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On Quasiperiodic Solutions to Van der Pol Equation

By

Yoshitane SHINOHARA, Atsuhito KOHDA,
and Taketomo MITSUI

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§1. Introduction

T. Mitsui [2], one of the authors, has established a useful numero-analytical method of investigation of numerical solutions to two nonlinear quasiperiodic differential equations, that is, Duffing type and Van der Pol type. But he fails to estimate the norm of the Green function to Van der Pol equation.

In the present paper we correct his failures.

§2. Notations and Fundamental Theorem

A function $f(t)$ is said to be *quasiperiodic* with periods $\omega_1,\dots,\omega_m$ if $f(t)$ is represented as

$$f(t)=f_0(t, t,\dots, t) \tag{2.1}$$

for some continuous periodic function $f_0(u_1, u_2,\dots, u_m)$ with period ω_i in each u_i.

A linear differential operator

$$Lz=\frac{dz}{dt}-A(t)z \tag{2.2}$$

is said to be *almost periodic* (or *quasiperiodic*) if $A(t)$ is almost periodic (or quasiperiodic) matrix.

An almost periodic operator L is said to be *regular* if and only if for any almost periodic function $f(t)$ the equation

$$Lz=f(t) \tag{2.3}$$

has at least one solution bounded for all $t\in J$, where J denotes the real line.

A quasiperiodic operator is said to be *regular* if it is regular as an almost periodic operator.

Let $\Phi(t)$ be the fundamental matrix of the linear homogeneous equation

$$Lz=0 \tag{2.4}$$

satisfying the initial condition $\Phi(0)=E$ (unit martix). Then we have

Proposition 1 ([1]). *L is regular if and only if there is a square matrix P such that*

(i) $P^2=P$,

(ii) $\|\Phi(t)P\Phi^{-1}(s)\| \leqq Ce^{-\sigma(t-s)}$ *for* $t\geqq s$,

(iii) $\|\Phi(t)(E-P)\Phi^{-1}(s)\| \leqq Ce^{-\sigma(s-t)}$ *for* $t<s$,

where C and σ are positive numbers.

Proposition 2 ([3]). *If a quasiperiodic operator L with periods* $\omega_1, \omega_2,\dots, \omega_m$ *defined by* (2.2) *is regular, then for any quasiperiodic function* $f(t)$ *with periods* $\omega_1, \omega_2,\dots, \omega_m$ *the differential equation* (2.3) *possesses a unique quasiperiodic solution* $z=z(t)$ *with the same periods given by*

$$z(t)=\int_{-\infty}^{\infty} G(t, s)f(s)ds, \tag{2.5}$$

where

$$G(t, s)=\begin{cases} \Phi(t)P\Phi^{-1}(s) & \text{for } t\geqq s, \\ -\Phi(t)(E-P)\Phi^{-1}(s) & \text{for } t<s. \end{cases} \tag{2.6}$$

$G(t, s)$ is called a Green function for L, and satisfies the inequality

$$\|G(t, s)\| \leqq Ce^{-\sigma|t-s|}. \tag{2.7}$$

Our numerical analysis for the quasiperiodic oscillations is based on the following fundamental theorem.

Theorem 1 ([3]). *Given a nonlinear differential equation*

$$\frac{dz}{dt}=X(t, z), \tag{2.8}$$

where z and $X(t, z)$ *are vectors and* $X(t, z)$ *is quasiperiodic in t with periods* $\omega_1, \omega_2,\dots, \omega_m$ *and is continuously differentiable with respect to z belonging to a region* $\mathscr{D}$ *of z-space.*

Suppose that there is a quasiperiodic function $z_0(t)$ *with periods* $\omega_1, \omega_2,\dots, \omega_m$ *such that*

$$\begin{cases} z_0(t)\in\mathscr{D}, \\ \left\|\dfrac{dz_0}{dt}(t)-X[t, z_0(t)]\right\| \leqq r \end{cases}$$

for all $t\in J$. *Further suppose that there are a positive number* δ, *a nonnegative number* $\kappa<1$ *and a quasiperiodic matrix* $A(t)$ *with periods* $\omega_1, \omega_2,\dots, \omega_m$ *such that*

(i) *the quasiperiodic differential operator L defined by* (2.2) *is regular*;

$$(2.9)\quad \text{(ii)}\quad \begin{cases} \mathscr{D}_\delta=\{z;\ \|z-z_0(t)\|\leqq\delta \ \ \textit{for some}\ \ t\in J\}\subset\mathscr{D}, \\ \|\Psi(t,z)-A(t)\|\leqq\dfrac{\kappa}{M}\ \ \textit{whenever}\ \ \|z-z_0(t)\|\leqq\delta, \\ \dfrac{Mr}{1-\kappa}\leqq\delta. \end{cases}$$

Here $\Psi(t, z)$ *is the Jacobian matrix of* $X(t, z)$ *with respect to* z *and*

$$(2.10)\qquad M=\frac{2C}{\sigma},$$

where C *and* σ *are positive numbers satisfying* (2.7).

Then the given equation (2.8) *possesses a solution* $z=\hat{z}(t)$ *quasiperiodic in* t *with periods* $\omega_1, \omega_2,\dots, \omega_m$ *such that*

$$(2.11)\qquad \|z_0(t)-\hat{z}(t)\|\leqq\frac{Mr}{1-\kappa}$$

for all $t\in J$. *For the solution* $\hat{z}(t)$, *a quasiperiodic differential operator* $\hat{L}$ *defined by*

$$\hat{L}y=\frac{dy}{dt}-\Psi[t, \hat{z}(t)]y$$

is regular. Furthermore, to equation (2.8) *there is no other quasiperiodic solution belonging to* $\mathscr{D}_\delta$ *besides* $z=\hat{z}(t)$.

§3. Van der Pol equation

Consider a Van der Pol equation with a quasiperiodic forcing term such as

$$(3.1)\qquad \frac{d^2x}{dt^2}-2\lambda(1-x^2)\frac{dx}{dt}+x=a\cos\nu_1 t+b\cos\nu_2 t,$$

where λ is a positive parameter, $\nu_1=\dfrac{2\pi}{\omega_1}$, $\nu_2=\dfrac{2\pi}{\omega_2}$, ω_2/ω_1 is irrational and neither ν_1 nor ν_2 are equal to 1.

The equation (3.1) can be written into the vector form

$$(3.2)\qquad \frac{dz}{dt}=A(\lambda)z+\phi(t)+\lambda\eta(z),$$

where

$$z=\begin{pmatrix} x \\ y \end{pmatrix},\quad A(\lambda)=\begin{pmatrix} 0 & 1 \\ -1 & 2\lambda \end{pmatrix},\quad \phi(t)=\begin{pmatrix} 0 \\ a\cos\nu_1 t+b\cos\nu_2 t \end{pmatrix}$$

$$\eta(z)=\begin{pmatrix} 0 \\ -2x^2y \end{pmatrix}.$$

Consider a linear differential operator depending on λ such that

$$L(\lambda)w=\frac{dw}{dt}-A(\lambda)w, \tag{3.3}$$

then we have that $L(\lambda)$ is regular as a quasiperiodic operator for $\lambda<1$ and that the Green function for $L(\lambda)$ is given by

$$G(t, s)=\begin{cases} 0 \quad \textit{for } t\geqq s, \\ -e^{\lambda(t-s)}\begin{pmatrix} \cos\theta(t-s)-\frac{\lambda}{\theta}\sin\theta(t-s) & \frac{1}{\theta}\sin\theta(t-s) \\ -\frac{1}{\theta}\sin\theta(t-s) & \cos\theta(t-s)+\frac{\lambda}{\theta}\sin\theta(t-s) \end{pmatrix} \\ \qquad\qquad \textit{for } \quad t<s, \end{cases} \tag{3.4}$$

where $\theta=\sqrt{1-\lambda^2}$. Hence, for the ℓ_∞ norm, the Green function $G(t, s)$ satisfies the inequality

$$\|G(t, s)\|\leqq\frac{\sqrt{2+2\lambda}}{\theta}e^{-\lambda|t-s|}, \tag{3.5}$$

and the quasiperiodic solution to the linear equation

$$L(\lambda)z=\phi(t) \tag{3.6}$$

is given by $z=z_0(t;\lambda)=\begin{pmatrix} x_0(t;\lambda) \\ y_0(t;\lambda)\end{pmatrix}$, where

$$x_0(t;\lambda)=\frac{a}{(1-v_1^2)^2+4\lambda^2v_1^2}\{(1-v_1^2)\cos v_1t-2\lambda v_1\sin v_1t\}+$$
$$+\frac{b}{(1-v_2^2)^2+4\lambda^2v_2^2}\{(1-v_2^2)\cos v_2t-2\lambda v_2\sin v_2t\},$$

$$y_0(t;\lambda)=\frac{av_1}{(1-v_1^2)^2+4\lambda^2v_1^2}\{-(1-v_1^2)\sin v_1t-2\lambda v_1\cos v_1t\}+$$
$$+\frac{bv_2}{(1-v_2^2)^2+4\lambda^2v_2^2}\{-(1-v_2^2)\sin v_2t-2\lambda v_2\cos v_2t\}.$$

Define the constant number K by

$$K=\max\left(\frac{|a|}{|1-v_1^2|}+\frac{|b|}{|1-v_2^2|}, \frac{|a|v_1}{|1-v_1^2|}+\frac{|b|v_2}{|1-v_2^2|}\right), \tag{3.7}$$

then we have the following estimate

$$|x_0(t;\lambda)|, \quad |y_0(t;\lambda)|<K \tag{3.8}$$

for all $t \in J$ and $0<\lambda<1$. Using the estimate (3.8) for $x_0(t; \lambda)$ and $y_0(t; \lambda)$, we can estimate the residual function for $z_0(t; \lambda)$ in the following form:

$$\begin{aligned}&\left\|\frac{dz_0(t;\lambda)}{dt}-A(\lambda)z_0(t;\lambda)-\phi(t)-\lambda\eta(z_0(t;\lambda))\right\|\\&=\|-\lambda\eta(z_0(t;\lambda))\|\\&\leqq 2\lambda|x_0^2(t;\lambda)y_0(t;\lambda)|\\&\leqq 2\lambda K^3.\end{aligned}$$

Hence we can choose

$$r=2\lambda K^3. \tag{3.9}$$

Let $D_k=\{z;\ \|z\|\leqq 2K\}$, $D'=\bigcup_{t\in J}\{z;\ \|z-z_0(t;\lambda)\|\leqq K\}$. It is clear that $z_0(t;\lambda)\in D_k$ for any $t\in J$ and $D'\subset D_k$.

Let us denote the Jacobian matrix of the right-hand side of (3.2) with respect to z by $\Psi(z;\lambda)$. Then we have for $z\in D'$ the inequality

$$\begin{aligned}\|\Psi(z;\lambda)-A(\lambda)\|&\leqq 2\lambda(2|y|+|x|)|x|\\&\leqq 24\lambda K^2.\end{aligned} \tag{3.10}$$

By (2.10), the inequality (3.5) implies

$$M=\frac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}}. \tag{3.11}$$

In order to apply Theorem 1 to the present case, we have to check with the inequalities in (2.9). The question is "Is it possible to take a nonnegative number $\kappa<1$ satisfying the both inequalities

$$24\lambda K^2\leqq\frac{\lambda\sqrt{1-\lambda^2}}{2\sqrt{2+2\lambda}}\kappa, \tag{3.12}$$

$$\frac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}}\cdot 2\lambda K^3\leqq(1-\kappa)K \tag{3.13}$$

under the assumption $0<\lambda<1$?".

The answer is affirmative when the inequality

$$K\leqq\sqrt{\frac{\sqrt{1-\lambda^2}}{52\sqrt{2+2\lambda}}}$$

holds, because then we have the inequalities

$$\begin{cases} \dfrac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}} \cdot 24\lambda K^2 \leqq \dfrac{48}{52}, \\ \dfrac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}} \cdot 2\lambda K^2 \leqq \dfrac{4}{52}, \end{cases}$$

so that we can obtain such a nonnegative number $\kappa < 1$ that the both inequalities (3.12) and (3.13) hold. Hence we have

Theorem 2 ([2]). *If* $0<\lambda<1$ *holds, and if the constant*

$$K = \max\left(\frac{|a|}{|1-\nu_1^2|} + \frac{|b|}{|1-\nu_2^2|}, \frac{|a|\nu_1}{|1-\nu_1^2|} + \frac{|b|\nu_2}{|1-\nu_2^2|}\right)$$

satisfies the inequality

$$K \leqq \sqrt{\frac{\sqrt{1-\lambda^2}}{52\sqrt{2+2\lambda}}}, \tag{3.14}$$

then the given equation (3.1) *possesses a quasiperiodic solution* $z = \hat{z}(t)$ *with periods* ω_1, ω_2 *such that*

$$\|\hat{z}(t) - z_0(t;\, \lambda)\| \leqq K \qquad \text{for all } t \in J.$$

If the inequality (3.14) does not hold, in order to assure the existence of an exact quasiperiodic solution $\hat{z}(t)$ to (3.1) we have to compute a more accurate approximate solution than $z_0(t;\, \lambda)$. For this purpose we have considered an approximate quasiperiodic solution written in the form

$$x_m(t) = \alpha(0, 0) + \sum_{r=1}^{m} \sum_{|p|=r} \{\alpha_p \cos(p, \nu)t + \beta_p \sin(p, \nu)t\},$$

$$y_m(t) = \frac{d}{dt} x_m(t), \text{ where } (p, \nu) = p_1\nu_1 + p_2\nu_2,\ |p| = |p_1| + |p_2|,$$

and determined the unknown coefficients $\alpha(0, 0)$, α_p, β_p by means of Galerkin method. For the detail computations, see [2].

Thus for the computed Galerkin approximation

$$\bar{x}_m(t) = \bar{\alpha}(0, 0) + \sum_{r=1}^{m} \sum_{|p|=r} \{\bar{\alpha}_p \cos(p, \nu)t + \bar{\beta}_p \sin(p, \nu)t\}$$

we have the residual function

$$r(t) = \frac{d^2\bar{x}_m(t)}{dt^2} - 2\lambda(1 - \bar{x}_m^2(t))\frac{d\bar{x}_m(t)}{dt} + \bar{x}_m(t) - a\cos\nu_1 t - b\cos\nu_2 t$$

which can be expanded into the finite double Fourier series as follows;

$$r(t)=f(0, 0)+\sum_{r=1}^{3m}\sum_{|p|=r}\{f_p\cos(p, v)t+g_p\sin(p, v)t\}.$$

Put

$$r=|f(0, 0)|+\sum_{r=1}^{3m}\sum_{|p|=r}\{|f_p|+|g_p|\}, \tag{3.15}$$

then we have

$$|r(t)|\leqq r \qquad \textit{for all } t\in J.$$

Define

$$\Omega=|\bar{\alpha}(0, 0)|+\sum_{r=1}^{m}\sum_{|p|=r}(|\bar{\alpha}_p|+|\bar{\beta}_p|), \tag{3.16}$$

and

$$\Omega'=\sum_{r=1}^{m}\sum_{|p|=r}|(p, v)|(|\bar{\alpha}_p|+|\bar{\beta}_p|), \tag{3.17}$$

then we have the inequalities

$$\Omega\geqq\sup|\bar{x}_m(t)| \quad \text{and} \quad \Omega'\geqq\sup|\bar{y}_m(t)|.$$

For z which lies into the δ-neighbourhood of $\bar{z}_m(t)={}^t(\bar{x}_m(t), \bar{y}_m(t))$, we have

$$\|\Psi(z; \lambda)-A(\lambda)\|\leqq 2\lambda\{\Omega(2\Omega'+\Omega)+2(\Omega'+2\Omega)\delta+3\delta^2\}.$$

By (3.11), for the Green function $G(t, s)$, we have

$$M=\frac{2\sqrt{2}+2\lambda}{\lambda\sqrt{1-\lambda^2}}. \tag{3.18}$$

If there exist a non-negative number $\kappa<1$ and a positive number δ satisfying the both inequalities

$$2\lambda\{\Omega(2\Omega'+\Omega)+2(\Omega'+2\Omega)\delta+3\delta^2\}\leqq\frac{\lambda\sqrt{1-\lambda^2}}{2\sqrt{2}+2\lambda}\kappa, \tag{3.19}$$

$$\frac{r}{1-\kappa}\cdot\frac{2\sqrt{2}+2\lambda}{\lambda\sqrt{1-\lambda^2}}\leqq\delta, \tag{3.20}$$

then, by Theorem 1, the exact quasiperiodic solution $\hat{z}(t)={}^t(\hat{x}(t), \hat{y}(t))$ with periods ω_1 and ω_2 exists and the error estimation of $\bar{z}_m$ is given by

$$\|\bar{z}_m(t)-\hat{z}(t)\|\leqq\frac{r}{1-\kappa}\cdot\frac{2\sqrt{2}+2\lambda}{\lambda\sqrt{1-\lambda^2}}, \tag{3.21}$$

that is,

$$|\bar{x}_m(t)-\hat{x}(t)|,\quad \left|\frac{d}{dt}\bar{x}_m(t)-\frac{d}{dt}\hat{x}(t)\right| \leqq \frac{r}{1-\kappa}\cdot\frac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}}$$

for all $t\in J$.

§4. Numerical Results

In the previous paper [2], one of the authors has computed the quasiperiodic solutions to Van der Pol equation (3.1) with $v_1=\sqrt{2}$, $v_2=\sqrt{5}$. As for the case $\lambda=1/8$, $a=1/16$, $b=1/16$, Theorem 2 is not true, because the inequality (3.14) does not hold. Hence, by the Galerkin method we have computed

$$\begin{aligned}(4.1)\qquad \bar{x}_8(t)=2\{&-0.0277900\cos v_1t-0.0098059\sin v_1t\\ &-0.0076639\cos v_2t-0.0010672\sin v_2t\\ &-0.0000009\cos 3v_1t+0.0000005\sin 3v_1t\\ &-0.0000011\cos(2v_1+v_2)t+0.0000009\sin(2v_1+v_2)t\\ &+0.0000019\cos(2v_1-v_2)t-0.0000057\sin(2v_1-v_2)t\\ &-0.0000002\cos(v_1+2v_2)t+0.0000002\sin(v_1+2v_2)t\\ &-0.0000006\sin(v_1-2v_2)t\}.\end{aligned}$$

We can obtain M in (3.18) as

$$M=\frac{2\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}}=24.189726.$$

By (3.15), (3.16) and (3.17) we can obtain $r=1.9\times10^{-9}$, $\Omega=0.09267759$ and $\Omega'=0.1454338$, respectively. If we take $\delta=1/16$, we have

$$2\lambda(2\Omega'+\Omega+3\delta)(\Omega+\delta)\leqq 0.02215336\leqq\frac{\kappa}{M}$$

and

$$\kappa\geqq 0.02215336M=0.53588370.$$

Hence we can choose κ as 0.54, then we have

$$\frac{Mr}{1-\kappa}=9.9914\cdots\times10^{-8}<0.10\times10^{-6}<\delta.$$

Thus the Galerkin approximation (4.1) satisfies the both inequalities (3.19) and (3.20). Hence, from Theorem 1, we can assure that the exact quasiperiodic solution $\hat{x}(t)$ exists in the δ-neighbourhood of $\bar{x}_8(t)$ and we have an error estimation of $\bar{x}_8(t)$ as

$$|\bar{x}_8(t) - \hat{x}(t)| < 0.10 \times 10^{-6}.$$

Similar to the above case $\lambda=1/8$, $a=1/16$, $b=1/16$, we have computed the Galerkin approximations to the cases $\lambda=1/16$, $a=1/16$, $b=1/8$; $\lambda=1/4$, $a=1/16$, $b=1/16$; and $\lambda=1/2$, $a=1/16$, $b=1/16$. (See [2]).

For these cases we apply Theorem 1 and assure in each case that the exact quasiperiodic solution $\hat{x}(t)$ exists in the δ-neighbourhood of the Galerkin approximation. Moreover, we obtain the error bounds listed in the following Table which corrects Table 2 given in the previous paper [2].

Table. *Correct error bounds*

λ	a	b	m	Ω	Ω'	r	δ	κ	*error bouuds*
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	8	0.09267759	0.1454338	1.9×10^{-9}	$\frac{1}{16}$	0.54	0.10×10^{-6}
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	8	0.1045902	0.1752701	2.9×10^{-9}	$\frac{1}{16}$	0.63	0.37×10^{-6}
$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	8	0.08966011	0.1420448	2.4×10^{-9}	$\frac{3}{32}$	0.79	0.15×10^{-6}
$\frac{1}{2}$	$\frac{1}{16}$	$\frac{1}{16}$	8	0.06885278	0.1126341	3.0×10^{-9}	$\frac{3}{32}$	0.75	0.96×10^{-7}

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