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Geometric Characterization of Singular Points of Nonlinear Equations Involving Parameters

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§1. Introduction

We consider a point $(\hat{x}, \hat{B}) \in \Omega$ satisfying an n-dimensional nonlinear equation

$$F(x, B) = 0 \tag{1.1}$$

such that the rank of the Jacobian matrix $F_x(x, B)$ of $F(x, B)$ with respect to x is $n-1$ at $(x, B) = (\hat{x}, \hat{B})$, where $F(x, B)$ is defined in some region Ω of the (x, B)-space and $F(x, B)$ is $(d+2)$ times continuously differentiable with respect to (x, B) in Ω, and B is a parameter and we assume that the dimension of the parameter B is $(d+1)$, that is, $B = (B_1, B_2, \dots, B_{d+1})^T$ $(d \geq 0)$. Here $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$.

We call the point $(\hat{x}, \hat{B})$ above a "*singular point*" of the nonlinear equation (1.1). Especially, in the case $d=0$, that is, the dimension of the parameter B is one, $(\hat{x}, \hat{B})$ is called a "*turning point*" or "*fold point*", see [3], [4], [5], [6], [7]. Further, in the case $d=1$, that is, the dimension of the parameter B is two, $(\hat{x}, \hat{B})$ is called a "*cusp point*", see [1], [6], [7].

We shall show that the B-component $\hat{B}$ of such a singular point $(\hat{x}, \hat{B})$ is geometrically characterized as an extremum of some function which expresses a curve in the parameter space, that is, the B-space. Here $\hat{B}(=B(\hat{\sigma}))$ is called an "*extremum*" of a function $B(\sigma)$ if $\frac{dB}{d\sigma}(\hat{\sigma}) = 0$ and $\frac{d^2B}{d\sigma^2}(\hat{\sigma}) \neq 0$, where σ (scalar) is a real variable and $B(\sigma)$ is defined in some neighborhood of $\hat{\sigma}$ and is twice continuously differentiable with respect to σ in such a neighborhood.

When the dimension of the parameter B is ≤ 2, that is, $d \leq 1$, H. Kawakami [6] defined a singular point $(\hat{x}, \hat{B})$ as the B-component $\hat{B}$ of $(\hat{x}, \hat{B})$, where the component is an extremum of some function. Such a function coincides with the above one in the case $d=0$ but is different from that in the case $d=1$, and he proposed a method for computing it. But he did not give any condition for guaranteeing the isolatedness of a singular point and he did not describe anything about the case $d \geq 2$.

In the case $d \leq 1$, A. Spence and B. Werner [7] also considered the B-component $\hat{B}$ of a singular point $(\hat{x}, \hat{B})$ as an extremum of some function. In the case $d=0$, their characterization of a singular point is similar to ours, but in the case $d=1$, theirs is different from ours. And they also did not describe anything about the case $d\geq 2$.

In our case, on the other hand, we of course study the case $d\geq 2$ and we give a condition for guaranteeing the isolatedness of a singular point.

In this paper, in §2, we give geometric characterization of singular points of nonlinear equations involving parameters and we propose a method for computing them with high accuracy.

§2. Geometric Characterization of Singular Points of Nonlinear Equations Involving Parameters

We consider a singular point $(\hat{x}, \hat{B})\in\Omega$ of the nonlinear equation (1.1). In order to simplify the following argument, for the singular point $(\hat{x}, \hat{B})$, we assume that

$$n-1=\operatorname{rank} F_x(\hat{x}, \hat{B})=\operatorname{rank} F_0(\hat{x}, \hat{B}), \tag{2.1}$$

where $F_0(\hat{x}, \hat{B})$ is the $n\times(n-1)$ matrix obtained from $F_x(\hat{x}, \hat{B})$ by deleting the first column vector.

Now we define $n\times n$ matrices $X^{(m+1)}$ $(0\leq m\leq d)$ and n-dimensional vectors l_m $(1\leq m\leq d+1)$ by

$$X^{(m+1)}=\sum_{i=0}^{m} {}_mC_i X_x^{(i)} h_{m+1-i} \qquad (0\leq m\leq d) \tag{2.2}$$

and

$$l_m=\sum_{i=1}^{m} {}_mC_i X^{(i)} h_{m+1-i} \qquad (1\leq m\leq d+1) \tag{2.3}$$

respectively, where $X^{(0)}=F_x(x, B)$, and $X_x^{(i)}$ $(i=0, 1,\dots, d)$ are the derivatives of $X^{(i)}$ $(i=0, 1,\dots, d)$ with respect to x, respectively, and h_j $(j=1, 2,\dots, d+1)$ are n-dimensional vectors. Moreover, we define n-dimensional vectors $\mu_m (1\leq m\leq d+1)$ by

$$\mu_m=\sum_{j=0}^{m-1} {}_{m-1}C_j X^{(j+1)} h_{m-j} \qquad (1\leq m\leq d+1). \tag{2.4}$$

Now we consider the following equation

$$G(\boldsymbol{x}, B)=\begin{pmatrix} F(x, B) \\ X^{(0)}h_1 \\ X^{(0)}h_2+X^{(1)}h_1 \\ \vdots \\ \sum_{j=0}^{d-1} {}_{d-1}C_j X^{(j)} h_{d-j} \\ \psi(\boldsymbol{x}, B) \end{pmatrix}=\begin{pmatrix} F(x, B) \\ X^{(0)}h_1 \\ X^{(0)}h_2+l_1 \\ \vdots \\ X^{(0)}h_d+l_{d-1} \\ \psi(\boldsymbol{x}, B) \end{pmatrix}=0, \tag{2.5}$$

where $\boldsymbol{x}=(x, h_1, h_2,\dots, h_d)^T$, $x=(x_1,\dots, x_n)^T$, $h_j=(h_j^1, h_j^2,\dots, h_j^n)^T$ $(j=1, 2,\dots, d)$, $B=(B_1, B_2,\dots, B_{d+1})^T$, and $\psi(\boldsymbol{x}, B)=(h_1^1-1, h_2^1,\dots, h_d^1)^T$. Then the function $G(\boldsymbol{x}, B)$ defined by the equality (2.5) is a $\{(d+1)(n+1)-1\}$-dimensional vector and is twice continuously differentiable with respect to $(\boldsymbol{x}, B)$ due to the assumption on $F(x, B)$.

In particular, in the case $d=0$, the equation (2.5) becomes

$$G(\boldsymbol{x}, B)=F(x, B)=0 \tag{2.6}$$

since $\boldsymbol{x}=x$. Moreover, in the case $d=1$, the equation (2.5) becomes

$$G(\boldsymbol{x}, B)=\begin{pmatrix} F(x, B) \\ F_x(x, B)h_1 \\ h_1^1-1 \end{pmatrix}=0 \tag{2.7}$$

because $\boldsymbol{x}=(x, h_1)^T$ and $\psi(\boldsymbol{x}, B)=h_1^1-1$.

Assume that there exists a vector $(\hat{\boldsymbol{x}}, \hat{B})=(\hat{x}, \hat{h}_1,\dots, \hat{h}_d, \hat{B})^T$ such that the conditions

(2.8) $(\hat{x}, \hat{B})\in\Omega$ satisfies the equation (1.1) and the condition (2.1)

(2.9) $(\hat{\boldsymbol{x}}, \hat{B})$ satisfies the equation (2.5)

(2.10) $\operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_d)=n-1$

(2.11) $\operatorname{rank}(G_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{B}), G_{B_1}(\hat{\boldsymbol{x}}, \hat{B}),\dots, G_{B_{d+1}}(\hat{\boldsymbol{x}}, \hat{B}))=(d+1)(n+1)-1$

are satisfied, where $\hat{l}_d$ denotes the value of l_d at $(\boldsymbol{x}, B)=(\hat{\boldsymbol{x}}, \hat{B})$, and $G_{\boldsymbol{x}}(\boldsymbol{x}, B)$ denotes the derivative of $G(\boldsymbol{x}, B)$ with respect to $\boldsymbol{x}$, that is,

$$G_{\boldsymbol{x}}(\boldsymbol{x}, B)=\left(\begin{array}{ccccc} {}_0C_0X^{(0)} & & & & \mathbf{0} \\ {}_1C_0X^{(1)} & {}_1C_1X^{(0)} & & & \\ {}_2C_0X^{(2)} & {}_2C_1X^{(1)} & {}_2C_2X^{(0)} & & \\ \vdots & \vdots & \vdots & \ddots & \\ {}_dC_0X^{(d)} & {}_dC_1X^{(d-1)} & {}_dC_2X^{(d-2)} & \cdots & {}_dC_dX^{(0)} \\ \hline & 10\cdots\cdots0 & 00\cdots\cdots0 & & \\ 0 & 00\cdots\cdots0 & 10\cdots\cdots0 & 0 & 0 \\ & 0 & 0 & \ddots & 10\cdots\cdots0 \end{array}\right), \tag{2.12}$$

and $G_{B_i}(\boldsymbol{x}, B)$ $(i=1,2,\dots, d+1)$ denote the partial derivatives of $G(\boldsymbol{x}, B)$ with respect to B_i, respectively, that is,

$$(2.13)\qquad G_{B_i}(\boldsymbol{x}, B)=\begin{pmatrix} F_{B_i}(x, B) \\ X_{B_i}^{(0)}h_1 \\ X_{B_i}^{(0)}h_2+X_{B_i}^{(1)}h_1 \\ \vdots \\ \sum_{j=0}^{d-1} {}_{d-1}C_j X_{B_i}^{(j)}h_{d-j} \\ \boldsymbol{0} \end{pmatrix} \qquad (i=1, 2,\dots, d+1).$$

Here $F_{B_i}(x, B)$ and $X_{B_i}^{(q)}$ $(i=1, 2,\dots, d+1;\ q=0, 1,\dots, d-1)$ denote the partial derivatives of $F(x, B)$ and $X^{(q)}$ $(q=0, 1,\dots, d-1)$ with respect to B_i, respectively, and $\boldsymbol{0}$ denotes the d-dimensional zero vector.

By (2.8), (2.9) and (2.10), $\hat{z}=(\hat{\boldsymbol{x}}, \hat{h}_{d+1}, \hat{B})^T$ (where $\hat{h}_{d+1}$ is a solution of the equation $\hat{X}^{(0)}h_{d+1}+\hat{l}_d=0$, $h_{d+1}^1=0$) is certainly a solution of the system

$$(2.14)\qquad S(z)=\begin{pmatrix} F(x, B) \\ X^{(0)}h_1 \\ X^{(0)}h_2+X^{(1)}h_1 \\ \vdots \\ \sum_{j=0}^{d} {}_dC_j X^{(j)}h_{d+1-j} \\ \psi_{d+1}(z) \end{pmatrix} = \begin{pmatrix} F(x, B) \\ X^{(0)}h_1 \\ X^{(0)}h_2+l_1 \\ \vdots \\ X^{(0)}h_{d+1}+l_d \\ \psi_{d+1}(z) \end{pmatrix} =0,$$

where $\hat{X}^{(0)}=F_x(\hat{x}, \hat{B})$ and $\hat{l}_d$ denotes the value of l_d at $(\boldsymbol{x}, B)=(\hat{\boldsymbol{x}}, \hat{B})$, and $z=(\boldsymbol{x}, h_{d+1}, B)^T$, $\boldsymbol{x}=(x, h_1,\dots, h_d)^T$, $x=(x_1,\dots, x_n)^T$, $h_j=(h_j^1, h_j^2,\dots, h_j^n)^T$ $(j=1, 2,\dots, d+1)$ and $\psi_{d+1}(z)=(\psi(\boldsymbol{x}, B), h_{d+1}^1)^T=(h_1^1-1, h_2^1,\dots, h_d^1, h_{d+1}^1)^T$. For the solution $\hat{z}$, we have the following theorem.

Theorem.

The matrix $S'(\hat{z})$ is non-singular if and only if

$$(2.15)\qquad \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_{d+1})=n,$$

where $S'(z)$ denotes the Jacobian matrix of $S(z)$ with respect to z and $\hat{l}_{d+1}$ denotes the value of l_{d+1} at $z=\hat{z}$.

PROOF. Since $F(x, B)$ is $(d+2)$ times continuously differentiable with respect to (x, B) in Ω, $S(z)$ defined by the equality (2.14) is continuously differentiable with respect to z. Then we have

$$
(2.16)\qquad S'(z)=\left(\begin{array}{ccccc|c}
{}_0C_0X^{(0)} & & & & \mathbf{0} & \\
{}_1C_0X^{(1)} & {}_1C_1X^{(0)} & & & & \\
{}_2C_0X^{(2)} & {}_2C_1X^{(1)} & {}_2C_2X^{(0)} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
{}_{d+1}C_0X^{(d+1)} & {}_{d+1}C_1X^{(d)} & {}_{d+1}C_2X^{(d-1)} & \cdots & {}_{d+1}C_{d+1}X^{(0)} & S_{B_1}(z)\; S_{B_2}(z)\cdots\cdots S_{B_{d+1}}(z) \\
\boxed{0} & \begin{array}{c}10\cdots\cdots0\\00\cdots\cdots0\\ \boxed{0}\end{array} & \begin{array}{c}00\cdots\cdots0\\10\cdots\cdots0\\ \boxed{0}\end{array} & \ddots\; 0 & \begin{array}{c}\boxed{0}\\10\cdots\cdots0\end{array} &
\end{array}\right),
$$

where $S_{B_i}(z)$ $(i=1, 2,\dots, d+1)$ are $\{(d+2)n+d+1\}$-dimensional vectors defined by

$$
(2.17)\qquad S_{B_i}(z)=\begin{pmatrix} F_{B_i}(x, B) \\ X^{(0)}_{B_i}h_1 \\ X^{(0)}_{B_i}h_2+X^{(1)}_{B_i}h_1 \\ \vdots \\ \sum_{j=0}^{d} {}_dC_jX^{(j)}_{B_i}h_{d+1-j} \\ \boldsymbol{0} \end{pmatrix} \qquad (i=1, 2,\dots, d+1)\,,
$$

where $X^{(d)}_{B_i}$ $(i=1, 2,\dots, d+1)$ denote the partial derivatives of $X^{(d)}$ with respect to B_i, respectively, and $\boldsymbol{0}$ denotes the $(d+1)$-dimensional zero vector. From (2.16) it follows that for the solution $\hat{z}$, we have

$$
(2.18)\qquad \det S'(\hat{z})\neq 0 \text{ is equivalent to (2.15),}
$$

because

$$
\hat{X}^{(0)}\hat{h}_{m+1}+\hat{l}_m=\hat{X}^{(0)}\hat{h}_{m+1}+\sum_{i=1}^{m} {}_mC_i\hat{X}^{(i)}\hat{h}_{m+1-i}
$$

$$
=\sum_{i=0}^{m} {}_mC_i\hat{X}^{(i)}\hat{h}_{m+1-i}=0 \qquad (1\leq m\leq d),
$$

where $\hat{X}^{(i)}(i=0, 1,\dots, d)$ and $\hat{l}_j(j=1,2,\dots, d)$ denote the values of $X^{(i)}(i=0, 1,\dots, d)$ and l_j $(j=1, 2,\dots, d)$ at $(\boldsymbol{x}, B)=(\hat{\boldsymbol{x}}, \hat{B})$, respectively. This completes the proof.

Q. E. D.

Thus, if the condition (2.15) is satisfied, we can get an approximation to the solution $\hat{z}$ of (2.14) as accurately as we desire by applying the Newton method to the system (2.14). Hence we can also obtain a desired approximation to the singular point $(\hat{x}, \hat{B})$ of the equation (1.1). We call this singular point $(\hat{x}, \hat{B})$ satisfying $\det S'(\hat{z}) \neq 0$ an "*isolated singular point*".

From the conditions (2.9) and (2.11), due to the theorem on implicit function, we have the following results: The equation (2.5) defines a curve in some neighbourhood of $(\hat{\boldsymbol{x}}, \hat{B})$ in the $(\boldsymbol{x}, B)$-space. We denote such a curve by Γ. Then, taking some parameter σ, we can write the curve Γ in the form

(2.19) $\boldsymbol{x}=\boldsymbol{x}(\sigma)=(x(\sigma), h_1(\sigma),\dots, h_d(\sigma))^T$ and $B=B(\sigma)=(B_1(\sigma),\dots, B_{d+1}(\sigma))^T$

and we have

$$G(\boldsymbol{x}(\sigma), B(\sigma))=0 \tag{2.20}$$

for $(\boldsymbol{x}(\sigma), B(\sigma))$. Since $(\hat{\boldsymbol{x}}, \hat{B})$ is of course a point on the curve Γ, there exists one and only one $\hat{\sigma}$ corresponding to $(\hat{\boldsymbol{x}}, \hat{B})$, and we have

$$\hat{\boldsymbol{x}}=\boldsymbol{x}(\hat{\sigma}) \quad \text{and} \quad \hat{B}=B(\hat{\sigma}). \tag{2.21}$$

Further, $\boldsymbol{x}(\sigma)$ and $B(\sigma)$ defined by (2.19) are twice continuously differentiable with respect to σ because $G(\boldsymbol{x}, B)$ is twice continuously differentiable with respect to $(\boldsymbol{x}, B)$.

Then, differentiating the both sides of (2.20) with respect to σ, we have

$$G_{\boldsymbol{x}}\cdot\frac{d\boldsymbol{x}}{d\sigma}+\sum_{i=1}^{d+1} G_{B_i}\cdot\frac{dB_i}{d\sigma}=0, \tag{2.22}$$

where $G_{\boldsymbol{x}}$ and G_{B_i} $(i=1, 2,\dots, d+1)$ denote $G_{\boldsymbol{x}}(\boldsymbol{x}(\sigma), B(\sigma))$ and $G_{B_i}(\boldsymbol{x}(\sigma), B(\sigma))$ $(i=1, 2,\dots, d+1)$, respectively, and $\frac{d\boldsymbol{x}}{d\sigma}$ and $\frac{dB_i}{d\sigma}$ $(i=1, 2,\dots, d+1)$ are the derivatives of $\boldsymbol{x}(\sigma)$ and $B_i(\sigma)$ $(i=1, 2,\dots, d+1)$ with respect to σ, respectively, that is,

$$\frac{d\boldsymbol{x}}{d\sigma}=\frac{d\boldsymbol{x}}{d\sigma}(\sigma)=\left(\frac{dx}{d\sigma}(\sigma), \frac{dh_1}{d\sigma}(\sigma),\dots, \frac{dh_d}{d\sigma}(\sigma)\right)^T \quad \text{and} \quad \frac{dB_i}{d\sigma}=\frac{dB_i}{d\sigma}(\sigma) \quad (1\leq i\leq d+1).$$

Differentiating the both sides of (2.22) with respect to σ, we have

$$\begin{cases} \left(G_{\boldsymbol{xx}}\cdot\frac{d\boldsymbol{x}}{d\sigma}+\sum_{i=1}^{d+1} G_{\boldsymbol{x}B_i}\cdot\frac{dB_i}{d\sigma}\right)\frac{d\boldsymbol{x}}{d\sigma}+G_{\boldsymbol{x}}\cdot\frac{d^2\boldsymbol{x}}{d\sigma^2}+\sum_{i=1}^{d+1}\left(\frac{d}{d\sigma}G_{B_i}\right)\frac{dB_i}{d\sigma} \\ +\sum_{i=1}^{d+1} G_{B_i}\cdot\frac{d^2B_i}{d\sigma^2}=0, \end{cases} \tag{2.23}$$

where

$G_{\boldsymbol{xx}}$ denotes the second derivative of $G(\boldsymbol{x}, B)$ with respect to $\boldsymbol{x}$;

G_{xB_i} $(i=1, 2,\ldots, d+1)$ denote the partial derivatives of $G_x(x, B)$ with respect to B_i; $\frac{d^2x}{d\sigma^2}$ and $\frac{d^2B_i}{d\sigma^2}$ $(i=1, 2,\ldots, d+1)$ denote the second derivatives of $x(\sigma)$ and $B_i(\sigma)$ $(i=1, 2,\ldots, d+1)$ with respect to σ, respectively, that is,

$$\frac{d^2x}{d\sigma^2}=\frac{d^2x}{d\sigma^2}(\sigma)=\left(\frac{d^2x}{d\sigma^2}(\sigma),\ \frac{d^2h_1}{d\sigma^2}(\sigma),\ldots,\ \frac{d^2h_d}{d\sigma^2}(\sigma)\right)^T \quad \text{and} \quad \frac{d^2B_i}{d\sigma^2}=\frac{d^2B_i}{d\sigma^2}(\sigma)$$
$$(1\leq i\leq d+1);$$

$\frac{d}{d\sigma}G_{B_i}$ $(i=1, 2,\ldots, d+1)$ denote $\frac{d}{d\sigma}\{G_{B_i}(x(\sigma), B(\sigma))\}$ $(i=1, 2,\ldots, d+1)$.

We shall show that

$$\frac{dB}{d\sigma}(\hat{\sigma})=\left(\frac{dB_1}{d\sigma}(\hat{\sigma}),\ \frac{dB_2}{d\sigma}(\hat{\sigma}),\ldots,\ \frac{dB_{d+1}}{d\sigma}(\hat{\sigma})\right)^T=0 \tag{2.24}$$

and

$$\frac{d^2B}{d\sigma^2}(\hat{\sigma})=\left(\frac{d^2B_1}{d\sigma^2}(\hat{\sigma}),\ \frac{d^2B_2}{d\sigma^2}(\hat{\sigma}),\ldots,\ \frac{d^2B_{d+1}}{d\sigma^2}(\hat{\sigma})\right)^T\neq 0\,. \tag{2.25}$$

This implies that $\hat{B}=B(\hat{\sigma})$ is an extremum of the function $B(\sigma)$ which expresses a curve projected from the curve Γ into the parameter space, that is, the B-space. Hence it is sufficient to show that (2.24) and (2.25) hold.

From (2.8), (2.9) and (2.10), it follows that

$$\operatorname{rank} G_x(\hat{x},\ \hat{B})=(d+1)n-1. \tag{2.26}$$

By (2.11) and (2.26), we see that

$$\frac{dB}{d\sigma}(\hat{\sigma})=\left(\frac{dB_1}{d\sigma}(\hat{\sigma}),\ \frac{dB_2}{d\sigma}(\hat{\sigma}),\ldots,\ \frac{dB_{d+1}}{d\sigma}(\hat{\sigma})\right)^T=0\,. \tag{2.27}$$

This shows that the equality (2.24) holds. Next we will also show that (2.25) holds. From (2.27), at $(x, B)=(\hat{x}, \hat{B})$ (or at $\sigma=\hat{\sigma}$), it follows that both the equations (2.22) and (2.23) become

$$G_x(\hat{x},\ \hat{B})\frac{dx}{d\sigma}(\hat{\sigma})=0 \tag{2.28}$$

and

$$\begin{cases} \left\{G_{xx}(\hat{x},\ \hat{B})\frac{dx}{d\sigma}(\hat{\sigma})\right\}\frac{dx}{d\sigma}(\hat{\sigma})+G_x(\hat{x},\ \hat{B})\frac{d^2x}{d\sigma^2}(\hat{\sigma}) \\ +\sum_{i=1}^{d+1} G_{B_i}(\hat{x},\ \hat{B})\frac{d^2B_i}{d\sigma^2}(\hat{\sigma})=0\,, \end{cases} \tag{2.29}$$

respectively. We consider the vector $\{G_{xx}(\hat{x}, \hat{B})\frac{dx}{d\sigma}(\hat{\sigma})\}\frac{dx}{d\sigma}(\hat{\sigma})$. When we put

$$(2.30)\quad k_1(\sigma) = \frac{dx}{d\sigma}(\sigma),\ k_2(\sigma) = \frac{dh_1}{d\sigma}(\sigma),\ k_3(\sigma) = \frac{dh_2}{d\sigma}(\sigma), \dots, k_{d+1}(\sigma) = \frac{dh_d}{d\sigma}(\sigma),$$

from (2.12), we have

$$(2.31)\qquad G_x(x, B)\frac{dx}{d\sigma}(\sigma) = \begin{pmatrix} {}_0C_0X^{(0)}k_1(\sigma) \\ {}_1C_0X^{(0)}k_2(\sigma) + {}_1C_1X^{(1)}k_1(\sigma) \\ \sum_{i=0}^{2} {}_2C_iX^{(i)}k_{3-i}(\sigma) \\ \vdots \\ \sum_{i=0}^{d} {}_dC_iX^{(i)}k_{d+1-i}(\sigma) \\ \phi(k_1(\sigma), k_2(\sigma), \dots, k_{d+1}(\sigma)) \end{pmatrix},$$

where $k_i(\sigma) = (k_i^1(\sigma), k_i^2(\sigma), \dots, k_i^n(\sigma))^T$ $(i = 1, 2, \dots, d+1)$, and $\phi(k_1(\sigma), k_2(\sigma), \dots, k_{d+1}(\sigma)) = (k_2^1(\sigma), k_3^1(\sigma), \dots, k_{d+1}^1(\sigma))^T$.

We set

$$(2.32)\qquad \begin{cases} Y^{(0)}(\sigma) = F_x(x(\sigma), B(\sigma)), \\ Y^{(m+1)}(\sigma) = \sum_{i=0}^{m} {}_mC_iX_x^{(i)}(\sigma)k_{m+1-i}(\sigma) \qquad (0 \le m \le d), \end{cases}$$

where $k_1(\sigma) = \frac{dx}{d\sigma}(\sigma)$, $k_j(\sigma) = \frac{dh_{j-1}}{d\sigma}(\sigma)$ $(j = 2, 3, \dots, d+1)$, and $X_x^{(i)}(\sigma)$ $(i = 0, 1, \dots, d)$ denote the values of $X_x^{(i)}$ at $(x, B) = (x(\sigma), B(\sigma))$. Here $X_x^{(i)}$ $(i = 0, 1, \dots, d)$ are previously mentioned in (2.2). Then, the $\{(d+1)n + d\} \times (d+1)n$ matrix $G_{xx}(x(\sigma), B(\sigma))\frac{dx}{d\sigma}(\sigma)$ can be written in the form

$$(2.33)\qquad G_{xx}(x(\sigma), B(\sigma))\frac{dx}{d\sigma}(\sigma) = \left(\begin{array}{cc} {}_0C_0Y^{(1)}(\sigma) & 0 \\ {}_1C_0Y^{(2)}(\sigma) & {}_1C_1Y^{(1)}(\sigma) \\ {}_2C_0Y^{(3)}(\sigma) & {}_2C_1Y^{(2)}(\sigma) \\ \vdots & \vdots \\ {}_dC_0Y^{(d+1)}(\sigma) & {}_dC_1Y^{(d)}(\sigma) \\ \hline \multicolumn{2}{c}{0} \end{array}\right.$$

$$\left.\begin{matrix} 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \\ {}_2C_2Y^{(1)}(\sigma) & \cdots\cdots & 0 \\ \vdots & & \vdots \\ {}_dC_2Y^{(d-1)}(\sigma) & \cdots\cdots & {}_dC_dY^{(1)}(\sigma) \\ \hline & 0 & \end{matrix}\right),$$

from which it follows that

$$(2.34)\quad \left\{G_{xx}(x(\sigma),\ B(\sigma))\frac{dx}{d\sigma}(\sigma)\right\}\frac{dx}{d\sigma}(\sigma)=\begin{pmatrix} {}_0C_0Y^{(1)}(\sigma)k_1(\sigma) \\ {}_1C_0Y^{(1)}(\sigma)k_2(\sigma)+{}_1C_1Y^{(2)}(\sigma)k_1(\sigma) \\ \sum_{i=0}^{2}{}_2C_iY^{(i+1)}(\sigma)k_{3-i}(\sigma) \\ \vdots \\ \sum_{i=0}^{d}{}_dC_iY^{(i+1)}(\sigma)k_{d+1-i}(\sigma) \\ \boldsymbol{0} \end{pmatrix},$$

where $\boldsymbol{0}$ denotes the d-dimensional zero vector.

Assume that the parameter σ of the curve Γ is chosen so that $\frac{dx_1}{d\sigma}(\sigma)=1$. For example, we can take $\sigma=x_1$, where x_1 is the first component of $x=(x_1,\dots,x_n)^T$. Then $\left(x(\hat\sigma),\frac{dx}{d\sigma}(\hat\sigma),\ B(\hat\sigma)\right)^T=\left(x(\hat\sigma),\frac{dx}{d\sigma}(\hat\sigma),\ \frac{dh_1}{d\sigma}(\hat\sigma),\dots,\frac{dh_d}{d\sigma}(\hat\sigma),\ B(\hat\sigma)\right)^T$ is a solution of the system (2.14). In fact, in this case, we have

$$(2.35)\quad \hat x=x(\hat\sigma),\ \hat h_1=\frac{dx}{d\sigma}(\hat\sigma),\ \hat h_2=\frac{dh_1}{d\sigma}(\hat\sigma),\dots,\ \hat h_{d+1}=\frac{dh_d}{d\sigma}(\hat\sigma),\ \hat B=B(\hat\sigma),$$

where $\hat z=(\hat x,\ \hat h_1,\ \hat h_2,\dots,\ \hat h_{d+1},\ \hat B)^T$ is the previously mentioned solution of (2.14) which is referred to in Theorem. Therefore we have

$$(2.36)\qquad Y^{(0)}(\hat\sigma)=\hat X^{(0)},\ Y^{(1)}(\hat\sigma)=\hat X^{(1)},\dots,\ Y^{(d+1)}(\hat\sigma)=\hat X^{(d+1)},$$

where $\hat X^{(i)}$ $(i=0,1,2,\dots,d+1)$ denote the values of $X^{(i)}$ $(i=0,1,2,\dots,d+1)$ at $z=\hat z$. Then, by (2.4) and (2.36), we have the equalities

$$(2.37)\qquad \hat\mu_m=\sum_{i=0}^{m-1}{}_{m-1}C_iY^{(i+1)}(\hat\sigma)\hat k_{m-i}\qquad (1\le m\le d+1),$$

where $\hat\mu_m$ $(m=1,2,\dots,d+1)$ denote the values of μ_m $(m=1,2,\dots,d+1)$ at $z=\hat z$, and $\hat k_1=k_1(\hat\sigma)=\frac{dx}{d\sigma}(\hat\sigma)$, $\hat k_2=k_2(\hat\sigma)=\frac{dh_1}{d\sigma}(\hat\sigma),\dots,\ \hat k_{d+1}=k_{d+1}(\hat\sigma)=\frac{dh_d}{d\sigma}(\hat\sigma)$. Thus, the vector $\{G_{xx}(x(\hat\sigma),\ B(\hat\sigma))\frac{dx}{d\sigma}(\hat\sigma)\}\frac{dx}{d\sigma}(\hat\sigma)$ is of the form

(2.38) $$\{G_{xx}(x(\hat{\sigma}), B(\hat{\sigma}))\frac{dx}{d\sigma}(\hat{\sigma})\}\frac{dx}{d\sigma}(\hat{\sigma})=(\hat{\mu}_1, \hat{\mu}_2,\dots, \hat{\mu}_{d+1}, \boldsymbol{0})^T,$$

where $\boldsymbol{0}$ is the d-dimensional zero vector. When we put

$$\hat{\delta}=(\hat{\mu}_1, \hat{\mu}_2,\dots, \hat{\mu}_{d+1}, \boldsymbol{0})^T,$$

by (2.38), the equation (2.29) can be written in the form

(2.39) $$\hat{\delta}+G_x(\hat{x}, \hat{B})\frac{d^2x}{d\sigma^2}(\hat{\sigma})+\sum_{i=1}^{d+1} G_{B_i}(\hat{x}, \hat{B})\frac{d^2B_i}{d\sigma^2}(\hat{\sigma})=0.$$

From (2.11), (2.26) and (2.39) it follows that

(2.40) $$\begin{cases} \dfrac{d^2B}{d\sigma^2}(\hat{\sigma})=\left(\dfrac{d^2B_1}{d\sigma^2}(\hat{\sigma}), \dfrac{d^2B_2}{d\sigma^2}(\hat{\sigma}),\dots, \dfrac{d^2B_{d+1}}{d\sigma^2}(\hat{\sigma})\right)^T \neq 0 \text{ is} \\ \text{equivalent to rank } (\hat{\delta}, G_x(\hat{x}, \hat{B}))=(d+1)n. \end{cases}$$

Since

(2.41) $$\hat{\mu}_{i+1}+\sum_{j=0}^{i} {}_iC_j\hat{X}^{(i-j)}\hat{h}_{2+j}=0 \qquad (0\leq i\leq d-1)$$

and

(2.42) $$\hat{\mu}_{d+1}+\sum_{j=0}^{d-1} {}_dC_j\hat{X}^{(d-j)}\hat{h}_{2+j}=\hat{l}_{d+1},$$

we have

(2.43) $$\text{rank}(\hat{\delta}, G_x(\hat{x}, \hat{B}))=\text{rank}(\hat{\zeta}, G_x(\hat{x}, \hat{B})),$$

where $\hat{\zeta}=(0, 0,\dots, 0, \hat{l}_{d+1}, \boldsymbol{0})^T$. Here 0 is the n-dimensional zero vector and $\boldsymbol{0}$ is the d-dimensional zero vector. From (2.26), we have

(2.44) $$\text{rank}(\hat{\zeta}, G_x(\hat{x}, \hat{B}))=(d+1)n \text{ is equivalent to (2.15).}$$

Hence, by (2.40), (2.43) and (2.44), we have

(2.45) $$\begin{cases} \dfrac{d^2B}{d\sigma^2}(\hat{\sigma})=\left(\dfrac{d^2B_1}{d\sigma^2}(\hat{\sigma}), \dfrac{d^2B_2}{d\sigma^2}(\hat{\sigma}),\dots, \dfrac{d^2B_{d+1}}{d\sigma^2}(\hat{\sigma})\right)^T \neq 0 \text{ is} \\ \text{equivalent to (2.15).} \end{cases}$$

Thus, when the singular point $(\hat{x}, \hat{B})$ of the equation (1.1) satisfies the condition (2.15), the B-component $\hat{B}$ of the singular point $(\hat{x}, \hat{B})$ can be characterized as an extremum of the function $B(\sigma)$ which expresses a curve projected from the curve Γ into the B-space. From (2.45), due to Theorem, we also have

$$(2.46)\qquad \begin{cases} \det S'(\hat{z}) \neq 0 \text{ is equivalent to} \\ \dfrac{d^2B}{d\sigma^2}(\hat{\sigma}) = \left(\dfrac{d^2B_1}{d\sigma^2}(\hat{\sigma}),\ \dfrac{d^2B_2}{d\sigma^2}(\hat{\sigma}), \dots,\ \dfrac{d^2B_{d+1}}{d\sigma^2}(\hat{\sigma})\right)^T \neq 0 \end{cases}$$

for the solution $\hat{z}$ of (2.14), where $S'(z)$ is the Jacobian matrix of $S(z)$ defined by the equality (2.14) which is referred to in Theorem.

Especially, in the case $d=0$, both the equations (2.28) and (2.29) become

$$(2.47)\qquad F_x(\hat{x}, \hat{B})\frac{dx}{d\sigma}(\hat{\sigma}) = 0$$

and

$$(2.48)\qquad \left\{F_{xx}(\hat{x}, \hat{B})\frac{dx}{d\sigma}(\hat{\sigma})\right\}\frac{dx}{d\sigma}(\hat{\sigma}) + F_x(\hat{x}, \hat{B})\frac{d^2x}{d\sigma^2}(\hat{\sigma}) + \frac{d^2B}{d\sigma^2}(\hat{\sigma})F_B(\hat{x}, \hat{B}) = 0$$

respectively, because $G(\boldsymbol{x}, B) = F(x, B)$ from (2.6). As has been mentioned in the above argument, if the parameter σ of the curve Γ is chosen so that $\frac{dx_1}{d\sigma}(\sigma) = 1$, then the equation (2.48) can be rewritten in the form

$$(2.49)\qquad \hat{l}_1 + F_x(\hat{x}, \hat{B})\frac{d^2x}{d\sigma^2}(\hat{\sigma}) + \frac{d^2B}{d\sigma^2}(\hat{\sigma})F_B(\hat{x}, \hat{B}) = 0,$$

since $\hat{\mu}_1 = \hat{l}_1$. Therefore, if the condition $\operatorname{rank}(F_0(\hat{x}, \hat{B}), F_B(\hat{x}, \hat{B})) = n$ is satisfied, then we have

$$(2.50)\qquad \frac{d^2B}{d\sigma^2}(\hat{\sigma}) \neq 0 \text{ is equivalent to } \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_1) = n.$$

In this case, the system (2.14) is of the form

$$(2.51)\qquad S(z) = \begin{pmatrix} F(x, B) \\ F_x(x, B)h \\ h_1 - 1 \end{pmatrix} = 0,$$

where $z = (x, h, B)^T$, $x = (x_1, \dots, x_n)^T$, $h = (h_1, \dots, h_n)^T$. Since $\frac{dx}{d\sigma}(\hat{\sigma})$ is a solution of the equation

$$(2.52)\qquad \begin{cases} F_x(\hat{x}, \hat{B})h = 0, \\ h_1 - 1 = 0 \end{cases}$$

from (2.47), the system (2.51) certainly has a solution $\hat{z} = (\hat{x}, \hat{h}, \hat{B})^T$, where $\hat{h} = \frac{dx}{d\sigma}(\hat{\sigma})$. Due to Theorem, for this solution $\hat{z}$, we have

$$(2.53)\qquad \det S'(\hat{z}) \neq 0 \text{ is equivalent to } \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_1) = n$$

and also

$$\det S'(\hat{z}) \neq 0 \text{ is equivalent to } \frac{d^2 B}{d\sigma^2}(\hat{\sigma}) \neq 0 . \tag{2.54}$$

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References

[1] Seydel, R., Numerical computation of branch points in ordinary differential equations, *Numer. Math.*, **32** (1979), 51–68.

[2] Seydel, R., Numerical computation of branch points in nonlinear equations, *Numer. Math.*, **33** (1979), 339–352.

[3] Moore, G. and Spence, A., The calculation of turning points of nonlinear equations, *SIAM J. Numer. Anal.*, **17** (1980), 567–576.

[4] Pönish, G. and Schwetlick, H., Computing turning points of curves implicitly defined by nonlinear equations depending on a parameter, *Computing*, **26** (1981), 107–121.

[5] Melhem, R. G. and Rheinboldt, W. C., A comparison of methods for determining turning points of nonlinear equations, *Computing*, **29** (1982), 201–226.

[6] Kawakami, H., Bifurcations of periodic responces in forced dynamic nonlinear circuits: Computation of bifurcation values of the system parameters, to appear in *IEEE Trans. Circuits and Systems*.

[7] Spence, A. and Werner, B., Non-simple turning points and cusps, *IMA J. Numer. Anal.*, **2** (1982), 413–427.

[8] Yamamoto, N., Newton's method for singular problems and its application to boundary value problems, *J. Math. Tokushima Univ.*, **17** (1983), 27–88.