\Phi}_1(\pi)]v = 0 & (3.52) \end{cases}$$

have nontrivial solutions y and v respectively, where $y = (y_1, \dots, y_n)^T$ and $v = (v_1, \dots, v_n)^T$. Since rank $[E_n - \hat{\Phi}_1(2\pi)] = n - 2$ due to the assumption (3.6) and the solutions y and v are linearly independant, an arbitrary solution of the equation

$$[E_n - \hat{\Phi}_1(2\pi)]w = 0 \tag{3.53}$$

can be written in the form of a linear combination of y and v.

In this case, since $\hat{c} = X(\hat{x}(0), \hat{B})$ satisfies

$$[E_n - \hat{\Phi}_1(\pi)]\hat{c} = 0, \tag{3.54}$$

we can take $\hat{c}$ instead of y. Hence we can write an arbitrary solution of (3.53) in the following form:

$$a_1\hat{c} + a_2 v, \tag{3.55}$$

where a_1 and a_2 are arbitrary constants. Since a solution z of (3.50) is also a solution of (3.53), we can write z in the form

$$z = a_1\hat{c} + a_2 v. \tag{3.56}$$

However the assumption (3.6) shows that no solution of (3.50) is uniquely determined unless we add some additional condition. Therefore we must give one more condition.

Now, for the sake of simplicity, we assume that

$$\text{rank}\begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(2\pi) \\ 010 \cdots\cdots\cdots\cdots 0 \end{pmatrix} = \text{rank}\begin{pmatrix} \hat{D}_1(2\pi) \\ 10 \cdots 0 \end{pmatrix} = n. \tag{3.57}$$

As for the permissibility of the assumption (3.57), see Section 5.
Then, analogouly to Case (I), we give the condition

$$z_1 = a_1\hat{c}_1 + a_2 v_1 = 1 \tag{3.58}$$

as an additional condition. Then a solution of (3.50) satisfying the condition (3.58) is uniquely determined and we denote this solution by $\hat{z}$.

When we put $\hat{\hat{z}} = (\hat{z}, 0)^T$, $\hat{\hat{z}}$ is a solution of (3.47). That is, $\hat{H}(t) \equiv \hat{\Psi}(t)\hat{\hat{z}}$ is a 2π-periodic solution of the system

$$\frac{dk}{dt} = V_u(\hat{u}(t), \hat{B})k \tag{3.59}$$

satisfying the conditions

(3.60)
$$\begin{cases} k_1=1, \\ k_2=0, \end{cases}$$

where $k(0)=(k_1,\dots, k_n, k_{n+1})^T$. This tells us that in order to obtain the bifurcation point, we have only to seek for a periodic solution $(\hat{u}(t), \hat{k}(t), \hat{B})^T$ of the system

(3.61)
$$\begin{cases} \dfrac{du}{dt}=V(u, B), \\ \dfrac{dk}{dt}=V_u(u, B)k \end{cases}$$

satisfying the conditions

(3.62)
$$\begin{cases} x(0)-x(\pi)=0, \\ \tilde{k}(0)-\tilde{k}(2\pi)=0, \\ x_2-\beta=0, \\ k_1-1=0, \\ k_2=0, \end{cases}$$

where $u(t)=(x(t), \omega)^T$, $k(t)=(\tilde{k}(t), k_{n+1})^T$ and $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, $k(0)=(\tilde{k}(0), k_{n+1})^T$, $\tilde{k}(0)=(k_1,\dots, k_n)^T$. The above-mentioned argument shows that the problem (3.61)–(3.62) certainly has a solution $(\hat{u}(t), \hat{k}(t), \hat{B})^T$. Indeed, $\hat{x}(t)$ is π-periodic in t and $\hat{k}(t)=\hat{H}(t)=\hat{\Psi}(t)\hat{\tilde{z}}$, and the B-component $\hat{B}$ of this periodic solution is the desired bifurcation point.

Next we study the isolatedness of this solution.

Let $\left(\begin{pmatrix}\varphi(t, \boldsymbol{x})\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(t, \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T$ be a solution of (3.61) such that $\left(\begin{pmatrix}\varphi(0, \boldsymbol{x})\\ \omega\end{pmatrix}, \begin{pmatrix}\varphi_1(0, \boldsymbol{x})\\ k_{n+1}\end{pmatrix}\right)^T=\left(\begin{pmatrix}x(0)\\ \omega\end{pmatrix}, \begin{pmatrix}\tilde{k}(0)\\ k_{n+1}\end{pmatrix}\right)^T$, where $\boldsymbol{x}=(u(0), k(0), B)^T$, $u(0)=(x(0), \omega)^T$, $x(0)=(x_1,\dots, x_n)^T$, $k(0)=(\tilde{k}(0), k_{n+1})^T$, $\tilde{k}(0)=(k_1,\dots, k_n)^T$. Then, in this case, we consider the system

(3.63)
$$G(\boldsymbol{x})=\begin{pmatrix} \begin{pmatrix} \varphi(0, \boldsymbol{x})-\varphi(\pi, \boldsymbol{x}) \\ x_2-\beta \end{pmatrix} \\ \begin{pmatrix} \varphi_1(0, \boldsymbol{x})-\varphi_1(2\pi, \boldsymbol{x}) \\ k_2 \end{pmatrix} \\ k_1-1 \end{pmatrix}=0$$

instead of the system (3.28). The initial value $\hat{\boldsymbol{x}}=(\hat{u}(0), \hat{k}(0), \hat{B})^T$ of the above-mentioned solution $(\hat{u}(t), \hat{k}(t), \hat{B})^T$ of (3.61)–(3.62) is a solution of the system (3.63).

Analogously to Theorem 5, for the solution $\hat{x}$, we have

Theorem 6.
The matrix $G'(\hat{x})$ is non-singular if and only if

$$\text{(3.64)} \qquad \operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & \hat{\delta}' \\ 10\cdots 0 & 0 \end{pmatrix} = n+1,$$

where
$G'(x)$ denotes the Jacobian matrix of $G(x)$ with respect to x;
$\hat{\Psi}_1(t)$ and $\hat{\Psi}_2(t)$ are the $n\times(n+1)$ matrices such that

$$\hat{\Psi}(t) = \begin{pmatrix} \hat{\Psi}_1(t) \\ 0\cdots 01 \end{pmatrix} \quad and \quad \hat{\hat{\Psi}}_2(t) = \begin{pmatrix} \hat{\Psi}_2(t) \\ 0\cdots 00 \end{pmatrix}$$

respectively, where $(\hat{\Psi}(t), \hat{\hat{\Psi}}_2(t))^T$ is a solution ($2(n+1)\times(n+1)$ matrix) of the system

$$\text{(3.65)} \qquad \begin{cases} \dfrac{dz_1}{dt} = V_u(\hat{u}(t), \hat{B})z_1, \\ \dfrac{dz_2}{dt} = V_u(\hat{u}(t), \hat{B})z_2 + \{V_{uu}(\hat{u}(t), \hat{B})\hat{k}(t)\}z_1 \end{cases}$$

satisfying the initial condition $(\hat{\Psi}(0), \hat{\hat{\Psi}}_2(0))^T = (E_{n+1}, 0)^T$;
$\hat{\delta}' = -\hat{\Psi}_2(2\pi)\hat{\zeta}' - \hat{\xi}_2(2\pi)$, where $\hat{\zeta}'$ is a solution of the equation

$$\text{(3.66)} \qquad \begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix} \zeta' = \begin{pmatrix} \hat{\xi}_1(\pi) \\ 0 \end{pmatrix}$$

and $(\hat{\xi}_1(t), \hat{\xi}_2(t))^T$ $(\hat{\xi}_i(t) \equiv (\hat{\xi}_i(t), 0)^T \ (i=1, 2))$ is a solution of the system

$$\text{(3.67)} \qquad \begin{cases} \dfrac{d\eta_1}{dt} = V_u(\hat{u}(t), \hat{B})\eta_1 + V_B(\hat{u}(t), \hat{B}), \\ \dfrac{d\eta_2}{dt} = V_u(\hat{u}(t), \hat{B})\eta_2 + \{V_{uu}(\hat{u}(t), \hat{B})\eta_1 + V_{uB}(\hat{u}(t), \hat{B})\}\hat{k}(t) \end{cases}$$

satisfying the initial condition $(\hat{\xi}_1(0), \hat{\xi}_2(0))^T = (0, 0)^T$.

PROOF. For the solution $\hat{x}$, we have

$$\text{(3.68)} \quad G'(\hat{x}) = \begin{pmatrix} \begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix} -\hat{\xi}_1(\pi) \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\hat{\Psi}_2(2\pi) \\ 000\cdots\cdots\cdots\cdots 0 \end{pmatrix} & \begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(2\pi) \\ 010\cdots\cdots\cdots\cdots 0 \end{pmatrix} & \begin{pmatrix} -\hat{\xi}_2(2\pi) \\ 0 \end{pmatrix} \\ 000\cdots\cdots\cdots\cdots 0 & 100\cdots\cdots\cdots\cdots 0 & 0 \end{pmatrix},$$

from which it follows that

$$\text{(3.69)} \qquad \det G'(\hat{x}) \neq 0 \text{ is equivalent to (3.64).}$$

This completes the proof. Q. E. D.

(ii) When $A \neq 0$, due to the assumption (3.6), the equation

$$\text{(3.70)} \qquad \begin{pmatrix} E_{n,n+1} - \hat{\Psi}_1(\pi) \\ 010\cdots\cdots\cdots 0 \end{pmatrix} \tilde{z} = A \begin{pmatrix} \hat{\xi}_1(\pi) \\ 0 \end{pmatrix}$$

has one and only one solution $\hat{\tilde{z}}$, where $\tilde{z} = (z, z_{n+1})^T$, $z = (z_1, \ldots, z_n)^T$. The equation (3.70) can be rewritten in the following form:

$$\begin{cases} [E_n - \hat{\Phi}_1(\pi)]z - \dfrac{1}{2} z_{n+1} \hat{\Phi}_1(\pi)\hat{c} = A\hat{\xi}_1(\pi), & \text{(3.71)} \\ z_2 = 0. & \text{(3.72)} \end{cases}$$

Since $\hat{c} = X(\hat{x}(0), \hat{B})$ satisfies $[E_n - \hat{\Phi}_1(\pi)]\hat{c} = 0$, we have

$$\text{(3.73)} \qquad \frac{1}{2} \hat{\Phi}_1(\pi)\hat{c} = \frac{1}{2} \hat{c}$$

and

$$\text{(3.74)} \qquad \frac{1}{2} [E_n + \hat{\Phi}_1(\pi)]\hat{c} = \hat{c}.$$

From (3.73) it follows that the equation (3.71) is equivalent to the equation

$$\text{(3.75)} \qquad [E_n - \hat{\Phi}_1(\pi)]z - \frac{1}{2} z_{n+1}\hat{c} = A\hat{\xi}_1(\pi).$$

Hence, multiplying the both sides of (3.75) by the matrix $[E_n + \hat{\Phi}_1(\pi)]$ from the left, by (3.44), (3.45) and (3.74), we have

$$\text{(3.76)} \qquad \begin{cases} [E_{n,n+1} - \hat{\Psi}_1(2\pi)]\tilde{z} = [E_n - \hat{\Phi}_1(2\pi)]z - z_{n+1}\hat{c} \\ = [E_n + \hat{\Phi}_1(\pi)]\,[E_n - \hat{\Phi}_1(\pi)]z - \dfrac{1}{2} z_{n+1}[E_n + \hat{\Phi}_1(\pi)]\hat{c} \\ = A[E_n + \hat{\Phi}_1(\pi)]\hat{\xi}_1(\pi) = A\hat{\xi}_1(2\pi). \end{cases}$$

This shows that the solution $\hat{\tilde{z}}$ of (3.70) is a solution of (3.46). Therefore $\hat{\tilde{z}}$ becomes the initial value of a 2π-periodic solution $\hat{K}(t)$ of the system

$$\text{(3.77)} \qquad \frac{dK}{dt} = V_u(\hat{u}(t), \hat{B})K + A \cdot V_B(\hat{u}(t), \hat{B})$$

satisfying $h_2 = 0$, where $K = K(t) = (h(t), h_{n+1})^T$ and $h(0) = (h_1, \ldots, h_n)^T$. In fact,

$\hat{K}(0)=\hat{\hat{z}}$. Moreover, in this case, the periodic solution $\hat{K}(t)$ of (3.77) is a π-periodic in t, because $\hat{\hat{z}}$ is originally a solution of (3.70), and $X_x(\hat{x}(t), \hat{B})$, $X(\hat{x}(t), \hat{B})$ and $X_B(\hat{x}(t), \hat{B})$ are all π-periodic in t.

§ 4. Examples

In this section, in order to illustrate our theory and method mentioned in the preceding sections, we give some examples of singular points and bifurcations.

First, we consider a turning point and we compute it.

Example 1.

Let us consider an ω-periodic solution of the equation

$$\frac{d^2x}{d\tau^2}+\mu(1-Bx^2+x^4)\frac{dx}{d\tau}+x=0 \qquad (\mu=0.5). \tag{4.1}$$

Transforming τ to t by $\tau=\frac{\omega}{2\pi}t$ in (4.1), we have

$$\frac{d^2x}{dt^2}+\frac{\omega}{2\pi}\mu(1-Bx^2+x^4)\frac{dx}{dt}+\left(\frac{\omega}{2\pi}\right)^2 x=0. \tag{4.2}$$

The equation (4.2) can be rewritten in the form of a first order system as follows:

$$\begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-\left(\dfrac{\omega}{2\pi}\right)^2 x_1-\dfrac{\omega}{2\pi}\mu(1-Bx_1^2+x_1^4)x_2 . \end{cases} \tag{4.3}$$

As has been mentioned in the beginning of Section 2, in this example, we employ the condition

$$x_2(0)=0 \tag{4.4}$$

as an additional condition. Then we consider a 2π-periodic solution of (4.3) satisfying the condition (4.4). As has been mentioned in Section 2, we consider the equation

$$F(u(0), B)=\begin{pmatrix} x(0)-\varphi(2\pi, u(0), B) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0)-\varphi_1(2\pi, u(0), B) \\ x_2(0)-\varphi_2(2\pi, u(0), B) \\ x_2(0) \end{pmatrix}=0, \tag{4.5}$$

where $u(0)=(x(0), \omega)^T$, $x(0)=(x_1(0), x_2(0))^T$, and $\varphi(t, u(0), B)=(\varphi_1(t, u(0), B), \varphi_2(t, u(0), B))^T$ is a solution of (4.3) such that $\varphi(0, u(0), B)=x(0)$. This equation (4.5) has a singular point $(\hat{u}(0), \hat{B})$ satisfying the conditions (2.18) and (2.19), that is, a turning point. We compute it and the results of numerical computations are as follows:

$$(4.6)\qquad \begin{cases} \hat{x}_1(0)=1.68340\ 92229\ 706, & \hat{x}_2(0)=0.0, \\ \hat{\omega}=6.31609\ 14307\ 729, & \\ \hat{k}_1(0)=1.0, & \hat{k}_2(0)=0.0, \\ \hat{k}_3=-0.07838\ 76039\ 286, & \\ \hat{B}=2.82966\ 17309\ 151. & \end{cases}$$

Next, we study bifurcations of periodic solutions of autonomous systems involving parameters.

First, we consider a bifurcation point corresponding to Case (I) in Section 3.

Example 2.

Let us consider an ω-periodic solution of

$$(4.7)\qquad \frac{d^3x}{d\tau^3}-\{\mu(1-x^2)-a\}\frac{d^2x}{d\tau^2}-\{a\mu(1-x^2)-1\}\frac{dx}{d\tau}+ax=0 \qquad (\mu=0.1),$$

where a is a parameter. Transforming τ to t by $\tau=\frac{\omega}{2\pi}t$ in (4.7), we have

$$(4.8)\qquad \frac{d^3x}{dt^3}-\frac{\omega}{2\pi}\{\mu(1-x^2)-a\}\frac{d^2x}{dt^2} -\left(\frac{\omega}{2\pi}\right)^2\{a\mu(1-x^2)-1\}\frac{dx}{dt}+\left(\frac{\omega}{2\pi}\right)^3 ax=0.$$

The equation (4.8) is equivalent to the following first order system

$$(4.9)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=x_3, \\ \dfrac{dx_3}{dt}=-\left(\dfrac{\omega}{2\pi}\right)^3 ax_1+\left(\dfrac{\omega}{2\pi}\right)^2\{a\mu(1-x_1^2)-1\}x_2 \\ \qquad +\dfrac{\omega}{2\pi}\{\mu(1-x_1^2)-a\}x_3. \end{cases}$$

In this case, instead of the condition (2.3), we adopt the condition

$$(4.10) \qquad ax_2(0)+\frac{2\pi}{\omega}x_3(0)=0$$

as an additional condition.

We compute a bifurcation point and the results of numerical computations are as follows:

$$(4.11) \qquad \begin{cases} \hat{x}_1(0)=0.16744\ 12753\ 33655, & \hat{x}_2(0)=1.15096\ 04287\ 46062, \\ \hat{x}_3(0)=-0.15489\ 99077\ 90243, & \hat{\omega}=6.31292\ 32503\ 52323, \\ \hat{k}_1(0)=1.0, & \hat{k}_2(0)=0.04668\ 94093\ 61035, \\ \hat{k}_3(0)=-0.00628\ 36089\ 09721, & \hat{k}_4=0.14\times 10^{-17}, \\ \hat{a}=0.13394\ 92005\ 06121. & \end{cases}$$

Secondly, we consider a bifurcation point corresponding to Case (II) in Section 3.

Example 3.

Let us consider an ω-periodic solution of the equation

$$(4.12) \qquad \begin{aligned} &\frac{d^3x}{d\tau^3}-\{\mu(1+\gamma x-x^2)-a\}\frac{d^2x}{d\tau^2}-\{a\mu(1+\gamma x-x^2)-1\}\frac{dx}{d\tau} \\ &+ax=0 \qquad (\mu=0.4,\ a=0.46)\,, \end{aligned}$$

where γ is a parameter. We transform τ to t by $\tau=\frac{\omega}{2\pi}t$ in (4.12), then the equation (4.12) is rewritten in the form

$$(4.13) \qquad \begin{aligned} &\frac{d^3x}{dt^3}-\frac{\omega}{2\pi}\{\mu(1+\gamma x-x^2)-a\}\frac{d^2x}{dt^2}-\left(\frac{\omega}{2\pi}\right)^2\{a\mu(1+\gamma x-x^2)-1\}\frac{dx}{dt} \\ &+\left(\frac{\omega}{2\pi}\right)^3ax=0\,. \end{aligned}$$

The equation (4.13) is equivalent to the first order system

$$(4.14) \qquad \begin{cases} \dfrac{dx_1}{dt}=x_2\,, \\ \dfrac{dx_2}{dt}=x_3\,, \\ \dfrac{dx_3}{dt}=-\left(\dfrac{\omega}{2\pi}\right)^3ax_1+\left(\dfrac{\omega}{2\pi}\right)^2\{a\mu(1+\gamma x_1-x_1^2)-1\}x_2 \\ \qquad\qquad +\dfrac{\omega}{2\pi}\{\mu(1+\gamma x_1-x_1^2)-a\}x_3\,. \end{cases}$$

In this example, we add the same additional condition as in (4.10). We compute a bifurcation point and the results of numerical computations are as follows:

$$(4.15)\quad \begin{cases} \hat{x}_1(0)=0.01677\,65690, & \hat{x}_2(0)=1.59404\,43863, \\ \hat{x}_3(0)=-1.57257\,50454, & \hat{\omega}=13.47513\,13194, \\ \hat{k}_1(0)=1.0, & \hat{k}_2(0)=0.62703\,53520, \\ \hat{k}_3(0)=-0.61859\,01445, & \hat{k}_4=0.38\times 10^{-16}, \\ \hat{\gamma}=0.28392\,94943. & \end{cases}$$

§5. Appendix

We study in more detail how to choose an additional condition mentioned in the beginning of Section 2. Let $\hat{u}(t)=(\hat{x}(t), \hat{\omega})^T$ be a 2π-periodic solution of (2.4) at $B=\hat{B}$ and $\hat{\Phi}_1(t)$ be the fundamental matrix of (2.10) at $x=\hat{x}(t)$, $\omega=\hat{\omega}$ and $B=\hat{B}$ satisfying the initial condition $\hat{\Phi}_1(0)=E_n$.

At first, we assume that $\hat{\Phi}_1(2\pi)$ has a single eigenvalue one (we say that an eigenvalue one of $\hat{\Phi}_1(2\pi)$ is simple). As is well-known, $X(\hat{x}(t), \hat{B})$ is a 2π-periodic solution of (2.10). Therefore $X(\hat{x}(t), \hat{B})$ can be written in the form

$$(5.1)\qquad X(\hat{x}(t), \hat{B})=\hat{\Phi}_1(t)X(\hat{x}(0), \hat{B})$$

and so the n-dimensional vector $\hat{c}=X(\hat{x}(0), \hat{B})$ $(\neq 0)$ satisfies

$$(5.2)\qquad \hat{\Phi}_1(2\pi)\hat{c}=\hat{c} \qquad (\text{or } [E_n-\hat{\Phi}_1(2\pi)]\hat{c}=0).$$

That is, $\hat{c}$ is an eigenvector corresponding to the eigenvalue one. Since the eigenvalue one of $\hat{\Phi}_1(2\pi)$ is simple, it follows from (5.2) that

$$(5.3)\qquad \operatorname{rank}[E_n-\hat{\Phi}_1(2\pi)]=n-1<n=\operatorname{rank}[E_n-\hat{\Phi}_1(2\pi), \hat{c}]$$

and there exists a positive integer k $(1\leq k\leq n)$ such that

$$(5.4)\qquad \hat{c}_k\neq 0$$

and

$$(5.5)\qquad \operatorname{rank}[E_n-\hat{\Phi}_1(2\pi)]=\operatorname{rank}\hat{D}_{-k}=n-1,$$

where $\hat{c}=(\hat{c}_1, \hat{c}_2,\dots, \hat{c}_n)^T$ and $\hat{D}_{-k}$ is the $n\times(n-1)$ matrix obtained from $E_n-\hat{\Phi}_1(2\pi)$ by deleting the k-th column vector. Then, by (5.3) and (5.5), we have

$$(5.6)\qquad \operatorname{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}0\cdots 0 & 0 \end{pmatrix}=n+1,$$

where $\underset{\uparrow}{k}$ indicates the location of the k-th column vector of the $(n+1)\times(n+1)$ matrix $\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots010\cdots0 & 0 \end{pmatrix}$. This implies that we may employ the condition

$$x_k(0)-\beta=0 \tag{5.7}$$

as an additional condition, where $x(0)=(x_1(0),\dots, x_n(0))^T$ and β is a constant number. Consequently, in order to obtain the 2π-periodic solution $\hat{u}(t)=(\hat{x}(t), \hat{\omega})^T$ of (2.4) at $B=\hat{B}$, we consider the equation

$$S(u(0), \hat{B})=\begin{pmatrix} \varphi(0, u(0), \hat{B})-\varphi(2\pi, u(0), \hat{B}) \\ x_k(0)-\beta \end{pmatrix} = \begin{pmatrix} x(0)-\varphi(2\pi, u(0), \hat{B}) \\ x_k(0)-\beta \end{pmatrix}=0, \tag{5.8}$$

where $\varphi(t, u(0), B)$ is a solution of (2.2) such that $\varphi(0, u(0), B)=x(0)$. Evidently, the equation (5.8) has a solution $\hat{u}(0)=(\hat{x}(0), \hat{\omega})^T$. Denoting by $S_u(u(0), \hat{B})$ the Jacobian matrix of $S(u(0), \hat{B})$ with respect to $u(0)$, for the solution $\hat{u}(0)$, we have

$$S_u(\hat{u}(0), \hat{B})=\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & -\hat{c} \\ \underset{\substack{\uparrow \\ k}}{0\cdots010\cdots0} & 0 \end{pmatrix}. \tag{5.9}$$

Therefore, by (5.6), we have

$$\det S_u(\hat{u}(0), \hat{B})\neq 0. \tag{5.10}$$

This tells us that we can get an approximation to the solution $\hat{u}(0)$ of (5.8) as accurately as we desire by applying the Newton method to the equation (5.8).

Next, we consider the case where $\hat{\Phi}_1(2\pi)$ has a double eigenvalue one. We denote by $J(1)$ the Jordan block corresponding to the eigenvalue one of $\hat{\Phi}_1(2\pi)$. Then the following two cases are possibly considered:

$$\text{(i)} \qquad J(1)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.11}$$

and

$$\text{(ii)} \qquad J(1)=\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad (a\neq 0). \tag{5.12}$$

First, we study the case (i). In this case, we have

$$\text{rank}\,[E_n-\hat{\Phi}_1(2\pi)]=n-2<n-1=\text{rank}\,[E_n-\hat{\Phi}_1(2\pi),\ \hat{c}] \tag{5.13}$$

because $\hat{c}$ is an eigenvector corresponding to the eigenvalue one. On the other hand, there exist positive integers i and j $(1 \leq i, j \leq n,\ i \neq j)$ such that

$$\text{rank}\begin{pmatrix} \hat{c}_i & \hat{y}_i \\ \hat{c}_j & \hat{y}_j \end{pmatrix}=2 \tag{5.14}$$

and

$$\text{rank}\,[E_n-\hat{\Phi}_1(2\pi)]=\text{rank}\,\hat{D}_{-i,-j}=n-2, \tag{5.15}$$

where $\hat{c}=(\hat{c}_1, \hat{c}_2,\ldots, \hat{c}_n)^T$, and $\hat{y}=(\hat{y}_1, \hat{y}_2,\ldots, \hat{y}_n)^T$ is a solution of the equation

$$[E_n-\hat{\Phi}_1(2\pi)]y=0 \qquad (\text{where } y=(y_1, y_2,\ldots, y_n)^T) \tag{5.16}$$

and is linearly independent of $\hat{c}$, and $\hat{D}_{-i,-j}$ is the $n\times(n-2)$ matrix obtained from $E_n-\hat{\Phi}_1(2\pi)$ by deleting the i-th column vector and the j-th column vector. Then, by (5.13) and (5.15), we have

$$\text{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots\underset{\underset{i}{\uparrow}}{1}0\cdots0 & 0 \end{pmatrix}=\text{rank}\ \hat{V}_{-j}=n \tag{5.17}$$

and

$$\text{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots\underset{\underset{j}{\uparrow}}{1}0\cdots0 & 0 \end{pmatrix}=\text{rank}\ \hat{W}_{-i}=n, \tag{5.18}$$

where $\hat{V}_{-j}$ is the $(n+1)\times n$ matrix obtained from $\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots\underset{\underset{i}{\uparrow}}{1}0\cdots0 & 0 \end{pmatrix}$ by deleting the j-th column vector and $\hat{W}_{-i}$ is the $(n+1)\times n$ matrix obtained from $\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots\underset{\underset{j}{\uparrow}}{1}0\cdots0 & 0 \end{pmatrix}$ by deleting the i-th column vector.

Therefore, in this case, we employ either of the following

$$x_i(0)-\beta_1=0 \tag{5.19}$$

$$x_j(0)-\beta_2=0 \tag{5.20}$$

as an additional condition mentioned in the beginning of Section 2, where $x(0)=$

$(x_1(0),\dots, x_n(0))^T$ and β_1 and β_2 are constant numbers.

Secondly, we study the case (ii). In this case, due to (5.12), there exists a vector $\hat{v}$ such that $\hat{v}$ is linearly independent of $\hat{c}$ and satisfies

$$(5.21)\qquad \begin{cases} \hat{\Phi}_1(2\pi)\hat{c}=\hat{c}, \\ \hat{\Phi}_1(2\pi)\hat{v}=\hat{v}+a\hat{c}. \end{cases}$$

This implies that

$$(5.22)\qquad \operatorname{rank}[E_n-\hat{\Phi}_1(2\pi)]=\operatorname{rank}[E_n-\hat{\Phi}_1(2\pi),\ \hat{c}]=n-1.$$

From the first of (5.21) it follows that there exists a positive integer l $(1\le l\le n)$ such that

$$(5.23)\qquad \hat{c}_l\neq 0$$

and

$$(5.24)\qquad \operatorname{rank}[E_n-\hat{\Phi}_1(2\pi)]=\operatorname{rank}\hat{D}_{-l}=n-1,$$

where $\hat{c}=(\hat{c}_1, \hat{c}_2,\dots, \hat{c}_n)^T$ and $\hat{D}_{-l}$ is the $n\times(n-1)$ matrix obtained from $E_n-\hat{\Phi}_1(2\pi)$ by deleting the l-th column vector. From (5.21), (5.22) and (5.24) it follows that there exists a positive integer m $(1\le m\le n,\ m\neq l)$ such that

$$(5.25)\qquad \operatorname{rank}[\hat{D}_{-l,-m},\ \hat{c}]=n-1,$$

where $\hat{D}_{-l,-m}$ is the $n\times(n-2)$ matrix obtained from $E_n-\hat{\Phi}_1(2\pi)$ by deleting the l-th column vector and the m-th column vector. Then, by (5.24) and (5.25), we have

$$(5.26)\qquad \operatorname{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ m}}{1}0\cdots 0 & 0 \end{pmatrix}=\operatorname{rank}\hat{Z}_{-l}=n,$$

where $\hat{Z}_{-l}$ is the $(n+1)\times n$ matrix obtained from $\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ m}}{1}0\cdots 0 & 0 \end{pmatrix}$ by deleting the l-th column vector. Therefore, in this case, we employ the condition

$$(5.27)\qquad x_m(0)-\beta_3=0$$

as an additional condition mentioned in the beginning of Section 2, where $x(0)=(x_1(0),\dots, x_n(0))^T$ and β_3 is a constant number.

In either case, as is seen from the above, if $\hat{\Phi}_1(2\pi)$ has a double eigenvalue one, then there exist positive integers k and p $(1\le k,\ p\le n,\ k\neq p)$ such that

$$(5.28) \qquad \operatorname{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}0\cdots 0 & 0 \end{pmatrix} = \operatorname{rank}\ \hat{Y}_{-p} = n\,,$$

where $\hat{Y}_{-p}$ is the $(n+1)\times n$ matrix obtained from $\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ k}}{1}0\cdots 0 & 0 \end{pmatrix}$ by deleting the p-th column vector. In fact, in the case (i), $k=i$, $p=j$ or $k=j$, $p=i$ and in the case (ii), $k=m$, $p=l$. Therefore we employ the condition

$$(5.29) \qquad x_k(0)-\beta=0$$

as an additional condition mentioned in the beginning of Section 2, where $x(0)=(x_1(0),\dots, x_n(0))^T$ and β is a constant number.

In the present paper, in order to simplify notations and symbols used in Sections 2 and 3, we employ $k=2$ in (2.3) and $p=1$ in (2.13), (3.22) and (3.57). But, as is seen from the above, it is clear that the generality of the argument developed in Sections 2 and 3 is never lost if we employ such values for k and p.

Remark 4.

In the case (ii), we can make another choice of an additional condition. From (5.21), (5.22) and (5.24) it follows that

$$(5.30) \qquad \operatorname{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) & \hat{c} \\ 0\cdots 0\underset{\substack{\uparrow\\ l}}{1}0\cdots 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} E_n-\hat{\Phi}_1(2\pi) \\ 0\cdots 0\underset{\substack{\uparrow\\ l}}{1}0\cdots 0 \end{pmatrix} = n\,.$$

This tells us that we may employ the condition

$$(5.31) \qquad x_l(0)-\beta_4=0$$

instead of the condition (5.27), where $x(0)=(x_1(0),\dots, x_n(0))^T$ and β_4 is a constant number. Then, in this case, we may consider the equation

$$(5.32) \qquad H(\boldsymbol{x}) = \begin{pmatrix} \begin{pmatrix} \varphi(0,\boldsymbol{x})-\varphi(2\pi,\boldsymbol{x}) \\ x_l-\beta_4 \end{pmatrix} \\ \begin{pmatrix} \varphi_1(0,\boldsymbol{x})-\varphi_1(2\pi,\boldsymbol{x}) \\ k_l \end{pmatrix} \\ k_{n+1}-1 \end{pmatrix} = 0$$

instead of the equation (2.17), where $\boldsymbol{x}=(u(0), k(0), B)^T$, $u(0)=(x(0), \omega)^T$, $x(0)=$

$(x_1,\dots, x_n)^T$, $k(0)=(\tilde{k}(0), k_{n+1})^T$, $\tilde{k}(0)=(k_1,\dots, k_n)^T$.

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