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# *Newton's Method for Singular Problems and its Application to Boundary Value Problems*

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## §1. Introduction

When we compute a solution $x=\hat{x}$ of a nonlinear equation $F(x)=0$, it is difficult to get a highly accurate approximation to the solution $\hat{x}$ by applying the Newton method to the equation $F(x)=0$ in the case where the Jacobian matrix $F_x(x)$ of $F(x)$ is singular at $x=\hat{x}$.

In the present paper, we propose a useful method for overcoming the difficulty arising from the singularity of the Jacobian matrix $F_x(\hat{x})$. Our method is as follows.

Let us introduce a parameter in the equation $F(x)=0$ and consider an enlarged system consisting of the original equation $F(x)=0$ and additional equations involving the Jacobian matrix.

Then this enlarged system has an isolated solution which contains $\hat{x}$ in question and hence we can obtain an approximation to this isolated solution as accurately as we desire by the Newton method. Here a solution of a nonlinear equation is called to be "*isolated*" if the Jacobian matrix of the nonlinear equation is nonsingular at the solution.

H. Weber and W. Werner [18] have proposed a method similar to ours. When the dimension of $\mathrm{Ker}(F_x(\hat{x}))$ is one and the intersection $\mathrm{Ker}(F_x(\hat{x}))\cap\mathrm{Im}(F_x(\hat{x}))$ consists of the zero vector alone, they have considered an enlarged system similar to (2.3) in §2 and they have obtained a result similar to Theorem 1 in §2, where $\mathrm{Ker}(F_x(\hat{x}))$ denotes the kernel of $F_x(\hat{x})$ and $\mathrm{Im}(F_x(\hat{x}))$ denotes the image of $F_x(\hat{x})$. But, when the condition $\mathrm{Ker}(F_x(\hat{x}))\cap\mathrm{Im}(F_x(\hat{x}))=\{0\}$ does not hold, they have considered a complicated enlarged system instead of the one similar to (2.3).

On the other hand, in our case, we consider only the system (2.3) whether the condition $\mathrm{Ker}(F_x(\hat{x}))\cap\mathrm{Im}(F_x(\hat{x}))=\{0\}$ holds or not. Hence our method for introducing a parameter in the equation $F(x)=0$ seems to be more useful and convenient than theirs. For details, see Remark 3 in §2 and Example 2 in §5. Further, when the Jacobian matrix $F_x(x)$ has a high singularity at the solution, they did not describe anything. On the other hand, we can consider the solution with the Jacobian matrix having the high singularity and we give a condition for classifying solutions with singular Jacobian matrices.

In the present paper, we also consider singular points of a nonlinear equation $F(x, B)=0$, where $B$ is a parameter. Here a point $(\hat{x}, \hat{B})$ satisfying the equation $F(x, B)=0$ is called a "*singular point*" if the Jacobian matrix $F_x(x, B)$ of $F(x, B)$ with respect to $x$ is singular at $(x, B)=(\hat{x}, \hat{B})$. Our method mentioned above is also effective for singular points of the nonlinear equation $F(x, B)=0$ and we have results similar to ones obtained for a solution of the equation $F(x)=0$ with a singular Jacobian matrix.

When the dimension of the parameter $B$ is one, our method is the same as the one proposed by R. Seydel [12]. But he did not describe the case where the dimension of the parameter $B$ is greater than one and he did not give any condition for guaranteeing the isolatedness of a solution of an enlarged system. On the other hand, we consider such a high dimensional case and we give the necessary and sufficient condition for the isolatedness of the solution of the enlarged system.

Furthermore, our method can be applied to boundary value problems of ordinary differential equations involving parameters.

In §3, we consider bifurcations of periodic solutions of nonlinear periodic systems and in §4, we consider multi-point boundary value problems. We also give an existence theorem of an exact solution of a multi-point boundary value problem involving parameters.

In §5, in order to illustrate our theory and method, we present some examples of solutions of nonlinear equations with singular Jacobian matrices. Further, we also give some examples of singular points of nonlinear equations defined by solutions of boundary value problems of ordinary differential equations involving parameters. These examples show the usefulness of our theory and method.

## §2. Solutions of Nonlinear Equations

### 2.1. The Case of a Nonlinear Equation $F(x)=0$

We consider a solution $x=\hat{x}$ of an $n$-dimensional nonlinear equation

$$F(x)=0 \tag{2.1}$$

such that the rank of the Jacobian matrix $F_x(x)$ is $n-1$ at $x=\hat{x}$, where the function $F(x)$ is defined on some neighborhood $D$ of $\hat{x}$ in the $x$-space and $F(x)$ is continuously differentiable with respect to $x$ in $D$.

In order to simplify the following argument, we assume that $\operatorname{rank} F_x(\hat{x})=\operatorname{rank} F_0(\hat{x})=n-1$, where $F_0(\hat{x})$ is the $n\times(n-1)$ matrix obtained from $F_x(\hat{x})$ by deleting the first column vector.

Then there exists a positive integer $k$ $(1\leq k\leq n)$ such that

$$\operatorname{rank}(F_0(\hat{x}),\ e_k)=n, \tag{2.2}$$

where $e_k$ is the unit vector directed along the $x_k$ axis, that is,

$$e_k = (0, \dots, 0, \underset{\hat{k}}{1}, 0, \dots, 0)^T.$$

Here $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$.

Now, making use of the singularity of the Jacobian matrix $F_x(\hat{x})$, we consider the following three types of nonlinear systems consisting of the equation $F(x)=0$ and additional equations involving the Jacobian matrix.

In the first place, since the equation $\begin{cases} F_x(\hat{x})h = 0 \\ h_1 - 1 = 0 \end{cases}$ has a solution due to rank $F_0(\hat{x}) = n-1$, we introduce a parameter $B$ in the equation $F(x)=0$ and we consider the system

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - Be_k \\ F_x(x)h \\ h_1 - 1 \end{pmatrix} = 0, \tag{2.3}$$

where $\boldsymbol{x} = (x, h, B)^T$, $x = (x_1, \dots, x_n)^T$ and $h = (h_1, \dots, h_n)^T$. Then, from rank $F_0(\hat{x}) = n-1$, the system (2.3) has a solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0)^T$ and the $x$-component $\hat{x}$ of $\hat{\boldsymbol{x}}$ is the desired solution of (2.1). For this solution $\hat{\boldsymbol{x}}$, we have the following theorem.

**Theorem 1.**

*Assume that the function $F(x)$ is twice continuously differentiable with respect to $x$ in $D$. Then the solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0)^T$ of (2.3) is isolated (that is,* $\det G'(\hat{\boldsymbol{x}}) \neq 0$*) if and only if*

$$\operatorname{rank}(F_0(\hat{x}), \hat{l}) = n, \tag{2.4}$$

*where $G'(\boldsymbol{x})$ denotes the Jacobian martrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\hat{l} = \{F_{xx}(\hat{x})\hat{h}\}\hat{h}$. Here $F_{xx}(x)$ denotes the second derivative of $F(x)$ with respect to $x$.*

Proof. Since $F(x)$ is twice continuously differentiable with respect to $x$, the function $G(\boldsymbol{x})$ defined by the equality (2.3) is continuously differentiable with respect to $\boldsymbol{x}$ and we have

$$G'(\boldsymbol{x}) = \begin{pmatrix} F_x(x) & 0 & -e_k \\ F_{xx}(x)h & F_x(x) & 0 \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix}, \tag{2.5}$$

from which it follows that

$$\begin{cases} \text{the solution } \hat{\boldsymbol{x}} \text{ of (2.3) is isolated if and} \\ \text{only if } \operatorname{rank}(F_0(\hat{x}), \hat{l}) = n. \end{cases} \tag{2.6}$$

This completes the proof. Q.E.D.

Thus, if the condition (2.4) is satisfied, we can get an approximation to the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0)^T$ of (2.3) as accurately as we desire by the Newton method.

When rank $(F_0(\hat{x}), \hat{l})=n-1$, since the equation

$$\text{(2.7)} \qquad \begin{cases} F_x(\hat{x})\tilde{k}+\hat{l}=0, \\ \tilde{k}_1=0 \end{cases}$$

has a solution $\tilde{k}=(\tilde{k}_1,\dots, \tilde{k}_n)^T$, we introduce one more parameter and we consider the system

$$\text{(2.8)} \qquad G_1(\boldsymbol{x}_1) = \begin{pmatrix} F(x)-B_1e_k \\ F_x(x)h_1-B_2e_k \\ F_x(x)h_2+\{F_{xx}(x)h_1\}h_1 \\ h_1^1-1 \\ h_2^1 \end{pmatrix} =0,$$

where $\boldsymbol{x}_1=(x, h_1, h_2, B_1, B_2)^T$ and $h_i=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2)$. Evidently, this system (2.8) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$, where $\hat{h}_2$ is a solution of (2.7). For this solution $\hat{\boldsymbol{x}}_1$, we readily get the following theorem.

**Theorem 2.**

*Assume that the function $F(x)$ is three times continuously differentiable with respect to $x$ in $D$. Then the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$ of (2.8) is isolated (that is, $\det G_1'(\hat{\boldsymbol{x}}_1)\neq 0$) if and only if*

$$\text{(2.9)} \qquad \operatorname{rank}(F_0(\hat{x}), \hat{l}_2)=n,$$

*where $G_1'(\boldsymbol{x}_1)$ denotes the Jacobian matrix of $G_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$ and $X^{(0)}=F_x(x)$, $X^{(1)}=X_x^{(0)}h_1$, $X^{(2)}=X_x^{(0)}h_2+X_x^{(1)}h_1$, $\hat{l}_2=\hat{X}^{(2)}\hat{h}_1+2\hat{X}^{(1)}\hat{h}_2$. Here $X_x^{(j)}$ $(j=0, 1)$ are the derivatives of $X^{(j)}$ $(j=0, 1)$ with respect to $x$, respectively, and $\hat{X}^{(i)}$ $(i=0, 1, 2)$ and $\hat{X}_x^{(j)}$ $(j=0, 1)$ mean the values of $X^{(i)}$ $(i=0, 1, 2)$ and $X_x^{(j)}$ $(j=0, 1)$ at $x=\hat{x}$, $h_1=\hat{h}_1$ and $h_2=\hat{h}_2$, respectively.*

More generally, let us suppose that the function $F(x)$ is $(d+2)$ times continuously differentiable with respect to $x$ in $D$ $(d\geq 2)$ and we put

$$\text{(2.10)} \qquad X^{(i+1)}=\sum_{k=0}^{i} {}_iC_k X_x^{(k)}h_{i+1-k} \qquad (1\leq i\leq d)$$

and

$$\text{(2.11)} \qquad l_i=\sum_{k=1}^{i} {}_iC_k X^{(k)}h_{i+1-k} \qquad (1\leq i\leq d+1)$$

where $X_x^{(j)}$ $(j=0, 1,\dots, d)$ are the derivatives of $X^{(j)}$ $(j=0, 1,\dots, d)$ with respect to $x$, respectively.

If there exists a vector $\hat{y}_d=(\hat{x}, \hat{h}_1,..., \hat{h}_d)^T$ (where $\hat{x}$ is of course a solution of (2.1) and satisfies the condition (2.2)) such that the conditions

(2.12) $\hat{X}^{(0)}\hat{h}_1=0,\ \hat{h}_1^1-1=0$ and $\hat{X}^{(0)}\hat{h}_{j+1}+\hat{l}_j=0,\ \hat{h}_{j+1}^1=0 \qquad (j=1, 2,..., d-1),$

(2.13) $n-1=\operatorname{rank} F_0(\hat{x})=\operatorname{rank}(F_0(\hat{x}), \hat{l}_1)=\cdots=\operatorname{rank}(F_0(\hat{x}), \hat{l}_d)$

are satisfied, then we introduce $(d+1)$ parameters $B_1, B_2,..., B_{d+1}$ and we consider the system

$$(2.14) \qquad G_d(\boldsymbol{x}_d)=\begin{pmatrix} F(x)-B_1e_k \\ X^{(0)}h_1-B_2e_k \\ X^{(0)}h_2+X^{(1)}h_1-B_3e_k \\ \vdots \\ \sum_{i=0}^{d-1} {}_{d-1}C_iX^{(i)}h_{d-i}-B_{d+1}e_k \\ \sum_{i=0}^{d} {}_dC_iX^{(i)}h_{d+1-i} \\ \psi_d(\boldsymbol{x}_d) \end{pmatrix}=0,$$

where $\boldsymbol{x}_d=(x, h_1, h_2,..., h_{d+1}, B_1, B_2,..., B_{d+1})^T$, $h_i=(h_i^1, h_i^2,..., h_i^n)^T$ $(i=1, 2,..., d+1)$, $\psi_d(\boldsymbol{x}_d)=(h_1^1-1, h_2^1,..., h_{d+1}^1)^T$ and $\hat{X}^{(k)}$ $(k=0, 1, 2..., d)$ and $\hat{l}_i$ $(i=1, 2,..., d)$ mean the values of $X^{(k)}$ $(k=0, 1, 2,..., d)$ and $l_i$ $(i=1, 2,..., d)$ at $x=\hat{x}$, $h_1=\hat{h}_1,..., h_d=\hat{h}_d$, respectively. By (2.12) and (2.13), the system (2.14) has a solution $\hat{\boldsymbol{x}}_d=(\hat{y}_d, \hat{h}_{d+1}, \boldsymbol{0})^T$ (where $\hat{h}_{d+1}$ is a solution of the equation $\hat{X}^{(0)}h_{d+1}+\hat{l}_d=0$, $h_{d+1}^1=0$ and $\boldsymbol{0}$ is the $(d+1)$-dimensional zero vector) and for this solution $\hat{\boldsymbol{x}}_d$, we have

**Theorem 3.**

*The solution* $\hat{\boldsymbol{x}}_d=(\hat{x}, \hat{h}_1, \hat{h}_2,..., \hat{h}_{d+1}, 0, 0,..., 0)^T$ *of* (2.14) *is isolated* (*that is,* $\det G_d'(\hat{\boldsymbol{x}}_d)\neq 0$) *if and only if*

$$(2.15) \qquad \operatorname{rank}(F_0(\hat{x}), \hat{l}_{d+1})=n,$$

*where* $G_d'(\boldsymbol{x}_d)$ *denotes the Jacobian matrix of* $G_d(\boldsymbol{x}_d)$ *with respect to* $\boldsymbol{x}_d$ *and* $\hat{l}_{d+1}$ *means the value of* $l_{d+1}$ *at* $x=\hat{x}$, $h_1=\hat{h}_1$, $h_2=\hat{h}_2,..., h_{d+1}=\hat{h}_{d+1}$.

PROOF. Since $F(x)$ is $(d+2)$ times continuously differentiable with respect to $x$ in $D$, the function $G_d(\boldsymbol{x}_d)$ defined by the equality (2.14) is continuously differentiable with respect to $\boldsymbol{x}_d$ and we have

$$
(2.16) \qquad G'_d(\boldsymbol{x}_d)=\left(\begin{array}{cccc}
{}_0C_0X^{(0)} & 0 & 0 & \cdots \\
{}_1C_0X^{(1)} & {}_1C_1X^{(0)} & 0 & \cdots \\
{}_2C_0X^{(2)} & {}_2C_1X^{(1)} & {}_2C_2X^{(0)} & \cdots \\
\vdots & \vdots & \vdots & \\
{}_dC_0X^{(d)} & {}_dC_1X^{(d-1)} & {}_dC_2X^{(d-2)} & \cdots \\
{}_{d+1}C_0X^{(d+1)} & {}_{d+1}C_1X^{(d)} & {}_{d+1}C_2X^{(d-1)} & \cdots \\
 & 10\cdots0 & 00\cdots0 & \cdots \\
\boxed{0} & 00\cdots0 & 10\cdots0 & \cdots \\
 & \boxed{0} & \boxed{0} & \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
\cdots & 0 & 0 & -e_k & 0\ \cdots\ 0 \\
\cdots & 0 & 0 & 0 & -e_k\cdots\ 0 \\
\cdots & 0 & 0 & 0 & 0\ \cdots\ 0 \\
 & \vdots & \vdots & \vdots & \vdots\ \ddots\ \vdots \\
\cdots & {}_dC_dX^{(0)} & 0 & 0 & 0\ \cdots -e_k \\
\cdots & {}_{d+1}C_dX^{(1)} & {}_{d+1}C_{d+1}X^{(0)} & 0 & 0\ \cdots\ 0 \\
\cdots & \boxed{0} & \boxed{0} & 0 & 0\ \cdots\ 0 \\
\ddots & & & 0 & 0\ \cdots\ 0 \\
 & & & \vdots & \vdots\ \ \vdots\ \ \vdots \\
 & 10\cdots0 & 00\cdots0 & 0 & 0\ \cdots\ 0 \\
 & 00\cdots0 & 10\cdots0 & 0 & 0\ \cdots\ 0
\end{array}\right).
$$

Then, by (2.16), we see that

$$(2.17) \qquad \det G'_d(\hat{x}_d)\neq 0 \text{ is equivalent to rank } (F_0(\hat{x}),\ \hat{l}_{d+1})=n.$$

This completes the proof. Q.E.D.

Thus, if the condition (2.15) is satisfied, we can get an approximation to the solution $\hat{x}_d$ of (2.14) as accurately as we desire by the Newton method.

Particularly, when $n=1$, $F(x)$ is a scalar-function and, in this case, the conditions (2.13) and (2.15) become

$$(2.18) \qquad 0=F_x(\hat{x})=\hat{l}_1=\cdots=\hat{l}_{d-1}=\hat{l}_d=0 \quad \text{and} \quad \hat{l}_{d+1}\neq 0.$$

Of course, in this case, $x$, $h_1$, $h_2$,..., $h_{d+1}$ are scalars.

In the second place, since $\det F_x(\hat{x})=0$, we may consider the system

$$(2.19) \qquad H(\boldsymbol{x})=\begin{pmatrix} F(x)-Be_k \\ g(x) \end{pmatrix}=0$$

instead of the system (2.3), where $\boldsymbol{x}=(x, B)^T$ and $g(x)=\det F_x(x)$. Then $\hat{\boldsymbol{x}}=(\hat{x}, 0)^T$ is certainly a solution of (2.19) and for this solutin $\hat{\boldsymbol{x}}$, we have

**Theorem 4.**

*The solution $\hat{\boldsymbol{x}}=(\hat{x}, 0)^T$ of* (2.19) *is isolated if and only if the condition* (2.4) *is satisfied.*

PROOF. We denote by $H'(\boldsymbol{x})$ the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and we have

$$(2.20) \qquad H'(\boldsymbol{x})=\begin{pmatrix} F_x(x) & -e_k \\ \dfrac{\partial g(x)}{\partial x} & 0 \end{pmatrix},$$

where $\dfrac{\partial g(x)}{\partial x}=\left(\dfrac{\partial g(x)}{\partial x_1},\dots, \dfrac{\partial g(x)}{\partial x_n}\right)$. Then, for the solution $\hat{\boldsymbol{x}}$, we have

$$(2.21) \qquad \det H'(\hat{\boldsymbol{x}})=\begin{vmatrix} F_x(\hat{x}) & -e_k \\ \dfrac{\partial g(\hat{x})}{\partial x} & 0 \end{vmatrix}=\begin{vmatrix} 0 & F_0(\hat{x}) & -e_k \\ \hat{\eta} & \dfrac{\partial g}{\partial x_2}\cdots\dfrac{\partial g}{\partial x_n} & 0 \end{vmatrix},$$

where $\hat{\eta}=\dfrac{1}{\hat{h}_1}\cdot\det(\hat{l}, F_0(\hat{x}))$. Here $\hat{h}=(\hat{h}_1,\dots, \hat{h}_n)^T$ is the $h$-component of the solution $(\hat{x}, \hat{h}, 0)^T$ of (2.3). By (2.21), we easily get

$$(2.22) \qquad \det H'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to } \hat{\eta}\neq 0.$$

This completes the proof. Q.E.D.

When rank $(F_0(\hat{x}), \hat{l})=n-1$, since rank $H'(\hat{\boldsymbol{x}})=n$, we may consider the system

$$(2.23) \qquad H_1(\boldsymbol{x}_1)=\begin{pmatrix} F(x)-B_1e_k \\ g(x)-B_2 \\ g_1(x) \end{pmatrix}=0$$

instead of the system (2.8), where $B_1$ and $B_2$ are parameters, $\boldsymbol{x}_1=(x, B_1, B_2)^T$ and $g_1(x)=\det H'(\boldsymbol{x})$.
The system (2.23) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, 0, 0)^T$ and for this solution $\hat{\boldsymbol{x}}_1$, analogously to Theorem 2, we have

**Theorem 5.**

*The solution $\hat{\boldsymbol{x}}_1=(\hat{x}, 0, 0)^T$ of* (2.23) *is isolated if and only if the condition* (2.9) *is satisfied.*

More generally, if the conditions (2.12) and (2.13) are satisfied, then we may consider the system

$$H_d(\boldsymbol{x}_d)=\begin{pmatrix} F(x)-B_1e_k \\ g_0(x)-B_2 \\ g_1(x)-B_3 \\ \vdots \\ g_{d-1}(x)-B_{d+1} \\ g_d(x) \end{pmatrix}=0 \tag{2.24}$$

instead of the system (2.14), where $\boldsymbol{x}_d=(x, B_1,\dots, B_{d+1})^T$, $g_0(x)=\det F_x(x)$ and $g_i(x)=\det H'_{i-1}(\boldsymbol{x}_{i-1})$ $(i=1, 2,\dots, d)$. Here $H'_{i-1}(\boldsymbol{x}_{i-1})$ $(i=1, 2,\dots, d)$ are the Jacobian matrices of

$$H_{i-1}(\boldsymbol{x}_{i-1})=\begin{pmatrix} F(x)-B_1e_k \\ g_0(x)-B_2 \\ g_1(x)-B_3 \\ \vdots \\ g_{i-2}(x)-B_i \\ g_{i-1}(x) \end{pmatrix} \qquad (i=1, 2,\dots, d;\ H_0(\boldsymbol{x}_0)=H(\boldsymbol{x})), \tag{2.25}$$

with respect to $\boldsymbol{x}_{i-1}=(x, B_1,\dots, B_i)^T$, respectively.
Then the system (2.24) has a solution $\hat{x}_d=(\hat{x}, 0, 0,\dots, 0)^T$ and for this solution $\hat{x}_d$, analogously to Theorem 3, we get

**Theorem 6.**
*The solution* $\hat{x}_d=(\hat{x}, 0, 0,\dots, 0)^T$ *of* (2.24) *is isolated if and only if the condition* (2.15) *is satisfied.*

The third type of nonlinear systems is led from the second one. From (2.21), we have

$$\det H'(\hat{x})=(-1)\cdot(-1)^{k+(n+1)}\begin{vmatrix} F_{x,-k}(\hat{x}) \\ \dfrac{\partial g(\hat{x})}{\partial x} \end{vmatrix}, \tag{2.26}$$

where $F_{x,-k}(\hat{x})$ is the $(n-1)\times n$ matrix obtained from $F_x(\hat{x})$ by deleting the $k$-th row vector. Then we may consider the system

$$I(x)=\begin{pmatrix} F_{-k}(x) \\ g(x) \end{pmatrix}=0 \tag{2.27}$$

instead of the system (2.19), where $F_{-k}(x)$ is the $(n-1)$-dimensional vector obtained from $F(x)$ by deleting the $k$-th component. Of course, $\hat{x}$ is a solution of (2.27). Analogously to Theorem 4, we have

**Theorem 7.**
*The solution $\hat{x}$ of* (2.27) *is isolated if and only if the condition* (2.4) *is satisfied.*

When rank $(F_0(\hat{x}), \hat{l}) = n-1$, we may consider the system

$$I_1(x) = \begin{pmatrix} F_{-k}(x) \\ g_1(x) \end{pmatrix} = 0 \tag{2.28}$$

instead of the system (2.23). Further, if the conditions (2.12) and (2.13) are satisfied, then we may consider the system

$$I_d(x) = \begin{pmatrix} F_{-k}(x) \\ g_d(x) \end{pmatrix} = 0 \tag{2.29}$$

instead of the system (2.24). Of course, $\hat{x}$ is a solution of (2.29). Then, analogously to Theorem 6, we readily get

**Theorem 8.**
*The solution $\hat{x}$ of* (2.29) *is isolated if and only if the condition* (2.15) *is satisfied.*

**Remark 1.**
In the case of the first type, eliminating the parameter $B$ in the system (2.3), we have

$$\tilde{G}(\tilde{x}) = \begin{pmatrix} F_{-k}(x) \\ F_x(x)h \\ h_1 - 1 \end{pmatrix} = 0, \tag{2.30}$$

where $\tilde{x} = (x, h)^T$. Thus, we may consider the system (2.30) instead of the system (2.3).

**Remark 2.**
Since the $n$-th component $g_1(x)$ of $I_1(x)$ defined by the equality (2.28) is of the form

$$g_1(x) = \det H'(\boldsymbol{x}) = \begin{vmatrix} F_x(x) & -e_k \\ \dfrac{\partial g(x)}{\partial x} & 0 \end{vmatrix} = (-1)\cdot(-1)^{k+(n+1)} \begin{vmatrix} F_{x,-k}(x) \\ \dfrac{\partial g(x)}{\partial x} \end{vmatrix},$$

we may consider the system

$$\tilde{I}_1(x) = \begin{pmatrix} F_{-k}(x) \\ \tilde{g}_1(x) \end{pmatrix} = 0 \tag{2.31}$$

instead of the system (2.28), where $\tilde{g}_1(x)=\det\begin{pmatrix} F_{x,-k}(x) \\ \dfrac{\partial g(x)}{\partial x} \end{pmatrix}$.

Similarly, instead of the system (2.29), we may consider the system

$$\tilde{I}_d(x)=\begin{pmatrix} F_{-k}(x) \\ \tilde{g}_d(x) \end{pmatrix}=0, \tag{2.32}$$

where $\tilde{g}_d(x)=\det\begin{pmatrix} F_{x,-k}(x) \\ \dfrac{\partial g_{d-1}}{\partial x}(x) \end{pmatrix}$.

**Remark 3.**

When $\dim \operatorname{Ker}(F_x(\hat{x}))=1$ and $\operatorname{Ker}(F_x(\hat{x}))\cap \operatorname{Im}(F_x(\hat{x}))=\{0\}$, H. Weber and W. Werner [18] have considered the system

$$W(\boldsymbol{x})=\begin{pmatrix} F(x)+Bh \\ F_x(x)h \\ h^Th-1 \end{pmatrix}=0 \tag{2.33a}$$

instead of the system (2.3), where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1,\dots, x_n)^T$, $h=(h_1,\dots, h_n)^T$ and $B$ is a parameter. Evidently, the system (2.33a) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0)^T$ (where $\hat{h}$ is a solution of the equation $F_x(\hat{x})h=0$, $h^Th-1=0$) and for this solution $\hat{\boldsymbol{x}}$, they have obtained a result similar to Theorem 1. But, when the condition $\operatorname{Ker}(F_x(\hat{x}))\cap \operatorname{Im}(F_x(\hat{x}))=\{0\}$ does not hold, the solution $\hat{\boldsymbol{x}}$ of (2.33a) is not isolated. Then they have considered the system

$$\tilde{W}(\boldsymbol{x})=\begin{pmatrix} F_x(x)^TF(x)+Bh \\ F_x(x)h \\ h^Th-1 \end{pmatrix}=0 \tag{2.33b}$$

instead of the system (2.33a), where $F_x(x)^T$ denotes the transposed matrix of $F_x(x)$.

On the other hand, in our case, we consider only the system (2.3) whether the condition $\operatorname{Ker}(F_x(\hat{x}))\cap \operatorname{Im}(F_x(\hat{x}))=\{0\}$ holds or not. Hence our method seems to be simpler than theirs. Particularly, comparing the system (2.33b) with the system (2.3), it seems that our method is more convenient than theirs. Of course, we may adopt the condition $h_1-1=0$ instead of the condition $h^Th-1=0$ in the systems (2.33a) and (2.33b) when $\operatorname{rank} F_x(\hat{x})=\operatorname{rank} F_0(\hat{x})=n-1$.

Now, we consider the case where $\operatorname{rank} F_x(\hat{x})=n-d$ $(1<d\le n)$. For the sake of simplicity, we assume that

$$n-d=\operatorname{rank} F_x(\hat{x})=\operatorname{rank} F_d(\hat{x}), \tag{2.34}$$

where $F_d(\hat{x})$ is the $n\times(n-d)$ matrix obtained from $F_x(\hat{x})$ by deleting the first column vector through the $d$-th column vector.

Then there exist $d$ positive integers $k_1, k_2,\dots, k_d$ $(1\le k_1, k_2,\dots, k_d\le n)$ such that

$$\text{rank}\,(F_d(\hat{x}),\ e_{k_1},\ e_{k_2},\dots,\ e_{k_d})=n, \tag{2.35}$$

where $e_{k_i}=(0,\dots,0,\underset{\hat{k}_i}{1},0,\dots,0)^T$ $(i=1, 2,\dots, d)$.

Then we introduce $d$ parameters $B_1, B_2,\dots, B_d$ in the equation $F(x)=0$ and we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x)-B_1e_{k_1}-B_2e_{k_2}-\cdots-B_de_{k_d} \\ F_x(x)h \\ \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_d \end{pmatrix}-\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \end{pmatrix}=0, \tag{2.36}$$

where $\boldsymbol{x}=(x, h, B_1,\dots, B_d)^T$, $x=(x_1,\dots, x_n)^T$, $h=(h_1,\dots, h_n)^T$ and $a=(a_1,\dots, a_d)^T$ is a $d$-dimensional non-zero constant vector. Then the condition (2.34) implies that the system (2.36) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0, 0,\dots, 0)^T$. For this solution $\hat{\boldsymbol{x}}$, we have

**Theorem 9.**

*Assume that the function $F(x)$ is twice continuously differentiable with respect to $x$ in $D$. Then the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0, 0,\dots, 0)^T$ of (2.36) is isolated if and only if*

$$\text{rank}\,(F_d(\hat{x}),\ \hat{m}_1,\ \hat{m}_2,\dots,\ \hat{m}_d)=n, \tag{2.37}$$

*where $\hat{m}_i=\{F_{xx}(\hat{x})\hat{h}\}\hat{h}^{(i)}$ $(1\le i\le d)$. Here $\hat{h}^{(i)}$ $(1\le i\le d)$ are solutions of the linear equations*

$$\begin{cases} F_x(\hat{x})h^{(i)}=0, \\ h_1^{(i)}=0, \\ \vdots \\ h_i^{(i)}=1, \\ \vdots \\ h_d^{(i)}=0, \end{cases} \qquad (\text{where } h^{(i)}=(h_1^{(i)}, h_2^{(i)},\dots, h_n^{(i)})^T) \tag{2.38}$$

*$(1\le i\le d)$, respectively.*

We must choose a $d$-dimensional vector $a(\neq 0)$ so that (2.37) is satisfied. But, in case of

$$\text{rank}\,(F_d(\hat{x}),\ \hat{m}_1,\ \hat{m}_2,\dots,\ \hat{m}_d)<n, \tag{2.39}$$

we regard the system $G(\boldsymbol{x})=0$ as the original equation $F(x)=0$ and we repeat the above-mentioned process for the system $G(\boldsymbol{x})=0$.

### 2.2 The Case of a Nonlinear Equation $F(x, B)=0$

We consider a point $(x, B)=(\hat{x}, \hat{B})$ satisfying a nonlinear equation

$$F(x, B)=0 \tag{2.40}$$

whose left member's Jacobian matrix with respect to $x$ is singular at $(\hat{x}, \hat{B})$, where $x$ and $F(x, B)$ are $n$-dimensional vectors and $F(x, B)$ is defined on some neighborhood $\Omega$ of $(\hat{x}, \hat{B})$ in the $(x, B)$-space and $F(x, B)$ is continuously differentiable with respect to $x$ and $B$ in $\Omega$. Here $B$ is a parameter and we assume that the dimension of the parameter $B$ is $m$ $(m \geq 1)$.

The point $(\hat{x}, \hat{B})$ mentioned above is called a "*singular point*" of the nonlinear equation $F(x, B)=0$.

At first, let us suppose that the values of $(m-1)$ components of $B$ are given and one and only one component of $B$ is unknown. For the sake of simplicity, we denote this unknown component of $B$ by $B$ and this is called the parameter $B$ with dimension one.

Now, we assume that

$$\operatorname{rank} F_x(\hat{x}, \hat{B})=n-1<n=\operatorname{rank}(F_x(\hat{x}, \hat{B}), F_B(\hat{x}, \hat{B})), \tag{2.41}$$

where $F_x(x, B)$ denotes the Jacobian matrix of $F(x, B)$ with respect to $x$ and $F_B(x, B)$ denotes the partial derivative of $F(x, B)$ with respect to $B$.
Further, in order to simplify the following argument, we assume that

$$\begin{cases} n-1=\operatorname{rank} F_x(\hat{x}, \hat{B})=\operatorname{rank} F_0(\hat{x}, \hat{B}) \\ \qquad <n=\operatorname{rank}(F_0(\hat{x}, \hat{B}), F_B(\hat{x}, \hat{B})), \end{cases} \tag{2.42}$$

where $F_0(\hat{x}, \hat{B})$ is the $n \times (n-1)$ matrix obtained from $F_x(\hat{x}, \hat{B})$ by deleting the first column vector.

Here we consider the following problem:

(2.43) *Look for a singular point* $(\hat{x}, \hat{B}) \in \Omega$ *satisfying* (2.42).

Since the equation $F_x(\hat{x}, \hat{B})h=0$ has a nontrivial solution due to $\operatorname{rank} F_x(\hat{x}, \hat{B})= n-1$, we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x, B) \\ F_x(x, B)h \\ h_1-1 \end{pmatrix}=0, \tag{2.44}$$

where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1, \ldots, x_n)^T$ and $h=(h_1, \ldots, h_n)^T$.
The system (2.44) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, \hat{B})^T$ (where $\hat{h}$ is a solution of the equation $F_x(\hat{x}, \hat{B})h=0$, $h_1-1=0$).

Then, the problem (2.43) is reduced to the problem of finding a solution of the system (2.44), because $(\hat{x}, \hat{B})$ is the desired singular point, where $\hat{x}$ and $\hat{B}$ are the $x$-component and the $B$-component of the solution $\hat{\boldsymbol{x}}$ of (2.44), respectively. Hence we have only to consider the system (2.44).

Analogously to Theorem 1, for the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, \hat{B})^T$ of (2.44), we have

**Theorem 10.**

*Assume that the function $F(x, B)$ is twice continuously differentiable with respect to $x$ and $B$ in $\Omega$. Then the solution $\hat{\boldsymbol{x}}$ of the system (2.44) is isolated if and only if*

$$\text{rank}\,(F_0(\hat{x}, \hat{B}), \hat{l})=n, \tag{2.45}$$

*where $\hat{l}=\{F_{xx}(\hat{x}, \hat{B})\hat{h}\}\hat{h}$. Here $F_{xx}(x, B)$ denotes the second derivative of $F(x, B)$ with respect to $x$.*

PROOF. Since $F(x, B)$ is twice continuously differentiable with respect to $x$ and $B$ in $\Omega$, $G(\boldsymbol{x})$ is continuously differentiable with respect to $\boldsymbol{x}$. We denote by $G'(\boldsymbol{x})$ the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and we have

$$G'(\boldsymbol{x})=\begin{pmatrix} F_x(x, B) & 0 & F_B(x, B) \\ F_{xx}(x, B)h & F_x(x, B) & F_{xB}(x, B)h \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix}, \tag{2.46}$$

where $F_{xB}(x, B)$ denotes the partial derivative of $F_x(x, B)$ with respect to $B$. By (2.42) and (2.46) we see that

$$\det G'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to (2.45).} \tag{2.47}$$

This completes the proof. Q. E. D.

Thus, if the condition (2.45) is satisfied, then we can get an approximation to the solution $\hat{\boldsymbol{x}}$ of (2.44) as accurately as we desire by the Newton method. Then we can also obtain a desired approximation to the singular point $(\hat{x}, \hat{B})$ of the equation $F(x, B)=0$ satisfying (2.42).

This singular point $(\hat{x}, \hat{B})$ is called a "*turning point*".

When $\text{rank}\,(F_0(\hat{x}, \hat{B}), \hat{l})=n-1$, since the equation

$$\begin{cases} F_x(\hat{x}, \hat{B})k+\hat{l}=0, \\ k_1=0 \end{cases}$$

has a solution $k=(k_1,\dots, k_n)^T$, we suppose that besides the previous unknown component of the parameter $B$, one of the given $(m-1)$ components of $B$ is unknown, that is, two components of $B$ are unknown and we consider the system

$$(2.48)\qquad G_1(\boldsymbol{x}_1)=\begin{pmatrix} F(x, B) \\ F_x(x, B)h_1 \\ F_x(x, B)h_2+l_1 \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0,$$

where $\boldsymbol{x}_1=(x, h_1, h_2, B)^T$, $h_i=(h_i^1,\ldots, h_i^n)^T$ $(i=1, 2)$, $B=(B_1, B_2)^T$ and $l_1=\{F_{xx}(x, B)h_1\}h_1$. Here we write two unknown components of the parameter $B$ as $B_1$ and $B_2$. For brevity, we denote this parameter $B$ by $B=(B_1, B_2)^T$ and this is called the parameter $B$ with dimension two.

Besides the conditions

$$(2.49)\qquad \begin{cases} n-1=\operatorname{rank} F_x(\hat{x}, \hat{B})=\operatorname{rank} F_0(\hat{x}, \hat{B}) \\ \qquad <n=\operatorname{rank}(F_0(\hat{x}, \hat{B}), F_{B_1}(\hat{x}, \hat{B})) \end{cases}$$

and

$$(2.50)\qquad n-1=\operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_1),$$

we assume that

$$(2.51)\qquad \operatorname{rank}\begin{pmatrix} F_x(\hat{x}, \hat{B}) & 0 & F_{B_1}(\hat{x}, \hat{B}) & F_{B_2}(\hat{x}, \hat{B}) \\ F_{xx}(\hat{x}, \hat{B})\hat{h}_1 & F_x(\hat{x}, \hat{B}) & F_{xB_1}(\hat{x}, \hat{B})\hat{h}_1 & F_{xB_2}(\hat{x}, \hat{B})\hat{h}_1 \end{pmatrix}=2n,$$

where $\hat{l}_1=\{F_{xx}(\hat{x}, \hat{B})\hat{h}_1\}\hat{h}_1$, and $F_{B_i}(x, B)$ and $F_{xB_i}(x, B)$ $(i=1, 2)$ are the partial derivatives of $F(x, B)$ and $F_x(x, B)$ with respect to $B_i$ $(i=1, 2)$, respectively.

From (2.49) and (2.50), the system (2.48) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{B})^T$ and for this solution $\hat{\boldsymbol{x}}_1$, analogously to Theorem 2, we easily get

**Theorem 11.**

*Assume that $F(x, B)$ is three times continuously differentiable with respect to $x$ and $B$ in $\Omega$. Then the solution $\hat{\boldsymbol{x}}_1$ of (2.48) is isolated if and only if*

$$(2.52)\qquad \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_2)=n,$$

*where $\hat{X}^{(0)}=F_x(\hat{x}, \hat{B})$, $\hat{X}^{(1)}=\hat{X}_x^{(0)}\hat{h}_1$, $\hat{X}^{(2)}=\hat{X}_x^{(0)}\hat{h}_2+\hat{X}_x^{(1)}\hat{h}_1$ and $\hat{l}_2=\hat{X}^{(2)}\hat{h}_1+2\hat{X}^{(1)}\hat{h}_2$. Here $X_x^{(i)}$ $(i=0, 1)$ are the derivatives of $X^{(i)}$ $(i=0, 1)$ with respect to $x$, respectively, and $\hat{X}^{(i)}$ $(i=0, 1, 2)$ and $\hat{X}_x^{(j)}$ $(j=0, 1)$ mean the values of $X^{(i)}$ $(i=0, 1, 2)$ and $X_x^{(j)}$ $(j=0, 1)$ at $x=\hat{x}$, $h_1=\hat{h}_1$, $h_2=\hat{h}_2$ and $B=\hat{B}$, respectively.*

This singular point $(\hat{x}, \hat{B})$ is called a "*cusp point*", where $\hat{x}$ and $\hat{B}$ are the $x$-component and the $B$-component of the solution $\hat{\boldsymbol{x}}_1$ of (2.48), respectively.

More generally, we suppose that $(d+1)$ components of the parameter $B$ are unknown $(2 \leq d \leq m-1)$ and we write these $(d+1)$ unknown components as $B_1, B_2, \dots, B_{d+1}$ and, for brevity, we denote this parameter $B$ by $B=(B_1, B_2, \dots, B_{d+1})^T$ and this is called the parameter $B$ with dimension $d+1$. Further, let us suppose that $F(x, B)$ is $(d+2)$ times continuously differentiable with respect to $x$ and $B$ in $\Omega$ $(2 \leq d \leq m-1)$.

Put

$$X^{(i+1)} = \sum_{k=0}^{i} {}_iC_k X_x^{(k)} h_{i+1-k} \qquad (1 \leq i \leq d) \tag{2.53}$$

and

$$l_i = \sum_{k=1}^{i} {}_iC_k X^{(k)} h_{i+1-k} \qquad (1 \leq i \leq d+1), \tag{2.54}$$

where $X^{(0)} = F_x(x, B)$, $X^{(1)} = X_x^{(0)} h_1$, $X^{(2)} = X_x^{(0)} h_2 + X_x^{(1)} h_1$. Here $X_x^{(j)}$ $(j=0, 1, \dots, d)$ are the derivatives of $X^{(j)}$ $(j=0, 1, \dots, d)$ with respect to $x$, respectively.

If there exists a vector $(\hat{y}_d, \hat{B})^T$ (where $\hat{y}_d = (\hat{x}, \hat{h}_1, \dots, \hat{h}_d)^T$, $\hat{B} = (\hat{B}_1, \dots, \hat{B}_{d+1})^T$ and $(\hat{x}, \hat{B})$ satisfies (2.40)) such that the conditions

$$\hat{X}^{(0)} \hat{h}_1 = 0,\ \hat{h}_1^1 - 1 = 0 \text{ and } \hat{X}^{(0)} \hat{h}_{j+1} + \hat{l}_j = 0,\ \hat{h}_{j+1}^1 = 0 \quad (j=1, 2, \dots, d-1), \tag{2.55}$$

$$n-1 = \operatorname{rank} F_0(\hat{x}, \hat{B}) = \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_1) = \cdots = \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_d) \tag{2.56}$$

and

$$\operatorname{rank}\begin{pmatrix} {}_0C_0\hat{X}^{(0)} & & & & 0 & \hat{U}_1 & \hat{U}_2 \cdots \hat{U}_{d+1} \\ {}_1C_0\hat{X}^{(1)} & {}_1C_1\hat{X}^{(0)} & & & & & \\ {}_2C_0\hat{X}^{(2)} & {}_2C_1\hat{X}^{(1)} & {}_2C_2\hat{X}^{(0)} & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ {}_dC_0\hat{X}^{(d)} & {}_dC_1\hat{X}^{(d-1)} & {}_dC_2\hat{X}^{(d-2)} & \cdots & {}_dC_d\hat{X}^{(0)} & & \end{pmatrix} = (d+1)n \tag{2.57}$$

are satisfied, then we consider the system

$$G_d(\boldsymbol{x}_d) = \begin{pmatrix} F(x, B) \\ X^{(0)} h_1 \\ X^{(0)} h_2 + X^{(1)} h_1 \\ \vdots \\ \sum_{k=0}^{d} {}_dC_k X^{(k)} h_{d+1-k} \\ \psi_d(\boldsymbol{x}_d) \end{pmatrix} = 0, \tag{2.58}$$

where $\boldsymbol{x}_d = (x, h_1, \dots, h_{d+1}, B)^T$, $x = (x_1, \dots, x_n)^T$, $h_i = (h_i^1, \dots, h_i^n)^T$ $(i=1, 2, \dots, d+1)$, $B = (B_1, \dots, B_{d+1})^T$, $\psi_d(\boldsymbol{x}_d) = (h_1^1 - 1, h_2^1, \dots, h_{d+1}^1)^T$ and $\hat{U}_i$ $(i=1, 2, \dots, d+1)$ are the following $n(d+1)$-dimensional vectors:

$$
(2.59) \qquad \hat{U}_i = \begin{pmatrix} F_{B_i}(\hat{x}, \hat{B}) \\ \hat{X}^{(0)}_{B_i}\hat{h}_1 \\ \hat{X}^{(0)}_{B_i}\hat{h}_2 + \hat{X}^{(1)}_{B_i}\hat{h}_1 \\ \vdots \\ \sum_{k=0}^{d-1} {}_{d-1}C_k \hat{X}^{(k)}_{B_i}\hat{h}_{d-k} \end{pmatrix} \qquad (i=1, 2,\dots, d+1).
$$

Here, $X^{(k)}_{B_i}$ $(k=0, 1,\dots, d;\ i=1, 2,\dots, d+1)$ are the partial derivatives of $X^{(k)}$ with respect to $B_i$, respectively, and $\hat{X}^{(k)}$ $(k=0, 1,\dots, d)$ and $\hat{l}_j (j=1, 2,\dots, d)$ mean the values of $X^{(k)}$ and $l_j$ at $x=\hat{x}$, $h_1=\hat{h}_1,\dots, h_d=\hat{h}_d$ and $B=\hat{B}$, respectively.

Then, by (2.55) and (2.56), the system (2.58) has a solution $\hat{\mathfrak{x}}_d=(\hat{\mathfrak{y}}_d, \hat{h}_{d+1}, \hat{B})^T$ (where $\hat{h}_{d+1}$ is a solution of the equation $\hat{X}^{(0)}h_{d+1}+\hat{l}_d=0$, $h^1_{d+1}=0$) and for this solution $\hat{\mathfrak{x}}_d$, analogously to Theorem 3, we readily get

**Theorem 12.**

*The solution* $\hat{\mathfrak{x}}_d=(\hat{\mathfrak{y}}_d, \hat{h}_{d+1}, \hat{B})^T$ *of* (2.58) *is isolated if and only if*

$$
(2.60) \qquad \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l}_{d+1})=n,
$$

*where* $\hat{l}_{d+1}$ *means the value of* $l_{d+1}$ *at* $x=\hat{x}$, $h_1=\hat{h}_1,\dots, h_{d+1}=\hat{h}_{d+1}$ *and* $B=\hat{B}$.

**Remark 4.**

When the dimension of the parameter $B$ is one, R. Seydel [11], [12] has considered the enlarged system (2.44), but he did not give the condition (2.45). When the dimension of the parameter $B$ is greater than one, the present paper is the first to consider singular points and to give a method for calculating them as far as the author is aware.

**Remark 5.**

J. P. Abbott [10] and H. Kawakami [23] have independently considered the system

$$
(2.61) \qquad H(\boldsymbol{x}) = \begin{pmatrix} F(x, B) \\ g(x, B) \end{pmatrix} = 0
$$

instead of the system (2.44) since $\det F_x(\hat{x}, \hat{B})=0$, where $\boldsymbol{x}=(x, B)^T$ and $g(x, B)=\det F_x(x, B)$.

The system (2.61) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{B})^T$ and for this solution $\hat{\boldsymbol{x}}$, we have

$$
(2.62) \qquad \begin{cases} \text{the matrix } H'(\hat{\boldsymbol{x}}) \text{ is non-singular if and only if} \\ \operatorname{rank}(F_0(\hat{x}, \hat{B}), \hat{l})=n, \end{cases}
$$

where $H'(\boldsymbol{x})$ denotes the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\hat{l}$ is the vector given in (2.45).

This tells us that the condition rank $(F_0(\hat{x}, \hat{B}), \hat{l})=n$ is essentially important also when we discuss whether the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{B})^T$ of (2.61) is isolated or not.

The proof of (2.62) is as follows: By $F\in C^2[\Omega]$, we have

$$H'(\boldsymbol{x})=\begin{pmatrix} F_x(x, B) & F_B(x, B) \\ \dfrac{\partial g(x, B)}{\partial x} & \dfrac{\partial g(x, B)}{\partial B} \end{pmatrix}, \tag{2.63}$$

where $\dfrac{\partial g}{\partial x}=\left(\dfrac{\partial g}{\partial x_1},\dots, \dfrac{\partial g}{\partial x_n}\right)$ and $\dfrac{\partial g}{\partial B}$ denotes the partial derivative of $g(x, B)$ with respect to $B$.

Then, for the solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{B})^T$ of (2.61), we have

$$\det H'(\hat{\boldsymbol{x}})=\begin{vmatrix} F_x(\hat{x}, \hat{B}) & F_B(\hat{x}, \hat{B}) \\ \dfrac{\partial g(\hat{x}, \hat{B})}{\partial x} & \dfrac{\partial g(\hat{x}, \hat{B})}{\partial B} \end{vmatrix} \tag{2.64}$$

$$=\begin{vmatrix} 0 & F_0(\hat{x}, \hat{B}) & F_B(\hat{x}, \hat{B}) \\ \hat{\eta} & \dfrac{\partial g}{\partial x_2}\cdots\dfrac{\partial g}{\partial x_n} & \dfrac{\partial g}{\partial B} \end{vmatrix},$$

where $\hat{\eta}=\dfrac{1}{\hat{h}_1}\cdot\det(\hat{l}, F_0(\hat{x}, \hat{B}))$. Here $\hat{h}$ is the $h$-component of the solution $(\hat{x}, \hat{h}, \hat{B})^T$ of (2.44). By (2.64), we have

$$\det H'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to } \hat{\eta}\neq 0. \tag{2.65}$$

Thus (2.62) follows from (2.65).

**Remark 6.**

When rank $(F_0(\hat{x}, \hat{B}), \hat{l})=n-1$, since rank $H'(\hat{\boldsymbol{x}})=n$, we may consider the system

$$H_1(\boldsymbol{x}_1)=\begin{pmatrix} F(x, B) \\ g(x, B) \\ g_1(x, B) \end{pmatrix}=0 \tag{2.66}$$

instead of the system (2.48), where the dimension of the parameter $B$ is two and $\boldsymbol{x}_1=(x, B)^T$, $B=(B_1, B_2)^T$ and $g_1(x, B)=\det H'(\boldsymbol{x}_1)$. Then the system (2.66) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{B})^T$ from rank $(F_0(\hat{x}, \hat{B}), \hat{l})=n-1$. We denote by $H_1'(\boldsymbol{x}_1)$ the Jacobian matrix of $H_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$ and for the solution $\hat{\boldsymbol{x}}_1$ of (2.66), we have

$$\begin{cases} \text{the matrix } H_1'(\hat{\boldsymbol{x}}_1) \text{ is non-singular if and only if} \\ \text{rank}\,(F_0(\hat{x}, \hat{B}), \hat{l}_2)=n, \end{cases} \tag{2.67}$$

where $\hat{l}_2$ is the vector given in (2.52).

When rank $(F_0(\hat{x}, \hat{B}), \hat{l}_2)=n-1$, by increasing the number of unknown components of the parameter $B$, we have a result similar to Theorem 12.

## §3. Periodic Solutions of Periodic Systems Involving Parameters

### 3.1. Singular Points Different from Bifurcation Points

We consider a $2\pi$-periodic solution $x(t)$ of an $n$-dimensional periodic system

$$\frac{dx}{dt}=X(x, B, t) \tag{3.1}$$

whose first variation equation with respect to $x=x(t)$

$$\frac{dh}{dt}=X_x(x(t), B, t)h \tag{3.2}$$

has a characteristic multiplier one, where $X(x, B, t)$ is periodic in $t$ of period $2\pi$ and is continuously differentiable with respect to $(x, B)$ in the region $\Delta\times R$. Here $\Delta$ is a given region of the $(x, B)$-space and $R$ is the real line and $B$ is a parameter and $X_x(x, B, t)$ denotes the Jacobian matrix of $X(x, B, t)$ with respect to $x$.

We assume that $X(x, B, t)$ and its first partial derivatives with respect to $(x, B)$ are all continuous on the region $\Delta\times R$.

Let $\varphi(t, x(0), B)$ be a solution of (3.1) at a given parameter $B$ such that $\varphi(0, x(0), B)=x(0)$. Of course, $(x(0), B)\in\Delta$. Then we consider the equation

$$\begin{aligned} F(x(0), B)&=\varphi(0, x(0), B)-\varphi(2\pi, x(0), B)\\ &=x(0)-\varphi(2\pi, x(0), B)=0. \end{aligned} \tag{3.3}$$

The $x(0)$-component $\tilde{x}(0)$ of $(\tilde{x}(0), \tilde{B})$ satisfying (3.3) is the initial value of a $2\pi$-periodic solution of (3.1) at $B=\tilde{B}$. That is, the solution of (3.1) at $B=\tilde{B}$ through $\tilde{x}(0)$ at $t=0$ is $2\pi$-periodic in $t$.

By the assumption, the function $F(x(0), B)$ defined by the equality (3.3) is continuously differentiable with respect to $(x(0), B)$ in $\Delta$ and we denote by $F_x(x(0), B)$ the Jacobian matrix of $F(x(0), B)$ with respect to $x(0)$. Then we have

$$F_x(x(0), B)=E_n-\Phi(2\pi), \tag{3.4}$$

where $E_n$ is the $n\times n$ unit matrix and $\Phi(t)$ is the fundamental matrix of (3.2) at $x=x(t)=\varphi(t, x(0), B)$ and the given $B$ satisfying the initial condition $\Phi(0)=E_n$.

Now, we assume that there exists a point $(\hat{x}(0), \hat{B})\in\Delta$ satisfying the equation (3.3) and also satisfying

$$\operatorname{rank} F_x(\hat{x}(0), \hat{B})=\operatorname{rank}[E_n-\hat{\Phi}(2\pi)]=n-1, \tag{3.5}$$

where $\hat{\Phi}(t)$ is the fundamental matrix of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying the initial condition $\hat{\Phi}(0)=E_n$. Here $\hat{x}(t)$ is a solution of (3.1) at $B=\hat{B}$ through $\hat{x}(0)$ at $t=0$. Indeed, $\hat{x}(t)$ is a $2\pi$-periodic solution of (3.1) at $B=\hat{B}$.
The condition (3.5) means that the first variation equation of (3.1) with respect to $x=\hat{x}(t)$ (at $B=\hat{B}$) has a characteristic multiplier one and it tells us that $(\hat{x}(0), \hat{B})$ is a singular point of the nonlinear equation $F(x(0), B)=0$. Hence our theory and method developed for singular points of a nonlinear equation $F(x, B)=0$ in Section 2.2 are also effective for ones of the nonlinear equation $F(x(0), B)=0$.

At first, let us suppose that the dimension of the parameter $B$ is one. Further, in order to simplify the following argument, we assume that

$$\text{rank}\,[E_n-\hat{\Phi}(2\pi)]=\text{rank}\,\hat{D}_1(2\pi)=n-1, \tag{3.6}$$

where $\hat{D}_1(2\pi)$ is the $n\times(n-1)$ matrix obtained from $E_n-\hat{\Phi}(2\pi)$ by deleting the first column vector.

As has been mentioned in Section 2.2, we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x(0), B) \\ F_x(x(0), B)k \\ k_1-1 \end{pmatrix}=0, \tag{3.7}$$

where $\boldsymbol{x}=(x(0), k, B)^T$ and $k=(k_1,\dots, k_n)^T$. By the condition (3.6), the system (3.7) has a solution $\hat{\boldsymbol{x}}=(\hat{x}(0), \hat{k}, \hat{B})^T$, where $\hat{k}$ is a solution of the equation $[E_n-\hat{\Phi}(2\pi)]k=0$, $k_1-1=0$. In fact, $(\hat{x}(0), \hat{k})^T$ is the initial value of a $2\pi$-periodic solution $(\hat{x}(t), \hat{h}(t))^T$ of the system

$$\begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\[2mm] \dfrac{dh}{dt}=X_x(x, B, t)h \end{cases} \tag{3.8}$$

at $B=\hat{B}$ satisfying the condition

$$h_1-1=0, \tag{3.9}$$

where $h(0)=(h_1,\dots, h_n)^T$. Indeed, $\hat{h}(t)=\hat{\Phi}(t)\hat{k}$.
Hence, the system (3.7) can be rewritten in the following way.

Let $(\varphi(t, \boldsymbol{x}), \varphi_1(t, \boldsymbol{x}))^T$ be a solution of (3.8) such that $(\varphi(0, \boldsymbol{x}), \varphi_1(0, \boldsymbol{x}))^T=(x(0), h(0))^T$, where $\boldsymbol{x}=(x(0), h(0), B)^T$. Then the system (3.7) is equivalent to the system

$$G(\boldsymbol{x})=\begin{pmatrix} \varphi(0, \boldsymbol{x})-\varphi(2\pi, \boldsymbol{x}) \\ \varphi_1(0, \boldsymbol{x})-\varphi_1(2\pi, \boldsymbol{x}) \\ h_1-1 \end{pmatrix}=0. \tag{3.10}$$

For the solution $\hat{x}$ of the system (3.10) (or (3.7)), analogously to Theorem 10, we have

**Theorem 13.**

*Assume that $X(x, B, t)$ is twice continuously differentiable with respect to $(x, B)$ in the region $\Delta \times R$ and $X(x, B, t)$ and its first and second partial derivatives with respect to $(x, B)$ are all continuous on the region $\Delta \times R$.*

*If the conditions*

$$\text{(3.11)} \qquad \begin{cases} n-1=\operatorname{rank}[E_n-\hat{\Phi}(2\pi)]=\operatorname{rank}\hat{D}_1(2\pi) \\ \qquad <n=\operatorname{rank}[\hat{D}_1(2\pi),\ \hat{\xi}_1(2\pi)] \end{cases}$$

*are satisfied, then the solution $\hat{x}$ of (3.10) (or (3.7)) is isolated if and only if*

$$\text{(3.12)} \qquad \operatorname{rank}(\hat{D}_1(2\pi),\ \hat{l}(2\pi))=n,$$

*where $\hat{\xi}_1(2\pi)=\hat{\Phi}(2\pi)\int_0^{2\pi}\hat{\Phi}^{-1}(s)X_B(\hat{x}(s), \hat{B}, s)\,ds$, $\hat{l}(2\pi)=-\hat{\Phi}_2(2\pi)\hat{h}(0)$ (of course, $\hat{h}(0)=\hat{k}$). Here $(\hat{\Phi}(t), \hat{\Phi}_2(t))^T$ is a solution ($2n\times n$ matrix) of the first variation equation of (3.8) with respect to $(x, h)^T=(\hat{x}(t), \hat{h}(t))^T$ (at $B=\hat{B}$) satisfying the initial condition $(\hat{\Phi}(0), \hat{\Phi}_2(0))^T=(E_n, 0)^T$ (0 is the $n\times n$ zero matrix) and $X_B(x, B, t)$ is the partial derivative of $X(x, B, t)$ with respect to $B$.*

PROOF. By the assumption of the theorem, the function $G(\boldsymbol{x})$ is continuously differentiable with respect to $\boldsymbol{x}$ and we denote the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ by $G'(\boldsymbol{x})$. Then we get

$$\text{(3.13)} \qquad G'(\boldsymbol{x})=\begin{pmatrix} E_n-\Phi(2\pi) & 0 & -\xi_1(2\pi) \\ -\Phi_2(2\pi) & E_n-\Phi(2\pi) & -\xi_2(2\pi) \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix},$$

where $(\xi_1(t), \xi_2(t))^T$ is a solution of the system

$$\text{(3.14)} \qquad \begin{cases} \dfrac{d\zeta_1}{dt}=X_x(x(t), B, t)\zeta_1+X_B(x(t), B, t), \\ \dfrac{d\zeta_2}{dt}=X_x(x(t), B, t)\zeta_2+\{X_{xx}(x(t), B, t)\zeta_1+X_{xB}(x(t), B, t)\}h(t) \end{cases}$$

satisfying the initial condition $(\xi_1(0), \xi_2(0))^T=(0, 0)^T$ and $(\Phi(t), \Phi_2(t))^T$ is a solution ($2n\times n$ matrix) of the first variation equation of (3.8) with respect to $(x, h)^T=(x(t), h(t))^T$ (at the given $B$) satisfying the initial condition $(\Phi(0), \Phi_2(0))^T=(E_n, 0)^T$. Here $X_{xx}(x, B, t)$ is the second derivative of $X(x, B, t)$ with respect to $x$ and $X_{xB}(x, B, t)$ is the partial derivative of $X_x(x, B, t)$ with respect to $B$. Thus, for the solution $\hat{\boldsymbol{x}}$ of (3.10) (or (3.7)), we have

$$\text{(3.15)} \qquad \det G'(\hat{x}) = \begin{vmatrix} E_n - \hat{\Phi}(2\pi) & 0 & -\hat{\xi}_1(2\pi) \\ -\hat{\Phi}_2(2\pi) & E_n - \hat{\Phi}(2\pi) & -\hat{\xi}_2(2\pi) \\ 00\cdots0 & 10\cdots0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \hat{D}_1(2\pi) & 0 & 0 & -\hat{\xi}_1(2\pi) \\ \hat{l}(2\pi) & \hat{D}_2(2\pi) & 0 & \hat{D}_1(2\pi) & -\hat{\xi}_2(2\pi) \\ 0 & 0\cdots0 & 1 & 0\cdots0 & 0 \end{vmatrix},$$

from which, by (3.11), it follows that

$$\text{(3.16)} \qquad \det G'(\hat{x}) \neq 0 \text{ is equivalent to (3.12),}$$

where $\hat{D}_2(2\pi)$ is the $n\times(n-1)$ matrix obtained from $-\hat{\Phi}_2(2\pi)$ by deleting the first column vector.

This completes the proof. Q. E. D.

The $2\pi$-periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (3.8)–(3.9) satisfying $\det G'(\hat{x}) \neq 0$ is called to be "isolated". In order to obtain an approximation to the isolated periodic solution of (3.8)–(3.9), we applied the Urabe-Galerkin method to (3.8)–(3.9) and we obtained a highly accurate approximation and, moreover, we gave a sharp error bound for it. For details, see [24] and [25].

When $\operatorname{rank}(\hat{D}_1(2\pi), \hat{l}(2\pi)) = n-1$, since the equation

$$\text{(3.17)} \qquad \begin{cases} [E_n - \hat{\Phi}(2\pi)]\tilde{k} + \hat{l}(2\pi) = 0, \\ \tilde{k}_1 = 0 \end{cases}$$

has a solution $\tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_n)^T$, analogously to (2.48) in Section 2.2, we consider the system

$$\text{(3.18)} \qquad G_1(\boldsymbol{x}_1) = \begin{pmatrix} F(x(0), B) \\ F_x(x(0), B)k_1 \\ F_x(x(0), B)k_2 + l_1 \\ k_1^1 - 1 \\ k_2^1 \end{pmatrix} = 0,$$

where the dimension of the parameter $B$ is two and $\boldsymbol{x}_1 = (x(0), k_1, k_2, B)^T$, $k_i = (k_i^1, k_i^2, \dots, k_i^n)^T$ $(i=1, 2)$, $B = (B_1, B_2)^T$ and $l_1 = \{F_{xx}(x(0), B)k_1\}k_1 = -\Phi_2(2\pi)k_1$. Here $F_{xx}(x(0), B)$ denotes the second derivative of $F(x(0), B)$ with respect to $x(0)$. Since $\operatorname{rank}\hat{D}_1(2\pi) = \operatorname{rank}(\hat{D}_1(2\pi), \hat{l}(2\pi)) = n-1$, the system (3.18) has a solution $\hat{x}_1 = (\hat{x}(0), \hat{k}_1, \hat{k}_2, \hat{B})^T$ and $(\hat{x}(0), \hat{k}_1, \hat{k}_2)^T$ is really the initial value of a $2\pi$-periodic solution $(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t))^T$ of the system

$$(3.19)\qquad \begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\ \dfrac{dh_1}{dt}=X_x(x, B, t)h_1, \\ \dfrac{dh_2}{dt}=X_x(x, B, t)h_2+\{X_{xx}(x, B, t)h_1\}h_1 \end{cases}$$

at $B=\hat{B}$ satisfying the conditions

$$(3.20)\qquad \begin{cases} h_1^1-1=0, \\ h_2^1=0, \end{cases}$$

where $h_i(0)=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2)$. Indeed, $\hat{h}_1(t)=\hat{\Phi}(t)\hat{k}_1$ and $\hat{h}_2(t)=\hat{\Phi}_2(t)\hat{k}_1+\hat{\Phi}(t)\hat{k}_2$.

Thus the system (3.18) can be rewritten in the following way.

Let $(\varphi(t, \boldsymbol{x}_1), \varphi_1(t, \boldsymbol{x}_1), \varphi_2(t, \boldsymbol{x}_1))^T$ be a solution of (3.19) with $(\varphi(0, \boldsymbol{x}_1), \varphi_1(0, \boldsymbol{x}_1), \varphi_2(0, \boldsymbol{x}_1))^T=(x(0), h_1(0), h_2(0))^T$, where $\boldsymbol{x}_1=(x(0), h_1(0), h_2(0), B)^T$. Then the system (3.18) is equivalent to the system

$$(3.21)\qquad G_1(\boldsymbol{x}_1)=\begin{pmatrix} \varphi(0, \boldsymbol{x}_1)-\varphi(2\pi, \boldsymbol{x}_1) \\ \varphi_1(0, \boldsymbol{x}_1)-\varphi_1(2\pi, \boldsymbol{x}_1) \\ \varphi_2(0, \boldsymbol{x}_1)-\varphi_2(2\pi, \boldsymbol{x}_1) \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0.$$

For the solution $\hat{\boldsymbol{x}}_1$ of (3.21) (or (3.18)), similarly to Theorem 11, we have

**Theorem 14.**

*Assume that $X(x, B, t)$ is three times continuously differentiable with respect to $(x, B)$ in the region $\Delta\times R$ and $X(x, B, t)$ and its first, second and third partial derivatives with respect to $(x, B)$ are all coutinuous on the region $\Delta\times R$ and that the conditions*

$$(3.22)\qquad \operatorname{rank}(\hat{D}_1(2\pi), \hat{\xi}_{11}(2\pi))=n$$

and

$$(3.23)\qquad \operatorname{rank}\begin{pmatrix} \hat{D}_1(2\pi) & 0 & \hat{\xi}_{11}(2\pi) & \hat{\xi}_{12}(2\pi) \\ \hat{D}_2(2\pi) & \hat{D}_1(2\pi) & \hat{\xi}_{21}(2\pi) & \hat{\xi}_{22}(2\pi) \end{pmatrix}=2n$$

*are satisfied. Then the solution $\hat{\boldsymbol{x}}_1$ of* (3.21) (*or* (3.18)) *is isolated if and only if*

$$(3.24)\qquad \operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_2(2\pi))=n,$$

*where* $\hat{\xi}_i(t)=(\hat{\xi}_{1i}(t), \hat{\xi}_{2i}(t))^T$ $(i=1, 2)$ *are solutions of*

$$(3.25)\quad \begin{cases} \dfrac{dz_1}{dt}=X_x(\hat{x}(t), \hat{B}, t)z_1+X_{B_i}(\hat{x}(t), \hat{B}, t), \\ \dfrac{dz_2}{dt}=X_x(\hat{x}(t), \hat{B}, t)z_2+\{X_{xx}(\hat{x}(t), \hat{B}, t)z_1+X_{xB_i}(\hat{x}(t), \hat{B}, t)\}\hat{h}_1(t) \end{cases}$$

*satisfying the initial condition* $\hat{\xi}_i(0)=(0, 0)^T$ $(i=1, 2)$, *respectively, and*

$$(3.26)\quad \hat{l}_2(2\pi)=-\hat{\Phi}_3(2\pi)\hat{h}_1(0)-2\hat{\Phi}_2(2\pi)\hat{h}_2(0).$$

*Here* $(\hat{\Phi}(t), \hat{\Phi}_2(t), \hat{\Phi}_3(t))^T$ *is a solution* $(3n\times n$ *matrix) of the first variation equation of* (3.19) *with respect to* $(x, h_1, h_2)^T=(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t))^T$ *(at* $B=\hat{B}$*) such that* $(\hat{\Phi}(0), \hat{\Phi}_2(0), \hat{\Phi}_3(0))^T=(E_n, 0, 0)^T$.

The $2\pi$-periodic solution $(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t), \hat{B})^T$ of (3.19)–(3.20) satisfying $\det G_1'(\hat{\mathfrak{x}}_1)\neq 0$ is also called to be "*isolated*".

More generally, we suppose that the dimension of the parameter $B$ is $(d+1)$ and $X(x, B, t)$ is $(d+2)$ times continuously differentiable with respect to $(x, B)$ in the region $\Delta\times R$ $(d\geq 2)$ and $X(x, B, t)$ and its first, second,..., $(d+2)$-th partial derivatives with respect to $(x, B)$ are all continuous on the region $\Delta\times R$.

Putting

$$(3.27)\quad X^{(i+1)}=\sum_{k=0}^{i} {}_iC_k X_x^{(k)}h_{i+1-k} \qquad (1\leq i\leq d),$$

we consider the system

$$(3.28)\quad \begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\ \dfrac{dh_1}{dt}=X^{(0)}h_1, \\ \dfrac{dh_2}{dt}=X^{(0)}h_2+X^{(1)}h_1, \\ \vdots \\ \dfrac{dh_{d+1}}{dt}=\sum_{k=0}^{d} {}_dC_k X^{(k)}h_{d+1-k}, \end{cases}$$

where $X^{(0)}=X_x(x, B, t)$, $X^{(1)}=X_x^{(0)}h_1$. Here $X_x^{(k)}$ $(0\leq k\leq d)$ are the derivatives of $X^{(k)}$ $(0\leq k\leq d)$ with respect to $x$, respectively.

Now, we assume that there exists a $2\pi$-periodic solution $(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t),\ldots, \hat{h}_{d+1}(t), \hat{B})^T$ of (3.28) satisfying the conditions

$$(3.29)\quad \begin{cases} h_1^1-1=0, \\ h_2^1=0, \\ \vdots \\ h_{d+1}^1=0 \end{cases}$$

and, moreover, we assume that for the solution $(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t),\dots, \hat{h}_{d+1}(t), \hat{B})^T$, the conditions

$$(3.30)\qquad \begin{cases} n-1=\operatorname{rank}\hat{D}_1(2\pi)=\operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_1(2\pi))=\cdots\cdots \\ \qquad =\operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_d(2\pi)) \\ \qquad <n=\operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_{d+1}(2\pi)) \end{cases}$$

and

$$(3.31)\qquad \operatorname{rank}\begin{pmatrix} {}_0C_0\hat{D}_1(2\pi) & & & & & \hat{\xi}_{11}(2\pi) & \hat{\xi}_{12}(2\pi) & \cdots\hat{\xi}_{1d+1}(2\pi) \\ {}_1C_0\hat{D}_2(2\pi) & {}_1C_1\hat{D}_1(2\pi) & & 0 & & \hat{\xi}_{21}(2\pi) & \hat{\xi}_{22}(2\pi) & \cdots\hat{\xi}_{2d+1}(2\pi) \\ {}_2C_0\hat{D}_3(2\pi) & {}_2C_1\hat{D}_2(2\pi) & {}_2C_2\hat{D}_1(2\pi) & & & \hat{\xi}_{31}(2\pi) & \hat{\xi}_{32}(2\pi) & \cdots\hat{\xi}_{3d+1}(2\pi) \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ {}_dC_0\hat{D}_{d+1}(2\pi) & {}_dC_1\hat{D}_d(2\pi) & {}_dC_2\hat{D}_{d-1}(2\pi) & \cdots & {}_dC_d\hat{D}_1(2\pi) & \hat{\xi}_{d+11}(2\pi) & \hat{\xi}_{d+12}(2\pi) & \cdots\hat{\xi}_{d+1d+1}(2\pi) \end{pmatrix}=(d+1)n$$

are satisfied, where $h_i(0)=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2,\dots, d+1)$ and

$$(3.32)\qquad \hat{l}_i(2\pi)=-\sum_{k=1}^{i}{}_iC_k\hat{\Phi}_{k+1}(2\pi)\hat{h}_{i+1-k}(0) \qquad (1\leq i\leq d+1)$$

and $\hat{D}_1(2\pi), \hat{D}_2(2\pi),\dots, \hat{D}_{d+1}(2\pi)$ are the $n\times(n-1)$ matrices obtained from $E_n-\hat{\Phi}(2\pi)$, $-\hat{\Phi}_2(2\pi),\dots, -\hat{\Phi}_{d+1}(2\pi)$ by deleting the first column vectors, respectively. Here $(\hat{\Phi}(t), \hat{\Phi}_2(t),\dots, \hat{\Phi}_{d+2}(t))^T$ is a solution ($(d+2)n\times n$ matrix) of the first variation equation of (3.28) with respect to $(x, h_1, h_2,\dots, h_{d+1})^T=(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t),\dots, \hat{h}_{d+1}(t))^T$ (at $B=\hat{B}$) such that $(\hat{\Phi}(0), \hat{\Phi}_2(0),\dots, \hat{\Phi}_{d+2}(0))^T=(E_n, 0,\dots, 0)^T$, and $\hat{\xi}_i(t)=(\hat{\xi}_{1i}(t), \hat{\xi}_{2i}(t),\dots, \hat{\xi}_{d+1i}(t))^T$ $(i=1, 2,\dots, d+1)$ are solutions of

$$(3.33)\qquad \begin{cases} \dfrac{dz_1}{dt}=\hat{X}^{(0)}z_1+X_{B_i}(\hat{x}(t), \hat{B}, t), \\ \dfrac{dz_2}{dt}=\hat{X}^{(0)}z_2+\hat{X}^{(1)}z_1+\hat{X}_{B_i}^{(0)}\hat{h}_1(t), \\ \vdots \\ \dfrac{dz_{d+1}}{dt}=\sum_{k=0}^{d}{}_dC_k\hat{X}^{(k)}z_{d+1-k}+\sum_{k=0}^{d-1}{}_{d-1}C_k\hat{X}_{B_i}^{(k)}\hat{h}_{d-k}(t) \end{cases}$$

satisfying the initial condition $\hat{\xi}_i(0)=(0, 0,\dots, 0)^T$ $(i=1, 2,\dots, d+1)$, respectively, where $\hat{X}^{(k)}$ $(k=0, 1,\dots, d)$ and $\hat{X}_{B_i}^{(j)}$ $(j=0, 1,\dots, d-1;\ i=1, 2,\dots, d+1)$ are the values of $X^{(k)}$ and $X_{B_i}^{(j)}$ at $x=\hat{x}(t)$, $h_1=\hat{h}_1(t),\ldots, h_{d+1}=\hat{h}_{d+1}(t)$ and $B=\hat{B}$, respectively. Here $X_{B_i}^{(j)}$ $(j=0, 1,\ldots, d-1;\ i=1,2,\ldots, d+1)$ are the partial derivatives of $X^{(j)}$ with respect to $B_i$, respectively.

Let $(\varphi(t, \boldsymbol{x}_d), \varphi_1(t, \boldsymbol{x}_d),\ldots, \varphi_{d+1}(t, \boldsymbol{x}_d))^T$ be a solution of (3.28) such that $(\varphi(0, \boldsymbol{x}_d), \varphi_1(0, \boldsymbol{x}_d),\ldots, \varphi_{d+1}(0, \boldsymbol{x}_d))^T=(x(0), h_1(0),\ldots, h_{d+1}(0))^T$ and let us consider the system

$$G_d(\boldsymbol{x}_d)=\begin{pmatrix} \varphi(0, \boldsymbol{x}_d)-\varphi(2\pi, \boldsymbol{x}_d) \\ \varphi_1(0, \boldsymbol{x}_d)-\varphi_1(2\pi, \boldsymbol{x}_d) \\ \vdots \\ \varphi_{d+1}(0, \boldsymbol{x}_d)-\varphi_{d+1}(2\pi, \boldsymbol{x}_d) \\ \psi_d(\boldsymbol{x}_d) \end{pmatrix}=0, \tag{3.34}$$

where $\boldsymbol{x}_d=(x(0), h_1(0),\ldots, h_{d+1}(0), B)^T$, $h_i(0)=(h_i^1, h_i^2,\ldots, h_i^n)^T$ $(i=1, 2,\ldots, d+1)$, $B=(B_1,\ldots, B_{d+1})^T$ and $\psi_d(\boldsymbol{x}_d)=(h_1^1-1, h_2^1,\ldots, h_{d+1}^1)^T$. Then the initial value $\hat{\boldsymbol{x}}_d=(\hat{x}(0), \hat{h}_1(0),\ldots, \hat{h}_{d+1}(0), \hat{B})^T$ of the $2\pi$-periodic solution $(\hat{x}(t), \hat{h}_1(t),\ldots, \hat{h}_{d+1}(t), \hat{B})^T$ of (3.28)–(3.29) is of course a solution of the system (3.34). Further, by (3.30) and (3.31), for this solution $\hat{\boldsymbol{x}}_d$, we readily get

**Theorem 15.**

*The solution* $\hat{\boldsymbol{x}}_d$ *of* (3.34) *is isolated.*

The $2\pi$-periodic solution $(\hat{x}(t), \hat{h}_1(t),\ldots, \hat{h}_{d+1}(t), \hat{B})^T$ of (3.28)–(3.29) satisfying $\det G_d'(\hat{\boldsymbol{x}}_d)\neq 0$ is also called to be "*isolated*", where $G_d'(\boldsymbol{x}_d)$ denotes the Jacobian matrix of $G_d(\boldsymbol{x}_d)$ with respect to $\boldsymbol{x}_d$.

Concerning the method for computing a highly accurate approximation to the isolated $2\pi$-periodic solution of (3.28)–(3.29), see [24] and [25].

**Remark 7.**

As is seen from Theorem 12 in Section 2.2, for the solution $\hat{\boldsymbol{x}}_d$ of (3.34), we have

$$\begin{cases} \text{the matrix } G_d'(\hat{\boldsymbol{x}}_d) \text{ is non-singular if and only if} \\ \operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_{d+1}(2\pi))=n. \end{cases} \tag{3.35}$$

This shows that the condition $\operatorname{rank}(\hat{D}_1(2\pi), \hat{l}_{d+1}(2\pi))=n$ plays an important role when we investigate whether a solution of (3.34) is isolated or not.

**Remark 8.**

Analogously to (2.61) in Section 2.2, we may consider the system

$$H(\boldsymbol{x})=\begin{pmatrix} \varphi(0, \boldsymbol{x})-\varphi(2\pi, \boldsymbol{x}) \\ g(\boldsymbol{x}) \end{pmatrix}=0 \tag{3.36}$$

instead of the system (3.10), where $\boldsymbol{x}=(x(0), B)^T$ and $\varphi(t, \boldsymbol{x})$ is a solution of (3.1) at a given $B$ such that $\varphi(0, \boldsymbol{x})=x(0)$ and $g(\boldsymbol{x})=\det[E_n-\Phi(2\pi)]$.

In this case, under the same assumptions as in Theorem 13, the system (3.36) has a solution $\hat{\boldsymbol{x}}=(\hat{x}(0), \hat{B})^T$ and for this solution $\hat{\boldsymbol{x}}$, we have

$$(3.37)\qquad \begin{cases} \text{the matrix } H'(\hat{\boldsymbol{x}}) \text{ is non-singular if and only if} \\ \text{the condition (3.12) is satisfied,} \end{cases}$$

where $H'(\boldsymbol{x})$ denotes the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. For the proof of (3.37), see the one of (2.62) in Remark 5.

### 3.2. Bifurcations of Periodic Solutions

We consider bifurcations of periodic solutions of periodic systems involving a parameter. In this section, we assume that the right-hand member $X(x, B, t)$ of the periodic system (3.1) is twice continuously differentiable with respect to $(x, B)$ in the region $\Delta \times R$ and $X(x, B, t)$ and its first and second partial derivatives with respect to $(x, B)$ are all continuous on the region $\Delta \times R$ and that the dimension of the parameter $B$ is one.

We classify bifurcation problems into the following two cases.

**Case (I)**

*Concerning the right-hand member* $X(x, B, t)$ *of the periodic system* (3.1), *we assume that for any* $t$

$$(3.38)\qquad \begin{cases} X(x_0(t+\pi), B, t+\pi) = -X(x_0(t), B, t),\ X_B(x_0(t+\pi), B, t+\pi) = \\ -X_B(x_0(t), B, t) \quad \text{and} \\ X_x(x_0(t+\pi), B, t+\pi) = X_x(x_0(t), B, t) \end{cases}$$

*for an arbitrary* $2\pi$*-periodic function* $x_0(t)$ *which satisfies both* $x_0(t+\pi) = -x_0(t)$ *for any* $t$ *and* $(x_0(t), B) \in \Delta$ *for all* $t$, *where* $X_x(x, B, t)$ *is the Jacobian matrix of* $X(x, B, t)$ *with respect to* $x$ *and* $X_B(x, B, t)$ *is the partial derivative of* $X(x, B, t)$ *with respect to* $B$.

*Let* $B=\hat{B}$ *be a bifurcation point and* $x=\hat{x}(t)$ *be a* $2\pi$*-periodic solution of* (3.1) *at* $B=\hat{B}$ *satisfying* $\hat{x}(t+\pi) = -\hat{x}(t)$ *for any* $t$. *When* $\hat{\Phi}(t)$ *is the fundamental matrix of* (3.2) *at* $x=\hat{x}(t)$ *and* $B=\hat{B}$ *satisfying the initial condition* $\hat{\Phi}(0)=E_n$, *we moreover assume that*

$$(3.39)\qquad \begin{cases} n-1 = \text{rank}\,[E_n - \hat{\Phi}(\pi)] = \text{rank}\,\hat{D}_1(\pi), \\ n-1 = \text{rank}\,[E_n - \hat{\Phi}(2\pi)] = \text{rank}\,\hat{D}_1(2\pi) = \text{rank}\,[\hat{D}_1(2\pi),\ \hat{\xi}_1(2\pi)] \quad \textit{and} \\ n = \text{rank}\,[E_n + \hat{\Phi}(\pi)], \end{cases}$$

*where* $\hat{D}_1(t)$ *is the* $n \times (n-1)$ *matrix obtained from* $E_n - \hat{\Phi}(t)$ *by deleting the first column vector and*

(3.40) $$\hat{\xi}_1(t)=\hat{\Phi}(t)\int_0^t \hat{\Phi}^{-1}(s)X_B(\hat{x}(s),\ \hat{B},\ s)\,ds.$$

**Case (II)**

*We assume that*

(3.41) $$\begin{cases} X(x_0(t),\ B,\ t),\ X_B(x_0(t),\ B,\ t) \text{ and } X_x(x_0(t),\ B,\ t) \text{ are all periodic in } t \text{ of} \\ \text{period } \pi \text{ for any } \pi\text{-periodic function } x_0(t) \text{ satisfying } (x_0(t),\ B)\in\Delta \text{ for all } t. \end{cases}$$

*Let $x=\hat{x}(t)$ be a $\pi$-periodic solution of* (3.1) *at $B=\hat{B}$ and $\hat{\Phi}(t)$ be the fundamental matrix of* (3.2) *at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying the initial condition $\hat{\Phi}(0)=E_n$. Further, we assume that*

(3.42) $$\begin{cases} n-1=\operatorname{rank}[E_n+\hat{\Phi}(\pi)]=\operatorname{rank}\hat{A}_1(\pi), \\ n-1=\operatorname{rank}[E_n-\hat{\Phi}(2\pi)]=\operatorname{rank}\hat{D}_1(2\pi)=\operatorname{rank}[\hat{D}_1(2\pi),\ \hat{\xi}_1(2\pi)] \quad \text{and} \\ n=\operatorname{rank}[E_n-\hat{\Phi}(\pi)], \end{cases}$$

*where $\hat{A}_1(t)$ and $\hat{D}_1(t)$ are the $n\times(n-1)$ matrices obtained from $E_n+\hat{\Phi}(t)$ and $E_n-\hat{\Phi}(t)$ by deleting the first column vectors, respectively, and $\hat{\xi}_1(t)$ is the vector defined by* (3.40).

First, we consider Case (I).

**Case (I).** From the assumption (3.38), we have

(3.43) $$\hat{\Phi}(t+\pi)=\hat{\Phi}(t)\hat{\Phi}(\pi)$$

for the fundamental matrix $\hat{\Phi}(t)$ and we have

(3.44) $$\hat{\xi}_1(2\pi)=-[E_n-\hat{\Phi}(\pi)]\hat{\xi}_1(\pi)$$

since $\hat{\xi}_1(t)=\hat{\Phi}(t)\int_0^t \hat{\Phi}^{-1}(s)X_B(\hat{x}(s),\ \hat{B},\ s)\,ds$.

Now, we consider the equation

(3.45) $$[E_n-\hat{\Phi}(2\pi)]k=A\hat{\xi}_1(2\pi)$$

for any constant number $A$. From (3.43) and (3.44), we have

(3.46) $$\begin{cases} [E_n+\hat{\Phi}(\pi)]\,[E_n-\hat{\Phi}(\pi)]k=[E_n-\hat{\Phi}(\pi)]\,[E_n+\hat{\Phi}(\pi)]k \\ =[E_n-\hat{\Phi}(2\pi)]k=A\hat{\xi}_1(2\pi)=-[E_n-\hat{\Phi}(\pi)]A\hat{\xi}_1(\pi) \end{cases}$$

since $\hat{\Phi}(2\pi)=\hat{\Phi}(\pi)^2$.

(i) When $A=0$, the equation (3.46) becomes

(3.47) $$[E_n-\hat{\Phi}(2\pi)]k=[E_n+\hat{\Phi}(\pi)]\,[E_n-\hat{\Phi}(\pi)]k=0.$$

By the assumption (3.39), the equation

$$(3.48)\qquad \begin{cases} [E_n-\hat{\Phi}(\pi)]y=0, \\ y_1-1=0 \end{cases}$$

has a solution $\hat{y}$, where $y=(y_1,\dots, y_n)^T$. This solution $\hat{y}$ is also a solution of (3.47). In fact,

$$(3.49)\qquad [E_n-\hat{\Phi}(2\pi)]\hat{y}=[E_n+\hat{\Phi}(\pi)]\,[E_n-\hat{\Phi}(\pi)]\hat{y}=[E_n+\hat{\Phi}(\pi)]0=0.$$

Since rank $[E_n-\hat{\Phi}(2\pi)]=n-1$ by the assumption (3.39), an arbitrary solution $k$ of the equation (3.47) is of the form

$$(3.50)\qquad k=c\hat{y},$$

where $c$ is a constant number. Thus the solution $\hat{y}$ becomes the initial value of a $2\pi$-periodic solution of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$. But, since $\hat{y}$ is a solution of (3.48) and $X_x(\hat{x}(t), \hat{B}, t)$ is periodic in $t$ of period $\pi$ by the assumption (3.38), $\hat{y}$ becomes the initial value of a $\pi$-periodic solution of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$. That is, $\hat{h}(t)\equiv \hat{\Phi}(t)\hat{y}$ is a $\pi$-periodic solution of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$.

Consequently, in order to obtain the bifurcation point, we find a periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of the system

$$(3.51)\qquad \begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\ \dfrac{dh}{dt}=X_x(x, B, t)h \end{cases}$$

satisfying the conditions

$$(3.52)\qquad \begin{cases} x(0)+x(\pi)=0, \\ h(0)-h(\pi)=0, \\ h_1-1=0, \end{cases}$$

where $x(0)=(x_1,\dots, x_n)^T$ and $h(0)=(h_1,\dots, h_n)^T$. As is shown in the above arguments, (3.51)–(3.52) certainly has a periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ and the $B$-component $\hat{B}$ of this periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ is really the desired bifurcation point and, of course, $(\hat{x}(t), \hat{h}(t))^T$ is the periodic solution of (3.51) at the bifurcation point $\hat{B}$.

Next we study the isolatedness of the periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (3.51)–(3.52).

Let $(\varphi(t, \boldsymbol{x}), \varphi_1(t, \boldsymbol{x}))^T$ be a solution of (3.51) such that $(\varphi(0, \boldsymbol{x}), \varphi_1(0, \boldsymbol{x}))^T=(x(0), h(0))^T$ and let us consider the system

(3.53)
$$F(\boldsymbol{x})=\begin{pmatrix} \varphi(0,\ \boldsymbol{x})+\varphi(\pi,\ \boldsymbol{x}) \\ \varphi_1(0,\ \boldsymbol{x})-\varphi_1(\pi,\ \boldsymbol{x}) \\ h_1-1 \end{pmatrix}=0,$$

where $\boldsymbol{x}=(x(0),\ h(0),\ B)^T$, $x(0)=(x_1,\dots,x_n)^T$ and $h(0)=(h_1,\dots,h_n)^T$. Then the initial value $\hat{\boldsymbol{x}}=(\hat{x}(0),\ \hat{h}(0),\ \hat{B})^T$ of the periodic solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$ of (3.51)-(3.52) is certainly a solution of the system (3.53) and for the solution $\hat{\boldsymbol{x}}$, we have

(3.54)
$$F'(\hat{\boldsymbol{x}})=\begin{pmatrix} E_n+\hat{\Phi}(\pi) & 0 & \hat{\xi}_1(\pi) \\ -\hat{\Phi}_2(\pi) & E_n-\hat{\Phi}(\pi) & -\hat{\xi}_2(\pi) \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix},$$

where $F'(\boldsymbol{x})$ denotes the Jacobian matrix of $F(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $(\hat{\Phi}(t),\ \hat{\Phi}_2(t))^T$ is a solution ($2n\times n$ matrix) of the first variation equation of (3.51) with respect to $(x,\ h)^T=(\hat{x}(t),\ \hat{h}(t))^T$ (at $B=\hat{B}$) satisfying the initial condition $(\hat{\Phi}(0),\ \hat{\Phi}_2(0))^T=(E_n,\ 0)^T$ and $(\hat{\xi}_1(t),\ \hat{\xi}_2(t))^T$ is a solution of

(3.55)
$$\begin{cases} \dfrac{dz_1}{dt}=X_x(\hat{x}(t),\ \hat{B},\ t)z_1+X_B(\hat{x}(t),\ \hat{B},\ t), \\ \dfrac{dz_2}{dt}=X_x(\hat{x}(t),\ \hat{B},\ t)z_2+\{X_{xx}(\hat{x}(t),\ \hat{B},\ t)z_1+X_{xB}(\hat{x}(t),\ \hat{B},\ t)\}\hat{h}(t) \end{cases}$$

satisfying the initial condition $(\hat{\xi}_1(0),\ \hat{\xi}_2(0))^T=(0,\ 0)^T$.
By (3.54) we see that

(3.56)
$$\det F'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to } \operatorname{rank}(\hat{D}_1(\pi),\ \hat{\delta})=n,$$

where $\hat{\delta}=-\hat{\Phi}_2(\pi)\hat{\zeta}-\hat{\xi}_2(\pi)$. Here $\hat{\zeta}$ is a solution of the linear equation

(3.57)
$$[E_n+\hat{\Phi}(\pi)]\zeta=-\hat{\xi}_1(\pi).$$

The periodic solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$ of (3.51)-(3.52) satisfying $\det F'(\hat{\boldsymbol{x}})\neq 0$ is called to be "*isolated*".

In order to obtain a highly accurate approximation to the isolated periodic solution of (3.51)-(3.52), we applied the Urabe-Galerkin method to (3.51)-(3.52) and we obtained such an accurate approximation. For details of the method, see [24] and [25].

(ii) When $A\neq 0$, since $\operatorname{rank}[E_n+\hat{\Phi}(\pi)]=n$ from the assumption (3.39), the equation

(3.58)
$$[E_n+\hat{\Phi}(\pi)]v=-A\hat{\xi}_1(\pi)$$

has one and only one solution $\hat{v}$ and this solution $\hat{v}$ is also a solution of (3.45) (or

(3.46)). Actually, by (3.38), this solution $\hat{v}$ becomes the initial value of a $2\pi$-periodic solution $\hat{h}(t)$ of the system

$$\frac{dh}{dt}=X_x(\hat{x}(t),\ \hat{B},\ t)h+A\cdot X_B(\hat{x}(t),\ \hat{B},\ t) \tag{3.59}$$

satisfying $\hat{h}(t+\pi)=-\hat{h}(t)$ for any $t$. Indeed, $\hat{h}(0)=\hat{v}$.

Secondly, we consider Case (II).

**Case (II).** From the assumption (3.41), analogously to Case (I), we have

$$\hat{\Phi}(t+\pi)=\hat{\Phi}(t)\hat{\Phi}(\pi) \tag{3.60}$$

for the fundamental matrix $\hat{\Phi}(t)$. By (3.60), for $\hat{\xi}_1(2\pi)$, in this case, we have

$$\hat{\xi}_1(2\pi)=[E_n+\hat{\Phi}(\pi)]\hat{\xi}_1(\pi). \tag{3.61}$$

Similarly to Case (I), we consider the equation

$$[E_n-\hat{\Phi}(2\pi)]k=A\hat{\xi}_1(2\pi) \tag{3.62}$$

for any constant number $A$. Taking account of (3.60) and (3.61), the equation (3.62) can be rewritten in the following form:

$$\begin{cases} [E_n-\hat{\Phi}(\pi)]\,[E_n+\hat{\Phi}(\pi)]k=[E_n+\hat{\Phi}(\pi)]\,[E_n-\hat{\Phi}(\pi)]k \\ =[E_n-\hat{\Phi}(2\pi)]k=A\hat{\xi}_1(2\pi)=[E_n+\hat{\Phi}(\pi)]A\hat{\xi}_1(\pi). \end{cases} \tag{3.63}$$

(i) When $A=0$, since $\operatorname{rank}[E_n+\hat{\Phi}(\pi)]=n-1$ from the assumption (3.42), the equation

$$\begin{cases} [E_n+\hat{\Phi}(\pi)]y=0, \\ y_1-1=0 \end{cases} \qquad (\text{where } y=(y_1,\dots,y_n)^T) \tag{3.64}$$

has a solution $\hat{y}$ and this solution $\hat{y}$ satisfies the equation

$$[E_n-\hat{\Phi}(2\pi)]k=0. \tag{3.65}$$

Then, by (3.42), an arbitrary solution $k$ of (3.65) can be written in the form

$$k=c\hat{y}, \tag{3.66}$$

where $c$ is a constant number. But, since $\hat{y}$ is originally a solution of (3.64) and $X_x(\hat{x}(t),\ \hat{B},\ t)$ is periodic in $t$ of period $\pi$ by the assumption (3.41), $\hat{y}$ becomes the initial value of a $2\pi$-periodic solution $h(t)$ of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying $h(t+\pi)=-h(t)$ for any $t$. That is, $\hat{h}(t)\equiv\hat{\Phi}(t)\hat{y}$ is a $2\pi$-periodic solution of (3.2) at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying $\hat{h}(t+\pi)=-\hat{h}(t)$ for any $t$.

Consequently, in order to obtain the bifurcation point, we find a periodic solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$ of the system

(3.67)
$$\begin{cases} \dfrac{dx}{dt} = X(x, B, t), \\ \dfrac{dh}{dt} = X_x(x, B, t)h \end{cases}$$

satisfying the conditions

(3.68)
$$\begin{cases} x(0) - x(\pi) = 0, \\ h(0) + h(\pi) = 0, \\ h_1 - 1 = 0, \end{cases}$$

where $x(0) = (x_1, \dots, x_n)^T$ and $h(0) = (h_1, \dots, h_n)^T$. Then, as is shown in the above argument, (3.67)–(3.68) has a periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ and the $B$-component $\hat{B}$ of this solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ is really the desired bifurcation point and, of course, $(\hat{x}(t), \hat{h}(t))^T$ is the periodic solution of (3.67) at the bifurcation point $\hat{B}$.

Analogously to Case (I)–(i), we consider the isolatedness of the solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (3.67)–(3.68).

Let $(\varphi(t, \boldsymbol{x}), \varphi_1(t, \boldsymbol{x}))^T$ be a solution of (3.67) with $(\varphi(0, \boldsymbol{x}), \varphi_1(0, \boldsymbol{x}))^T = (x(0), h(0))^T$ and, in this case, we consider the system

(3.69)
$$F(\boldsymbol{x}) = \begin{pmatrix} \varphi(0, \boldsymbol{x}) - \varphi(\pi, \boldsymbol{x}) \\ \varphi_1(0, \boldsymbol{x}) + \varphi_1(\pi, \boldsymbol{x}) \\ h_1 - 1 \end{pmatrix} = 0,$$

where $\boldsymbol{x} = (x(0), h(0), B)^T$, $x(0) = (x_1, \dots, x_n)^T$ and $h(0) = (h_1, \dots, h_n)^T$. Then, of course, the initial value $\hat{\boldsymbol{x}} = (\hat{x}(0), \hat{h}(0), \hat{B})^T$ of the periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (3.67)–(3.68) is a solution of the system (3.69) and for this solution $\hat{\boldsymbol{x}}$, we have

(3.70)
$$F'(\hat{\boldsymbol{x}}) = \begin{pmatrix} E_n - \hat{\Phi}(\pi) & 0 & -\hat{\xi}_1(\pi) \\ \hat{\Phi}_2(\pi) & E_n + \hat{\Phi}(\pi) & \hat{\xi}_2(\pi) \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix},$$

from which it follows that

(3.71) $\det F'(\hat{\boldsymbol{x}}) \neq 0$ is equivalent to $\operatorname{rank}(\hat{\Lambda}_1(\pi), \hat{\delta}') = n$,

where $F'(\boldsymbol{x})$ denotes the Jacobian matrix of $F(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\hat{\delta}' = \hat{\Phi}_2(\pi)\hat{\zeta}' + \hat{\xi}_2(\pi)$. Here $\hat{\zeta}'$ is a solution of the linear equation

(3.72)
$$[E_n - \hat{\Phi}(\pi)]\zeta' = \hat{\xi}_1(\pi).$$

The periodic solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (3.67)–(3.68) satisfying $\det F'(\hat{\boldsymbol{x}}) \neq 0$ is called to be "*isolated*".

Actually, when we considered bifurcations of periodic solutions of the Duffing equation, we used the Urabe-Galerkin method in order to get a highly accurate approximation to the isolated periodic solution of (3.67)–(3.68).

(ii) When $A \neq 0$, since rank $[E_n - \hat{\Phi}(\pi)] = n$ from the assumption (3.42), the equation

$$(3.73) \qquad [E_n - \hat{\Phi}(\pi)]v = A\hat{\xi}_1(\pi)$$

has one and only one solution $\hat{v}$ and this solution $\hat{v}$ is also a solution of (3.62). In fact, by (3.41), this solution $\hat{v}$ becomes the initial value of a $\pi$-periodic solution $\hat{h}(t)$ of the system

$$(3.74) \qquad \frac{dh}{dt} = X_x(\hat{x}(t), \hat{B}, t)h + A \cdot X_B(\hat{x}(t), \hat{B}, t).$$

Indeed, $\hat{h}(0) = \hat{v}$.

## §4. The Multi-Point Boundary Value Problems

The theory and method for singular problems mentioned in the preceding sections can also be applied to the following multi-point boundary value problem:

$$(4.1) \qquad \frac{dx}{dt} = X(x, B, t)$$

and

$$(4.2) \qquad \sum_{i=0}^{N} L_i x(t_i) = c,$$

where $x$ and $X(x, B, t)$ are $n$-dimensional vectors, $B$ is a parameter and

$$-1 = t_0 < t_1 < t_2 < \cdots\cdots < t_{N-1} < t_N = 1,$$

and $L_i$ $(i = 0, 1, 2, \ldots, N)$ are the given $n \times n$ matrices and $c$ is a given $n$-dimensional vector. Here $X(x, B, t)$ is continuously differentiable with respect to $(x, B)$ in the region $\tilde{\Delta}$, where $\tilde{\Delta}$ is a given region of the $(x, B, t)$-space intercepted by two hyperplanes $t = -1$ and $t = 1$.

We assume that $X(x, B, t)$ and its first partial derivatives with respect to $(x, B)$ are all continuous on the region $\tilde{\Delta}$.

At first, let us suppose that the dimension of the parameter $B$ is one.

Let $\varphi(t, x(-1), B)$ be a solution of (4.1) at a given $B$ such that $\varphi(-1, x(-1), B) = x(-1)$. Then we consider the equation

$$(4.3) \qquad F(x(-1), B) = \sum_{i=0}^{N} L_i \varphi(t_i, x(-1), B) - c = 0.$$

By the assumption, the function $F(x(-1), B)$ defined by the equality (4.3) is continuously differentiable with respect to $(x(-1), B)$ and we denote the Jacobian matrix of $F(x(-1), B)$ with respect to $x(-1)$ by $F_x(x(-1), B)$. Then we have

$$F_x(x(-1), B)=\sum_{i=0}^{N} L_i\Phi(t_i), \tag{4.4}$$

where $\Phi(t)$ is the fundamental matrix of the system

$$\frac{dh}{dt}=X_x(x, B, t)h \tag{4.5}$$

at $x=x(t)=\varphi(t, x(-1), B)$ and the given $B$ satisfying the initial condition $\Phi(-1)=E_n$.

Assume that there exists a point $(\hat{x}(-1), \hat{B})$ satisfying (4.3) and also satisfying

$$\operatorname{rank} F_x(\hat{x}(-1), \hat{B})=\operatorname{rank} \sum_{i=0}^{N} L_i\hat{\Phi}(t_i)=\operatorname{rank} \hat{G}_1=n-1, \tag{4.6}$$

where $(\hat{x}(t), \hat{B}, t)\in\tilde{\Delta}$ for any $t\in[-1, 1]$ and $\hat{\Phi}(t)$ is the fundamental matrix of (4.5) at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying the initial condition $\hat{\Phi}(-1)=E_n$ and $\hat{G}_1$ is the $n\times(n-1)$ matrix obtained from $\hat{G}=\sum_{i=0}^{N} L_i\hat{\Phi}(t_i)$ by deleting the first column vector. Here $\hat{x}(t)$ is a solution of (4.1) at $B=\hat{B}$ through $\hat{x}(-1)$ at $t=-1$. In fact, $\hat{x}(t)$ is a solution of (4.1)–(4.2) at $B=\hat{B}$.

By the assumption (4.6), the equation

$$\begin{cases} F_x(\hat{x}(-1), \hat{B})k=0, \\ k_1-1=0 \end{cases} \tag{4.7}$$

has a solution $\hat{k}$, where $k=(k_1,\dots, k_n)^T$.

Putting

$$\hat{h}(t)=\hat{\Phi}(t)\hat{k}, \tag{4.8}$$

$\hat{h}(t)$ is a solution of (4.5) at $x=\hat{x}(t)$ and $B=\hat{B}$ satisfying the conditions

$$\begin{cases} \sum_{i=0}^{N} L_i\hat{h}(t_i)=0, \\ \hat{h}_1-1=0, \end{cases} \tag{4.9}$$

where $\hat{h}(-1)=(\hat{h}_1,\dots, \hat{h}_n)^T=\hat{k}$.

Consequently, analogously to the case of periodic solutions in Section 3.1, we consider a solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of the system

$$\begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\ \dfrac{dh}{dt}=X_x(x, B, t)h \end{cases} \tag{4.10}$$

satisfying the conditions

$$
(4.11)\qquad \begin{cases} \sum_{i=0}^{N} L_i x(t_i) = c, \\ \sum_{i=0}^{N} L_i h(t_i) = 0, \\ h_1 - 1 = 0, \end{cases}
$$

where $h(-1)=(h_1,\dots, h_n)^T$.
As is shown in the above argument, (4.10)–(4.11) certainly has a solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$.

Now we consider the isolatedness of the solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$ of (4.10)–(4.11).

In this case, the system corresponding to (3.10) in Section 3.1 is of the following form:

$$
(4.12)\qquad H(\boldsymbol{x}) = \begin{pmatrix} \sum_{i=0}^{N} L_i \varphi(t_i,\ \boldsymbol{x}) - c \\ \sum_{i=0}^{N} L_i \varphi_1(t_i,\ \boldsymbol{x}) \\ h_1 - 1 \end{pmatrix} = 0,
$$

where $\boldsymbol{x}=(x(-1),\ h(-1),\ B)^T$, $x(-1)=(x_1,\dots, x_n)^T$, $h(-1)=(h_1,\dots, h_n)^T$, and $(\varphi(t,\ \boldsymbol{x}),\ \varphi_1(t,\ \boldsymbol{x}))^T$ is a solution of (4.10) such that $(\varphi(-1,\ \boldsymbol{x}),\ \varphi_1(-1,\ \boldsymbol{x}))^T = (x(-1),\ h(-1))^T$.

Of course, the initial value $\hat{\boldsymbol{x}}=(\hat{x}(-1),\ \hat{h}(-1),\ \hat{B})^T$ of the solution $(\hat{x}(t),\ \hat{h}(t),\ \hat{B})^T$ of (4.10)–(4.11) is a solution of the system (4.12) and for this solution $\hat{\boldsymbol{x}}$, analogously to Theorem 13, we have

**Theorem 16.**

*Assume that $X(x,\ B,\ t)$ is twice continuously differentiable with respect to $(x,\ B)$ in the region $\tilde{\Delta}$ and $X(x,\ B,\ t)$ and its first and second partial derivatives with respect to $(x,\ B)$ are all continuous on the region $\bar{\Delta}$.*

*If the conditions*

$$
(4.13)\qquad n-1 = \operatorname{rank} \hat{G} = \operatorname{rank} \hat{G}_1 < n = \operatorname{rank}(\hat{G}_1,\ \hat{\xi}_1)
$$

*are satisfied, then the solution $\hat{\boldsymbol{x}}$ of (4.12) is isolated if and only if*

$$
(4.14)\qquad \operatorname{rank}(\hat{G}_1,\ \hat{l}) = n,
$$

*where $\hat{\xi}_1 = \sum_{i=0}^{N} L_i \frac{\partial \varphi}{\partial B}(t_i,\ \hat{\boldsymbol{x}})$ and $\hat{l} = [\sum_{i=0}^{N} L_i \hat{\Phi}_2(t_i)]\hat{h}(-1)$.* Here *$\frac{\partial \varphi}{\partial B}$ is the partial derivative of $\varphi$ with respect to $B$, and $(\hat{\Phi}(t),\ \hat{\Phi}_2(t))^T$ is a solution ($2n \times n$ matrix)* *of the first variation equation of* (4.10) *with respect to* $(x, h)^T=(\hat{x}(t), \hat{h}(t))^T$ *(at* $B=\hat{B}$*) satisfying the initial condition* $(\hat{\Phi}(-1), \hat{\Phi}_2(-1))^T=(E_n, 0)^T$.

PROOF. By the assumption of the theorem, the function $H(\boldsymbol{x})$ defined by the equality (4.12) is continuously differentiable with respect to $\boldsymbol{x}$ and we denote by $H'(\boldsymbol{x})$ the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Then, for the solution $\hat{\boldsymbol{x}}$, we have

$$(4.15)\qquad H'(\hat{\boldsymbol{x}})=\begin{pmatrix} \sum_{i=0}^{N} L_i\hat{\Phi}(t_i) & 0 & \sum_{i=0}^{N} L_i\frac{\partial\varphi}{\partial B}(t_i, \hat{x}) \\ \sum_{i=0}^{N} L_i\hat{\Phi}_2(t_i) & \sum_{i=0}^{N} L_i\hat{\Phi}(t_i) & \sum_{i=0}^{N} L_i\frac{\partial\varphi_1}{\partial B}(t_i, \hat{x}) \\ 00\cdots\cdots0 & 10\cdots\cdots0 & 0 \end{pmatrix},$$

where $\frac{\partial\varphi_1}{\partial B}$ is the partial derivative of $\varphi_1$ with respect to $B$. By (4.13) and (4.15) we see that

$$(4.16)\qquad \det H'(\hat{\boldsymbol{x}})\neq 0 \text{ is equivalent to (4.14).}$$

This completes the proof. Q.E.D.

The solution $(\hat{x}(t), \hat{h}(t), \hat{B})^T$ of (4.10)–(4.11) satisfying $\det H'(\hat{x})\neq 0$ is called to be "*isolated*".

**Remark 9.**

$\left(\frac{\partial\varphi}{\partial B}(t, \hat{x}), \frac{\partial\varphi_1}{\partial B}(t, \hat{x})\right)^T$ is a solution of the system

$$(4.17)\qquad \begin{cases} \frac{dz_1}{dt}=X_x(\hat{x}(t), \hat{B}, t)z_1+X_B(\hat{x}(t), \hat{B}, t), \\ \frac{dz_2}{dt}=X_x(\hat{x}(t), \hat{B}, t)z_2+\{X_{xx}(\hat{x}(t), \hat{B}, t)z_1+X_{xB}(\hat{x}(t), \hat{B}, t)\}\hat{h}(t) \end{cases}$$

satisfying the initial condition $(z_1(-1), z_2(-1))^T=(0, 0)^T$, where $X_B(x, B, t)$ and $X_{xB}(x, B, t)$ are the partial derivatives of $X(x, B, t)$ and $X_x(x, B, t)$ with respect to $B$, respectively.

When $\operatorname{rank}(\hat{G}_1, \hat{l})=n-1$, the equation

$$(4.18)\qquad \begin{cases} \text{(i)}\quad \hat{G}k+\hat{l}=[\sum_{i=0}^{N} L_i\hat{\Phi}(t_i)]k+\hat{l}=0, \\ \text{(ii)}\quad k_1=0 \end{cases}$$

has a solution $\hat{k}$, where $k=(k_1,\dots, k_n)^T$.

Since $\hat{l}=[\sum_{i=0}^{N} L_i\hat{\Phi}_2(t_i)]\hat{h}(-1)$, we can rewrite (i) of (4.18) in the following form:

$$(4.19) \qquad \sum_{i=0}^{N} L_i[\hat{\Phi}(t_i)\hat{k}+\hat{\Phi}_2(t_i)\hat{h}(-1)]=0.$$

Putting

$$\hat{v}(t)=\hat{\Phi}(t)\hat{k}+\hat{\Phi}_2(t)\hat{h}(-1),$$

$\hat{v}(t)$ is a solution of the system

$$(4.20) \qquad \frac{dv}{dt}=X_x(\hat{x}(t),\ \hat{B},\ t)v+\{X_{xx}(\hat{x}(t),\ \hat{B},\ t)\hat{h}(t)\}\hat{h}(t)$$

satisfying the conditions

$$(4.21) \qquad \begin{cases} \sum_{i=0}^{N} L_i v(t_i)=0, \\ v_1=0, \end{cases}$$

where $v(-1)=(v_1,\ ..,\ v_n)^T$.

Then, let the dimension of the parameter $B$ be two and let us consider the following multi-point boundary value problem:

$$(4.22) \qquad \begin{cases} \frac{dx}{dt}=X(x,\ B,\ t), \\ \frac{dh_1}{dt}=X_x(x,\ B,\ t)h_1, \\ \frac{dh_2}{dt}=X_x(x,\ B,\ t)h_2+\{X_{xx}(x,\ B,\ t)h_1\}h_1 \end{cases}$$

and

$$(4.23) \qquad \begin{cases} \sum_{i=0}^{N} L_i x(t_i)=c, \\ \sum_{i=0}^{N} L_i h_1(t_i)=0, \\ \sum_{i=0}^{N} L_i h_2(t_i)=0, \\ h_1^1-1=0, \\ h_2^1=0, \end{cases}$$

where $x(-1)=(x_1,\dots,\ x_n)^T$, $h_i(-1)=(h_i^1,\ h_i^2,\dots,\ h_i^n)^T$ $(i=1,\ 2)$ and $B=(B_1,\ B_2)^T$. As is shown in the above argument, the problem (4.22)–(4.23) has a solution $(\hat{x}(t),\ \hat{h}_1(t),\ \hat{h}_2(t),\ \hat{B})^T$. In fact, $\hat{h}_1(t)=\hat{h}(t)$ and $\hat{h}_2(t)=\hat{v}(t)$.

Now we consider the isolatedness of the solution $(\hat{x}(t),\ \hat{h}_1(t),\ \hat{h}_2(t),\ \hat{B})^T$ of (4.22)–(4.23).

In this case, the system corresponding to (3.21) in Section 3.1 is of the following form:

$$(4.24)\qquad H_1(\boldsymbol{x}_1)=\begin{pmatrix}\sum_{i=0}^{N} L_i\varphi(t_i,\,\boldsymbol{x}_1)-c\\ \sum_{i=0}^{N} L_i\varphi_1(t_i,\,\boldsymbol{x}_1)\\ \sum_{i=0}^{N} L_i\varphi_2(t_i,\,\boldsymbol{x}_1)\\ h_1^1-1\\ h_2^1\end{pmatrix}=0,$$

where $\boldsymbol{x}_1=(x(-1),\,h_1(-1),\,h_2(-1),\,B)^T$, $x(-1)=(x_1,\dots,\,x_n)^T$, $h_i(-1)=(h_i^1,\,h_i^2,\dots,\,h_i^n)^T$ $(i=1,\,2)$, $B=(B_1,B_2)^T$, and $(\varphi(t,\,\boldsymbol{x}_1),\,\varphi_1(t,\,\boldsymbol{x}_1),\,\varphi_2(t,\,\boldsymbol{x}_1))^T$ is a solution of (4.22) such that $(\varphi(-1,\,\boldsymbol{x}_1),\,\varphi_1(-1,\,\boldsymbol{x}_1),\,\varphi_2(-1,\,\boldsymbol{x}_1))^T=(x(-1),\,h_1(-1),\,h_2(-1))^T$. Of course, the initial value $\hat{\boldsymbol{x}}_1=(\hat{x}(-1),\,\hat{h}_1(-1),\,\hat{h}_2(-1),\,\hat{B})^T$ of the solution $(\hat{x}(t),\,\hat{h}_1(t),\,\hat{h}_2(t),\,\hat{B})^T$ of (4.22)–(4.23) is a solution of the system (4.24) and for this solution $\hat{\boldsymbol{x}}_1$, similarly to Theorem 14, we easily get

**Theorem 17.**

*Assume that $X(x,\,B,\,t)$ is three times continuously differentiable with respect to $(x,\,B)$ in the region $\tilde{\Delta}$ and $X(x,\,B,\,t)$ and its first, second and third partial derivatives with respect to $(x,\,B)$ are all continuous on the region $\tilde{\Delta}$ and that the conditions*

$$(4.25)\qquad \operatorname{rank}(\hat{G}_1,\,\hat{\xi}_{11})=n$$

*and*

$$(4.26)\qquad \operatorname{rank}\begin{pmatrix}\hat{G}_1 & 0 & \hat{\xi}_{11} & \hat{\xi}_{12}\\ \hat{G}_2 & \hat{G}_1 & \hat{\xi}_{21} & \hat{\xi}_{22}\end{pmatrix}=2n$$

*are satisfied.*

*Then the solution $\hat{\boldsymbol{x}}_1$ of (4.24) is isolated if and only if*

$$(4.27)\qquad \operatorname{rank}(\hat{G}_1,\,\hat{l}_2)=n,$$

*where $\hat{\xi}_j=(\hat{\xi}_{j1},\,\hat{\xi}_{j2})^T=\left(\sum_{i=0}^{N} L_i\frac{\partial\varphi}{\partial B_j}(t_i,\,\hat{\boldsymbol{x}}_1),\,\sum_{i=0}^{N} L_i\frac{\partial\varphi_1}{\partial B_j}(t_i,\,\hat{\boldsymbol{x}}_1)\right)^T$ $(j=1,\,2)$, and $\hat{G}_1$ and $\hat{G}_2$ are the $n\times(n-1)$ matrices obtained from $\hat{G}=\sum_{i=0}^{N} L_i\hat{\Phi}(t_i)$ and $\sum_{i=0}^{N} L_i\hat{\Phi}_2(t_i)$ by deleting the first column vectors, respectively, and*

$$\hat{l}_2=[\sum_{i=0}^{N} L_i\hat{\Phi}_3(t_i)]\hat{h}_1(-1)+2[\sum_{i=0}^{N} L_i\hat{\Phi}_2(t_i)]\hat{h}_2(-1).$$

*Here* $\frac{\partial\varphi}{\partial B_j}$ *and* $\frac{\partial\varphi_1}{\partial B_j}$ $(j=1, 2)$ *are the partial derivatives of* $\varphi$ *and* $\varphi_1$ *with respect to* $B_j$ $(j=1, 2)$, *respectively, and* $(\hat{\Phi}(t), \hat{\Phi}_2(t), \hat{\Phi}_3(t))^T$ *is a solution* ($3n\times n$ *matrix*) *of the first variation equation of* (4.22) *with respect to* $(x, h_1, h_2)^T=(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t))^T$ (*at* $B=\hat{B}$) *satisfying the initial condition* $(\hat{\Phi}(-1), \hat{\Phi}_2(-1), \hat{\Phi}_3(-1))^T=(E_n, 0, 0)^T$.

More generally, we suppose that the dimension of the parameter $B$ is $(d+1)$ and $X(x, B, t)$ is $(d+2)$ times continuously differentiable with respect to $(x, B)$ in the region $\tilde{\Delta}$ $(d\geq 2)$ and $X(x, B, t)$ and its first, second,..., $(d+2)$-th partial derivatives with respect to $(x, B)$ are all continuous on the region $\tilde{\Delta}$.

Putting

$$X^{(i+1)}=\sum_{k=0}^{i} {}_iC_k X_x^{(k)} h_{i+1-k} \qquad (1\leq i\leq d), \tag{4.28}$$

we consider the system

$$\begin{cases} \dfrac{dx}{dt}=X(x, B, t), \\ \dfrac{dh_1}{dt}=X^{(0)}h_1, \\ \dfrac{dh_2}{dt}=X^{(0)}h_2+X^{(1)}h_1, \\ \vdots \\ \dfrac{dh_{d+1}}{dt}=\sum_{k=0}^{d} {}_dC_k X^{(k)} h_{d+1-k}, \end{cases} \tag{4.29}$$

where $X^{(0)}=X_x(x, B, t)$, $X^{(1)}=X_x^{(0)}h_1$. Here $X_x^{(k)}$ $(0\leq k\leq d)$ are the derivatives of $X^{(k)}$ $(0\leq k\leq d)$ with respect to $x$, respectively.

Now, we assume that there exists a solution $(\hat{x}(t), \hat{h}_1(t),\dots, \hat{h}_{d+1}(t), \hat{B})^T$ of (4.29) satisfying the conditions

$$\begin{cases} \sum_{i=0}^{N} L_i x(t_i)-c=0, \\ \sum_{i=0}^{N} L_i h_1(t_i)=0, \\ \vdots \\ \sum_{i=0}^{N} L_i h_{d+1}(t_i)=0, \\ h_1^1-1=0, \\ h_2^1=0, \\ \vdots \\ h_{d+1}^1=0 \end{cases} \tag{4.30}$$

and, moreover, we assume that for the solution $(\hat{x}(t), \hat{h}_1(t),\dots, \hat{h}_{d+1}(t), \hat{B})^T$, the conditions

$$(4.31)\qquad \begin{cases} n-1=\operatorname{rank}\hat{G}_1=\operatorname{rank}(\hat{G}_1,\hat{l}_1)=\cdots\cdots \\ \qquad =\operatorname{rank}(\hat{G}_1,\hat{l}_d)<n=\operatorname{rank}(\hat{G}_1,\hat{l}_{d+1}) \end{cases}$$

and

$$(4.32)\qquad \operatorname{rank}\begin{pmatrix} {}_0C_0\hat{G}_1 & & & & & \hat{\xi}_{11} & \hat{\xi}_{12} & \cdots\hat{\xi}_{1d+1} \\ {}_1C_0\hat{G}_2 & {}_1C_1\hat{G}_1 & & 0 & & \hat{\xi}_{21} & \hat{\xi}_{22} & \cdots\hat{\xi}_{2d+1} \\ {}_2C_0\hat{G}_3 & {}_2C_1\hat{G}_2 & {}_2C_2\hat{G}_1 & & & \hat{\xi}_{31} & \hat{\xi}_{32} & \cdots\hat{\xi}_{3d+1} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ {}_dC_0\hat{G}_{d+1} & {}_dC_1\hat{G}_d & {}_dC_2\hat{G}_{d-1} & \cdots & {}_dC_d\hat{G}_1 & \hat{\xi}_{d+11} & \hat{\xi}_{d+12} & \cdots\hat{\xi}_{d+1d+1} \end{pmatrix}=(d+1)n$$

are satisfied, where $h_i(-1)=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2,\dots, d+1)$ and

$$(4.33)\qquad \hat{l}_j=\sum_{k=1}^{j} {}_jC_k\Big[\sum_{i=0}^{N} L_i\hat{\Phi}_{k+1}(t_i)\Big]\hat{h}_{j+1-k}(-1)\qquad (1\le j\le d+1)$$

and $\hat{G}_1, \hat{G}_2,\dots, \hat{G}_{d+1}$ are the $n\times(n-1)$ matrices obtained from $\hat{G}=\sum_{i=0}^{N} L_i\hat{\Phi}(t_i)$, $\sum_{i=0}^{N} L_i\hat{\Phi}_2(t_i),\dots, \sum_{i=0}^{N} L_i\hat{\Phi}_{d+1}(t_i)$ by deleting the first column vectors, respectively. Here $(\hat{\Phi}(t), \hat{\Phi}_2(t),\dots, \hat{\Phi}_{d+2}(t))^T$ is a solution ($(d+2)n\times n$ matrix) of the first variation equation of (4.29) with respect to $(x, h_1,\dots, h_{d+1})^T=(\hat{x}(t), \hat{h}_1(t),\dots, \hat{h}_{d+1}(t))^T$ (at $B=\hat{B}$) satisfying the initial condition $(\hat{\Phi}(-1), \hat{\Phi}_2(-1),\dots, \hat{\Phi}_{d+2}(-1))^T=(E_n, 0,\dots, 0)^T$, and $\hat{\xi}_j=(\hat{\xi}_{1j}, \hat{\xi}_{2j},\dots, \hat{\xi}_{d+1j})^T$ $(j=1, 2,\dots, d+1)$ are $n(d+1)$-dimensional vectors defined by the following way.

Let $\hat{\eta}^{(j)}(t)=(\hat{\eta}_1^{(j)}(t), \hat{\eta}_2^{(j)}(t),\dots, \hat{\eta}_{d+1}^{(j)}(t))^T$ $(j=1, 2,\dots, d+1)$ be solutions of the systems

$$(4.34)\qquad \begin{cases} \dfrac{dz_1}{dt}=\hat{X}^{(0)}z_1+X_{B_j}(\hat{x}(t), \hat{B}, t), \\ \dfrac{dz_2}{dt}=\hat{X}^{(0)}z_2+\hat{X}^{(1)}z_1+\hat{X}_{B_j}^{(0)}\hat{h}_1(t), \qquad\qquad (j=1, 2,\dots, d+1) \\ \vdots \\ \dfrac{dz_{d+1}}{dt}=\displaystyle\sum_{k=0}^{d} {}_dC_k\hat{X}^{(k)}z_{d+1-k}+\sum_{k=0}^{d-1} {}_{d-1}C_k\hat{X}_{B_j}^{(k)}\hat{h}_{d-k}(t) \end{cases}$$

satisfying the initial condition $\hat{\eta}^{(j)}(-1)=(0, 0,\dots, 0)^T$ $(j=1, 2,\dots, d+1)$ respectively and we define the vectors $\hat{\xi}_j$ $(j=1, 2,\dots, d+1)$ by

$$\hat{\xi}_j=(\hat{\xi}_{1j}, \hat{\xi}_{2j},\dots, \hat{\xi}_{d+1j})^T=\Big(\sum_{i=0}^{N} L_i\hat{\eta}_1^{(j)}(t_i), \sum_{i=0}^{N} L_i\hat{\eta}_2^{(j)}(t_i),\dots \tag{4.35}$$
$$\dots, \sum_{i=0}^{N} L_i\hat{\eta}_{d+1}^{(j)}(t_i)\Big)^T \quad (j=1, 2,\dots, d+1) \text{ respectively,}$$

where $\hat{X}^{(k)}$ $(k=0, 1, 2,\dots, d)$ and $\hat{X}_{B_j}^{(m)}$ $(m=0, 1,\dots, d-1;\ j=1, 2,\dots, d+1)$ mean the values of $X^{(k)}$ and $X_{B_j}^{(m)}$ at $x=\hat{x}(t)$, $h_1=\hat{h}_1(t),\dots, h_{d+1}=\hat{h}_{d+1}(t)$ and $B=\hat{B}$, respectively. Here $X_{B_j}^{(m)}$ $(m=0, 1,\dots, d-1; j=1, 2,\dots, d+1)$ are the partial derivatives of $X^{(m)}$ with respect to $B_j$, respectively.

Now we consider the isolatedness of the solution $(\hat{x}(t), \hat{h}_1(t),\dots, \hat{h}_{d+1}(t), \hat{B})^T$ of the multi-point boundary value problem (4.29)–(4.30).

In this case, the system corresponding to (3.34) in Section 3.1 is of the following form:

$$H_d(\boldsymbol{x}_d)=\begin{pmatrix} \sum_{i=0}^{N} L_i\varphi(t_i, \boldsymbol{x}_d)-c \\ \sum_{i=0}^{N} L_i\varphi_1(t_i, \boldsymbol{x}_d) \\ \vdots \\ \sum_{i=0}^{N} L_i\varphi_{d+1}(t_i, \boldsymbol{x}_d) \\ \psi_d(\boldsymbol{x}_d) \end{pmatrix}=0, \tag{4.36}$$

where $\boldsymbol{x}_d=(x(-1), h_1(-1),\dots, h_{d+1}(-1), B)^T$, $h_j(-1)=(h_j^1, h_j^2,\dots, h_j^n)^T$ $(j=1, 2,\dots, d+1)$, $B=(B_1,\dots, B_{d+1})^T$, $\psi_d(\boldsymbol{x}_d)=(h_1^1-1, h_2^1,\dots, h_{d+1}^1)^T$, and $(\varphi(t, \boldsymbol{x}_d), \varphi_1(t, \boldsymbol{x}_d),\dots, \varphi_{d+1}(t, \boldsymbol{x}_d))^T$ is a solution of (4.29) such that $(\varphi(-1, \boldsymbol{x}_d), \varphi_1(-1, \boldsymbol{x}_d),\dots, \varphi_{d+1}(-1, \boldsymbol{x}_d))^T=(x(-1), h_1(-1),\dots, h_{d+1}(-1))^T$. Then, of course, the initial value $\hat{\boldsymbol{x}}_d=(\hat{x}(-1), \hat{h}_1(-1), \dots, \hat{h}_{d+1}(-1), \hat{B})^T$ of the solution $(\hat{x}(t), \hat{h}_1(t),\dots, \hat{h}_{d+1}(t), \hat{B})^T$ of the problem (4.29)–(4.30) certainly becomes a solution of the system (4.36). Further, by (4.31) and (4.32), analogously to Theorem 15, for this solution $\hat{\boldsymbol{x}}_d$, we have

**Theorem 18.**
*The solution* $\hat{\boldsymbol{x}}_d$ *of* (4.36) *is isolated.*

**Remark 10.**
In fact, the condition $\operatorname{rank}(\hat{G}_1, \hat{l}_{d+1})=n$ guarantees the isolatedness of the solution $\hat{\boldsymbol{x}}_d$ of the system (4.36).

As is seen from Theorem 12 in Section 2.2, for the solution $\hat{\boldsymbol{x}}_d$, we have

$$\begin{cases} \text{the matrix } H_d'(\hat{\boldsymbol{x}}_d) \text{ is non-singular if and only if} \\ \operatorname{rank}(\hat{G}_1, \hat{l}_{d+1})=n, \end{cases}$$

where $H_d'(\boldsymbol{x}_d)$ denotes the Jacobian matrix of $H_d(\boldsymbol{x}_d)$ with respect to $\boldsymbol{x}_d$.

The solution $(\hat{x}(t), \hat{h}_1(t), \ldots, \hat{h}_{d+1}(t), \hat{B})^T$ of the multi-point boundary value problem (4.29)–(4.30) satisfying $\det H'_d(\hat{x}_d) \neq 0$ is called to be "*isolated*".

Lastly, in order to estimate an error bound for an obtained approximation to the isolated solution of (4.29)–(4.30), we give an existence theorem analogous to Proposition 3 in [4].

We consider the multi-point boundary value problem

$$F(u) \equiv \begin{pmatrix} \dfrac{dx}{dt} - X(u, t) \\ f(u) \end{pmatrix} = 0, \tag{4.37}$$

where $x$ and $X(u, t)$ are $m$-dimensional vectors and $u = (x, B)^T$, $B = (B_1, \ldots, B_d)^T$ is a parameter and

$$f(u) \equiv \begin{pmatrix} \sum_{i=0}^{N} L_i x(t_i) - c \\ g(u) \end{pmatrix} = 0. \tag{4.38}$$

Here $L_i$ $(i = 0, 1, 2 \ldots, N)$ are the given $m \times m$ matrices and $c$ is a given $m$-dimensional vector and $g(u)$ is a suitable $d$-dimensional vector-function.

We assume that $X(u, t)$ is continuously differentiable with respect to $u$ in the region $\tilde{\Omega}$ and that $X(u, t)$ and its first partial derivatives with respect to $u$ are all continuous on the region $\tilde{\Omega}$, where $\tilde{\Omega}$ is a given region of the $(u, t)$-space intercepted by two hyperplanes $t = -1$ and $t = 1$.
Further, we assume that the vector-function $g(u)$ is continuously Fréchet differentiable with respect to $u$.

If we obtain a "*good*" approximate solution of (4.37), then we can guarantee the existence of an exact solution of (4.37) in a sufficiently small neighbourhood of this approximation by applying the following existence theorem:

**Theorem 19.**

*Assume that the equation* (4.37) *has an approximate solution* $u = \bar{u}(t)$ *in* $S$ *satisfying* $\det f'(\bar{u}(t))[\Psi(t)] \neq 0$, *where* $\Psi(t)$ *is the fundamental matrix of the homogeneous system*

$$\frac{dz}{dt} = \begin{pmatrix} X_u(\bar{u}(t), t) \\ \boxed{0} \end{pmatrix} z \tag{4.39}$$

*satisfying the initial condition* $\Psi(-1) = E_{m+d}$.

*Further, let* $\mu_1, \mu_{m+1}, \ldots, \mu_{m+d}$ *and* $r$ *be positive numbers such that*

$$\begin{cases} \mu_1 \geq \max(\|H_{11}\|_c, \|H_{21}\|_c), \\ \mu_{m+1} \geq \max(|H_{1m+1}|, |H_{2m+1}|), \\ \vdots \\ \mu_{m+d} \geq \max(|H_{1m+d}|, |H_{2m+d}|) \end{cases} \tag{4.40}$$

*and*

$$(4.41)\qquad r \geq |||F(\bar{u})||| = \left\| \frac{d\bar{x}}{dt} - X(\bar{u}(t), t) \right\|_c + \|f(\bar{u})\|_{m+d}.$$

*If there exist positive constants* $\delta_1, \delta_{m+1}, \ldots, \delta_{m+d}$ *and a non-negative constant* $\kappa < 1$ *such that*

$$(4.42)\qquad D'_\delta = \{u(t) = (x(t), B)^T;\ \|x(t) - \bar{x}(t)\|_c \leq \delta_1,\ |B_1 - \bar{B}_1| \leq \delta_{m+1}, \ldots$$
$$\ldots, |B_d - \bar{B}_d| \leq \delta_{m+d},\ u(t) \in C[I] \times R^d\} \subset S'$$
$$(\textit{where } I = [-1, 1]),$$

$$(4.43)\qquad \|X_u(u(t), t) - X_u(\bar{u}(t), t)\|_c + \|f'(u) - f'(\bar{u})\|_{m+d}$$
$$\leq \frac{\kappa}{\mu_1 + \mu_{m+1} + \cdots + \mu_{m+d}} \qquad \textit{on} \quad D'_\delta,$$

$$(4.44)\qquad \frac{\mu_1 r}{1-\kappa} \leq \delta_1,\ \frac{\mu_{m+1} r}{1-\kappa} \leq \delta_{m+1}, \ldots, \frac{\mu_{m+d} r}{1-\kappa} \leq \delta_{m+d},$$

*then the equation* (4.37) *has one and only one solution* $\hat{u}(t) = (\hat{x}(t), \hat{B})^T$ *in*

$$(4.45)\qquad D_\delta = \{u(t) = (x(t), B)^T;\ \|x(t) - \bar{x}(t)\|_c \leq \delta_1,\ |B_1 - \bar{B}_1| \leq \delta_{m+1}, \ldots$$
$$\ldots, |B_d - \bar{B}_d| \leq \delta_{m+d},\ u(t) \in C^1[I] \times R^d\}$$

*and for this solution* $\hat{u}(t)$ *we have*

$$(4.46)\qquad \|\hat{x}(t) - \bar{x}(t)\|_c \leq \frac{\mu_1 r}{1-\kappa},\ |\hat{B}_1 - \bar{B}_1| \leq \frac{\mu_{m+1} r}{1-\kappa}, \ldots, |\hat{B}_d - \bar{B}_d| \leq \frac{\mu_{m+d} r}{1-\kappa}.$$

This existence theorem is essentially the same as the one we have obtained for a periodic solution of periodic differential systems involving parameters. For details of the definitions and the notations of norms, operators, etc. appeared in the theorem, see [24] and [25].

**Remark 11.**
We can also obtain, for a solution of an autonomous system satisfying (4.38), an existence theorem similar to Theorem 19.

## §5. Examples

In this section, we give examples of solutions of nonlinear equations with singular Jacobian matrices and examples of singular points of nonlinear equations defined by solutions of boundary value problems of nonlinear ordinary differential equations involving parameters. Further, we give examples of bifurcations of solutions of boundary value problems of nonlinear ordinary differential equations involving parameters.

Now, we consider solutions of nonlinear equations with singular Jacobian matrices. First, we consider the case where rank $F_x(\hat{x}) = n-1$.

**Example 1** ([8]).

We consider the equation

$$F(x) = \begin{pmatrix} x_1^2 - x_2 + \alpha \\ -x_1 + x_2^2 + \alpha \end{pmatrix} = 0, \tag{5.1}$$

where $x = (x_1, x_2)^T$ and $\alpha = 0.25$. This equation (5.1) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2)^T = (0.5, 0.5)^T$ and the Jacobian matrix $F_x(x)$ of $F(x)$ with respect to $x$ is singular at the solution $\hat{x}$, that is, in this case,

$$\text{rank } F_x(\hat{x}) = \text{rank} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 1. \tag{5.2}$$

Then let us introduce a parameter $B$ in (5.1) and consider the system

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - Be_1 \\ F_x(x)h \\ h_1 - 1 \end{pmatrix} = 0, \tag{5.3}$$

where $\boldsymbol{x} = (x, h, B)^T$, $x = (x_1, x_2)^T$, $h = (h_1, h_2)^T$ and $e_1 = (1, 0)^T$. Then the system (5.3) has a solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0)^T$ (where $\hat{x} = (0.5, 0.5)^T$ and $\hat{h} = (1, 1)^T$) and for this solution $\hat{\boldsymbol{x}}$, we have

$$\det G'(\hat{\boldsymbol{x}}) \neq 0 \tag{5.4}$$

since

$$G'(\boldsymbol{x}) = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.5}$$

where $G'(\boldsymbol{x})$ is the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$.

**Example 2** ([18]).

Let us consider the equation

$$(5.6)\qquad F(x)=\begin{pmatrix} x_1^2-2x_1+\frac{1}{3}x_2^3+\frac{2}{3} \\ x_1^3-x_1x_2-2x_1+\frac{1}{2}x_2^2+\frac{3}{2} \end{pmatrix}=0,$$

where $x=(x_1, x_2)^T$. The equation (5.6) has a solution $\hat{x}=(\hat{x}_1, \hat{x}_2)^T=(1, 1)^T$ and for this solution $\hat{x}$ we have

$$(5.7)\qquad \operatorname{rank} F_x(\hat{x})=\operatorname{rank}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=1.$$

Since rank $(F_x(\hat{x}), e_2)=2$, we introduce a parameter $B$ in (5.6) and we consider the system

$$(5.8)\qquad G(\boldsymbol{x})=\begin{pmatrix} F(x)-Be_2 \\ F_x(x)h \\ h_1-1 \end{pmatrix}=0,$$

where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1, x_2)^T$, $h=(h_1, h_2)^T$ and $e_2=(0, 1)^T$. The system (5.8) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0)^T$ (where $\hat{x}=(\hat{x}_1, \hat{x}_2)^T=(1, 1)^T$ and $\hat{h}=(\hat{h}_1, \hat{h}_2)^T=(1, 0)^T$) and for this solution $\hat{\boldsymbol{x}}$, we have

$$(5.9)\qquad \det G'(\hat{\boldsymbol{x}})\neq 0$$

since

$$(5.10)\qquad G'(\hat{\boldsymbol{x}})=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 \\ 6 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

H. Weber and W. Werner [18] also considered this example. As is seen from Remark 3 in Section 2.1, their method for introducing a parameter in (5.6) is different from ours. In fact, in this example, they considered the system

$$(5.11)\qquad H(\boldsymbol{x})=\begin{pmatrix} F_x(x)^TF(x)+Bh \\ F_x(x)h \\ h^Th-1 \end{pmatrix}=0,$$

where $\boldsymbol{x}=(x, h, B)^T$, $x=(x_1, x_2)^T$, $h=(h_1, h_2)^T$ and $F_x(x)^T$ denotes the transposed matrix of $F_x(x)$ and $B$ is a parameter. Comparing the system (5.11) with the system

(5.8), it seems that our method is more useful and convenient than theirs. Of course, we may adopt the condition $h_1 - 1 = 0$ instead of the condition $h^T h - 1 = 0$ in (5.11).

**Example 3** ([15]).

Let us consider the equation

$$F(x) = \begin{pmatrix} x_1^3 + x_1 x_2 \\ x_2 + x_2^2 \end{pmatrix} = 0, \tag{5.12}$$

where $x = (x_1, x_2)^T$. Evidently, the equation (5.12) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2)^T = (0, 0)^T$ and for this solution $\hat{x}$ we have

$$\operatorname{rank} F_x(\hat{x}) = \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1. \tag{5.13}$$

In this case, since $\hat{l} = \{F_{xx}(\hat{x})\hat{h}\}\hat{h} = (0, 0)^T$, we have

$$\operatorname{rank}(F_x(\hat{x}), \hat{l}) = \operatorname{rank}(F_0(\hat{x}), \hat{l}) = 1, \tag{5.14}$$

where $\hat{h} = (1, 0)^T$ and $F_0(\hat{x}) = (0, 1)^T$.

Therefore we introduce two parameters $B_1$, $B_2$ and consider the system

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - B_1 e_1 \\ F_x(x) h_1 - B_2 e_1 \\ F_x(x) h_2 + \{F_{xx}(x) h_1\} h_1 \\ h_1^1 - 1 \\ h_2^1 \end{pmatrix} = 0, \tag{5.15}$$

where $\boldsymbol{x} = (x, h_1, h_2, B_1, B_2)^T$, $x = (x_1, x_2)^T$, $h_i = (h_i^1, h_i^2)^T$ $(i = 1, 2)$ and $e_1 = (1, 0)^T$. Then the system (5.15) has a solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$ (where $\hat{x} = (0, 0)^T$, $\hat{h}_1 = (1, 0)^T$ and $\hat{h}_2 = (0, 0)^T$) and for this solution $\hat{\boldsymbol{x}}$ we have

$$\det G'(\hat{\boldsymbol{x}}) \neq 0 \tag{5.16}$$

since

$$G'(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.17}$$

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{matrix}\Bigg)
$$

**Example 4** ([20]).
Let us consider the equation

$$
F(x)=\begin{pmatrix} 8x_1+x_2^2-12 \\ x_1^2+x_2-3 \end{pmatrix}=0, \tag{5.18}
$$

where $x=(x_1, x_2)^T$. The equation (5.18) has a solution $\hat{x}=(\hat{x}_1, \hat{x}_2)^T=(1, 2)^T$ and for this solution $\hat{x}$, the Jacobian matrix $F_x(\hat{x})$ has the same singularity as in Example 3. Then, considering the system (5.15) once more, $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}_1, \hat{h}_2, 0, 0)^T$ (where $\hat{x}=(1, 2)^T$, $\hat{h}_1=(1, -2)^T$ and $\hat{h}_2=(0, -2)^T$) is a solution of (5.15) and for this solution $\hat{\boldsymbol{x}}$ we have

$$
\det G'(\hat{\boldsymbol{x}})\neq 0 \tag{5.19}
$$

since

$$
G'(\hat{\boldsymbol{x}})=\begin{pmatrix}
8 & 4 & 0 & 0 & 0 & 0 & -1 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 8 & 4 & 0 & 0 & 0 & -1 \\
2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & -8 & 8 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. \tag{5.20}
$$

Secondly, we consider the case where rank $F_x(\hat{x})=n-2$.

**Example 5** ([16]).
Let us consider the equation

$$
F(x)=\begin{pmatrix} x_1^2+x_2^3 \\ x_2^2 \end{pmatrix}=0, \tag{5.21}
$$

where $x=(x_1, x_2)^T$. Obviously, the equation (5.21) has a solution $\hat{x}=(\hat{x}_1, \hat{x}_2)^T=(0, 0)^T$ and for this solution $\hat{x}$ we have

(5.22)
$$\operatorname{rank} F_x(\hat{x}) = \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Therefore we introduce two parameters $B_1$, $B_2$ in (5.21) and we consider the system

(5.23)
$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - B_1 e_1 - B_2 e_2 \\ F_x(x)h \\ h_1 - a_1 \\ h_2 - a_2 \end{pmatrix} = 0,$$

where $\boldsymbol{x} = (x, h, B_1, B_2),^T$ $x = (x_1, x_2),^T$ $h = (h_1, h_2)^T$, $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. In this example, we take $a_1 = a_2 = 1$. Then, $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0, 0)^T$ (where $\hat{x} = (0, 0)^T$ and $\hat{h} = (1, 1)^T$) is a solution of (5.23) and for this solution $\hat{\boldsymbol{x}}$ we have

(5.24)
$$\det G'(\hat{\boldsymbol{x}}) \neq 0$$

since

(5.25)
$$G'(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

**Example 6** ([7]).
We consider the equation

(5.26)
$$F(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = 0,$$

where $x = (x_1, x_2)^T$ and

(5.27)
$$\begin{cases} f_1(x_1, x_2) = x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4 - 3x_1^4 + 18x_1^2x_2^2 - 3x_2^4 - 2x_1^3 + 6x_1x_2^2 \\ \qquad + 3x_1^2x_2 - x_2^3 + 12x_1^2 - 12x_2^2 - 10x_1x_2 - 8x_1 + 8x_2, \\ f_2(x_1, x_2) = 5x_1^4x_2 - 10x_1^2x_2^3 + x_2^5 - 12x_1^3x_2 + 12x_1x_2^3 - x_1^3 + 3x_1x_2^2 \\ \qquad - 6x_1^2x_2 + 2x_2^3 + 5x_1^2 - 5x_2^2 + 24x_1x_2 - 8x_1 - 8x_2 + 4. \end{cases}$$

The equation (5.26) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2)^T = (2, 0)^T$ and for this solution $\hat{x}$ we have

$$\text{rank } F_x(\hat{x}) = \text{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{5.28}$$

Hence, this example is the same type as Example 5. Then we consider the system (5.23) again. In this case, we take $a_1 = 1$ and $a_2 = 0$ in (5.23). Evidently, $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}, 0, 0)^T$ (where $\hat{x} = (2, 0)^T$ and $\hat{h} = (1, 0)^T$) is a solution of (5.23) and for this solution $\hat{\boldsymbol{x}}$ we have

$$\det G'(\hat{\boldsymbol{x}}) \neq 0 \tag{5.29}$$

since

$$G'(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 16 & 2 & 0 & 0 & 0 & 0 \\ -2 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.30}$$

**Example 7** ([6]).
We consider the equation

$$F(x) = \begin{pmatrix} x_1 + x_2 + x_3 - 1 \\ 0.2x_1^3 + 0.5x_2^2 - x_3 + 0.5x_3^2 + 0.5 \\ x_1 + x_2 + 0.5x_3^2 - 0.5 \end{pmatrix} = 0, \tag{5.31}$$

where $x = (x_1, x_2, x_3)^T$. The equation (5.31) has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = (0, 0, 1)^T$ and for this solution $\hat{x}$ we have

$$\text{rank } F_x(\hat{x}) = \text{rank}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1. \tag{5.32}$$

Therefore we introduce two parameters $B_1$, $B_2$ in (5.31) and we consider the system

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x) - B_1 e_1 - B_2 e_2 \\ F_x(x)h \\ h_1 - a_1 \\ h_2 - a_2 \end{pmatrix} = 0, \tag{5.33}$$

where $\boldsymbol{x}=(x, h, B_1, B_2)^T$, $x=(x_1, x_2, x_3)^T$, $h=(h_1, h_2, h_3)^T$, $e_1=(1, 0, 0)^T$, $e_2=(0, 1, 0)^T$.

In this example, we take $a_1=0$ and $a_2=1$. Then the system (5.33) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0, 0)^T$ (where $\hat{x}=(0, 0, 1)^T$ and $\hat{h}=(0, 1, -1)^T$) and for this solution $\hat{\boldsymbol{x}}$ we have

$$\det G'(\hat{\boldsymbol{x}})\neq 0 \tag{5.34}$$

since

$$G'(\hat{\boldsymbol{x}})=\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.35}$$

**Example 8** ([21]).

Let us consider the equation

$$F(x)=\begin{pmatrix} x_1^2+x_2^2+3x_3+x_4-18 \\ x_1^2+x_2^2+\frac{1}{2}x_3^2-x_4-\frac{11}{2} \\ 2x_1+x_2^2+\frac{1}{2}x_3^2+2x_4-\frac{37}{2} \\ 2x_1+4x_2+\frac{1}{2}x_3^2-2x_4-\frac{13}{2} \end{pmatrix}=0, \tag{5.36}$$

where $x=(x_1, x_2, x_3, x_4)^T$. The equation (5.36) has a solution $\hat{x}=(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)^T=(1, 2, 3, 4)^T$ and for this solution $\hat{x}$ we have

$$\operatorname{rank} F_x(\hat{x})=\operatorname{rank}\begin{pmatrix} 2 & 4 & 3 & 1 \\ 2 & 4 & 3 & -1 \\ 2 & 4 & 3 & 2 \\ 2 & 4 & 3 & -2 \end{pmatrix}=2. \tag{5.37}$$

Therefore we introduce two parameters $B_1$, $B_2$ in (5.36) and we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x)-B_1e_1-B_2e_2 \\ F_x(x)h \\ h_1-a_1 \\ h_2-a_2 \end{pmatrix}=0, \tag{5.38}$$

where $\boldsymbol{x}=(x, h, B_1, B_2)^T$, $x=(x_1, x_2, x_3, x_4)^T$, $h=(h_1, h_2, h_3, h_4)^T$, $e_1=(1, 0, 0, 0)^T$ and $e_2=(0, 1, 0, 0)^T$.

In this example, we take $a_1=a_2=1$. Then the system (5.38) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, 0, 0)^T$ (where $\hat{x}=(1, 2, 3, 4)^T$ and $\hat{h}=(1, 1, -2, 0)^T$) and for this solution $\hat{\boldsymbol{x}}$

$$\det G'(\hat{\boldsymbol{x}})\neq 0 \tag{5.39}$$

since

$$G'(\hat{\boldsymbol{x}})=\begin{pmatrix} 2 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 4 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 4 & 3 & 1 & 0 & 0 \\ 2 & 2 & -2 & 0 & 2 & 4 & 3 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 2 & 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 & 4 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.40}$$

Next, we consider the case where $n=1$.

**Example 9.**

$F(x)=x^2=0$. Since $F_x(x)=2x$ and $F_{xx}(x)=2$, we have

$$\hat{l}_1=\{F_{xx}(\hat{x})\hat{h}_1\}\hat{h}_1=2\hat{h}_1^2=2\neq 0, \tag{5.41}$$

where $\hat{x}=0$ and $\hat{h}_1=1$.

Then, introducing a parameter in the equation $F(x)=0$, we consider the system

$$G(\boldsymbol{x})=\begin{pmatrix} F(x)-B \\ F_x(x)h \\ h-1 \end{pmatrix}=\begin{pmatrix} x^2-B \\ 2xh \\ h-1 \end{pmatrix}=0, \tag{5.42}$$

where $\boldsymbol{x}=(x, h, B)^T$ and $h=h_1$.
Let $G'(\boldsymbol{x})$ denote the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Then we have

$$(5.43)\qquad G'(\boldsymbol{x})=\begin{pmatrix} 2x & 0 & -1 \\ 2h & 2x & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

from which it follows that

$$(5.44)\qquad \det G'(\hat{\boldsymbol{x}})=\det\begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\neq 0$$

for a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}, \hat{B})^T=(0, 1, 0)^T$ of (5.42).

**Example 10.**

$F(x)=x^3=0$. Since $F_x(x)=3x^2$, $F_{xx}(x)=6x$ and $F_{xxx}(x)=6$, we have

$$(5.45)\qquad \begin{cases} \hat{l}_1=\{F_{xx}(\hat{x})\hat{h}_1\}\hat{h}_1=6\hat{x}\hat{h}_1^2=0, \\ \hat{l}_2=\sum_{i=1}^{2} {}_2C_i\hat{X}^{(i)}\hat{h}_{3-i}=2\hat{X}^{(1)}\hat{h}_2+\hat{X}^{(2)}\hat{h}_1 \\ \quad =\hat{X}^{(2)}\hat{h}_1=(6\hat{x}\hat{h}_2+6\hat{h}_1^2)\hat{h}_1=6\neq 0, \end{cases}$$

where $\hat{x}=0$, $\hat{h}_1=1$ and $\hat{h}_2=0$.
Thus, in this example, introducing two parameters $B_1$, $B_2$, we consider the system

$$(5.46)\qquad G_1(\boldsymbol{x}_1)=\begin{pmatrix} F(x)-B_1 \\ F_x(x)h_1-B_2 \\ F_x(x)h_2+\{F_{xx}(x)h_1\}h_1 \\ h_1-1 \\ h_2 \end{pmatrix}=\begin{pmatrix} x^3-B_1 \\ 3x^2h_1-B_2 \\ 3x^2h_2+6xh_1^2 \\ h_1-1 \\ h_2 \end{pmatrix}=0,$$

where $\boldsymbol{x}_1=(x, h_1, h_2, B_1, B_2)^T$.

Let $G_1'(\boldsymbol{x}_1)$ be the Jacobian matrix of $G_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$. Then we have

$$(5.47)\qquad G_1'(\boldsymbol{x}_1)=\begin{pmatrix} 3x^2 & 0 & 0 & -1 & 0 \\ 6xh_1 & 3x^2 & 0 & 0 & -1 \\ 6xh_2+6h_1^2 & 2\cdot 6xh_1 & 3x^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The system (5.46) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{B}_1, \hat{B}_2)^T=(0, 1, 0, 0, 0)^T$ and for this solution $\hat{\boldsymbol{x}}_1$ we have

$$\det G_1'(\hat{\boldsymbol{x}}_1)=\det\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \neq 0. \tag{5.48}$$

**Example 11.**

$F(x)=x^4=0$. Since $F_x(x)=4x^3$, $F_{xx}(x)=12x^2$, $F_{xxx}(x)=24x$ and $F_{xxxx}(x)=24$, we have

$$\begin{cases} \hat{l}_1=\{F_{xx}(\hat{x})\hat{h}_1\}\hat{h}_1=12\hat{x}^2\hat{h}_1^2=0, \\ \hat{l}_2=2\{F_{xx}(\hat{x})\hat{h}_1\}\hat{h}_2+\{F_{xx}(\hat{x})\hat{h}_2+F_{xxx}(\hat{x})\hat{h}_1^2\}\hat{h}_1=0, \\ \hat{l}_3=\sum_{i=1}^{3} {}_3C_i\hat{X}^{(i)}\hat{h}_{4-i}={}_3C_1\hat{X}^{(1)}\hat{h}_3+{}_3C_2\hat{X}^{(2)}\hat{h}_2+{}_3C_3\hat{X}^{(3)}\hat{h}_1 \\ \quad={}_3C_3\hat{X}^{(3)}\hat{h}_1=\{12\hat{x}^2\hat{h}_3+2\cdot 24\hat{x}\hat{h}_1\hat{h}_2+(24\hat{x}\hat{h}_2+24\hat{h}_1^2)\hat{h}_1\}\hat{h}_1 \\ \quad=24\neq 0, \end{cases} \tag{5.49}$$

where $\hat{x}=0$, $\hat{h}_1=1$, $\hat{h}_2=0$ and $\hat{h}_3=0$.

Then, in this example, introducing three parameters $B_1$, $B_2$, $B_3$, we consider the system

$$G_2(\boldsymbol{x}_2)=\begin{pmatrix} F(x)-B_1 \\ F_x(x)h_1-B_2 \\ F_x(x)h_2+\{F_{xx}(x)h_1\}h_1-B_3 \\ F_x(x)h_3+2\{F_{xx}(x)h_1\}h_2+\{F_{xx}(x)h_2+F_{xxx}(x)h_1^2\}h_1 \\ h_1-1 \\ h_2 \\ h_3 \end{pmatrix} \tag{5.50}$$

$$
= \begin{pmatrix} x^4 - B_1 \\ 4x^3 h_1 - B_2 \\ 4x^3 h_2 + 12x^2 h_1^2 - B_3 \\ 4x^3 h_3 + 2\cdot 12x^2 h_1 h_2 + (12x^2 h_2 + 24x h_1^2) h_1 \\ h_1 - 1 \\ h_2 \\ h_3 \end{pmatrix} = 0,
$$

where $\boldsymbol{x}_2 = (x, h_1, h_2, h_3, B_1, B_2, B_3)^T$.

We denote by $G_2'(\boldsymbol{x}_2)$ the Jacobian matrix of $G_2(\boldsymbol{x}_2)$ with respect to $\boldsymbol{x}_2$ and we have

$$
(5.51) \qquad G_2'(\boldsymbol{x}_2) = \begin{pmatrix} 4x^3 & 0 & 0 & 0 & -1 & 0 & 0 \\ 12x^2 h_1 & 4x^3 & 0 & 0 & 0 & -1 & 0 \\ (\sharp) & 2\cdot 12x^2 h_1 & 4x^3 & 0 & 0 & 0 & -1 \\ (\sharp\sharp) & 3\cdot(\sharp) & 3\cdot 12x^2 h_1 & 4x^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},
$$

where $(\sharp) = 12x^2 h_2 + 24x h_1^2$ and

$$
(\sharp\sharp) = 12x^2 h_3 + 2\cdot 24x h_1 h_2 + (24x h_2 + 24h_1^2) h_1.
$$

The system (5.50) has a solution $\hat{\boldsymbol{x}}_2 = (\hat{x}, \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{B}_1, \hat{B}_2, \hat{B}_3)^T = (0, 1, 0, 0, 0, 0, 0)^T$ and for this solution $\hat{\boldsymbol{x}}_2$ we have

$$
(5.52) \qquad \det G_2'(\hat{\boldsymbol{x}}_2) = \det \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \neq 0.
$$

Next, we consider an example of a turning point of a curve of the $(n+1)$-dimensional space (that is, the $(x, B)$-space).

**Example 12** ([12], [17]).
Let us consider the equation

$$
(5.53)\quad
\begin{cases}
F_1 = \dfrac{x_1 - x_3}{10000} + \dfrac{x_1 - x_2}{39} + \dfrac{x_1 + B}{51} = 0, \\[2ex]
F_2 = \dfrac{x_2 - x_6}{10} + \dfrac{x_2 - x_1}{39} + I(x_2) = 0, \\[2ex]
F_3 = \dfrac{x_3 - x_1}{10000} + \dfrac{x_3 - x_4}{25.5} = 0, \\[2ex]
F_4 = \dfrac{x_4 - x_3}{25.5} + \dfrac{x_4}{0.62} - x_5 + x_4 = 0, \\[2ex]
F_5 = \dfrac{x_5 - x_6}{13} + x_5 - x_4 + I(x_5) = 0, \\[2ex]
F_6 = \dfrac{x_6 - x_2}{10} + \dfrac{x_6 - x_5}{13} + \dfrac{x_6 - U(x_3 - x_1)}{0.201} = 0,
\end{cases}
$$

where

$$(5.54)\qquad I(y) = 5.6 \times 10^{-8}(e^{25y} - 1) \quad \text{and} \quad U(y) = 7.65 \times \tan^{-1}(1962y).$$

The equation (5.53) defines a curve in the $(x, B)$-space and this curve has turning points, where $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. We compute one of them. The results of numerical computations are as follows:

$$
(5.55)\quad
\begin{cases}
\hat{x}_1 = 0.49366\ 97072\ 217 \times 10^{-1}, & \hat{x}_2 = 0.54735\ 84097\ 315 \times 10^{0}, \\
\hat{x}_3 = 0.49447\ 20687\ 332 \times 10^{-1}, & \hat{x}_4 = 0.49447\ 41147\ 550 \times 10^{-1}, \\
\hat{x}_5 = 0.12920\ 13089\ 757 \times 10^{0}, & \hat{x}_6 = 0.11660\ 19652\ 671 \times 10^{1}, \\
\hat{B} = 0.60185\ 30125\ 713 \times 10^{0}.
\end{cases}
$$

The above results are equal to the ones given in [12] if we round off these results to nine decimal places.

Now we consider periodic solutions of periodic differential systems involving parameters. As has been mentioned in Section 3, in this case, the form of a nonlinear equation is implicit. But the theory and method used for investigating singular points of such a nonlinear equation implicit in form are the same as the ones used in the case where the form of a nonlinear equation is explicit.

**Example 13** ([23]).
Let us consider a $2\pi$-periodic solution of the Duffing equation

(5.56) $$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+x^3=B\cdot\cos t.$$

The equation (5.56) can be rewritten in the form of a first order system as follows:

(5.57) $$\begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-x_1^3-kx_2+B\cdot\cos t. \end{cases}$$

Then we consider the system (5.57) together with the periodic boundary conditions

(5.58) $$\begin{cases} x_1(0)-x_1(2\pi)=0, \\ x_2(0)-x_2(2\pi)=0. \end{cases}$$

Let $\varphi(t, x(0), \beta)=(\varphi_1(t, x(0), \beta), \varphi_2(t, x(0), \beta))^T$ be a solution of (5.57) with $\varphi(0, x(0), \beta)=x(0)$, where $x(0)=(x_1(0), x_2(0))^T$ and $\beta=(k, B)^T$. Then we consider the nonlinear equation

(5.59) $$F(x(0), \beta)=\begin{pmatrix} \varphi_1(0, x(0), \beta)-\varphi_1(2\pi, x(0), \beta) \\ \varphi_2(0, x(0), \beta)-\varphi_2(2\pi, x(0), \beta) \end{pmatrix} = \begin{pmatrix} x_1(0)-\varphi_1(2\pi, x(0), \beta) \\ x_2(0)-\varphi_2(2\pi, x(0), \beta) \end{pmatrix}=0.$$

This equation has turning points and cusp points.

In the first place, we put $k=0.2$ and $B$ is unknown. In this case, the equation (5.59) has turning points. We compute one of them and the results of numerical computations are as follows:

(5.60) $$\begin{cases} \hat{x}_1(0)=-0.65391\ 94080\ 07, \quad \hat{x}_2(0)=0.22397\ 44989\ 65, \\ \hat{B}=0.46139\ 11552\ 41. \end{cases}$$

In the second place, let $k$ and $B$ be (unknown) parameters. In this case, the equation (5.59) has cusp points. We compute one of them and the results of numerical computations are as follows:

(5.61) $$\begin{cases} \hat{x}_1(0)=-0.44363\ 29326, & \hat{x}_2(0)=0.78058\ 36056, \\ \hat{k}=0.57730\ 83634, & \hat{B}=0.62295\ 09169. \end{cases}$$

Next, we consider bifurcations of periodic solutions of (5.56). We take $k=0.2$ in (5.56) and $B$ is the only unknown parameter.

**Example 14.**

First, we consider bifurcations corresponding to Case (I) in Section 3.2. As has been mentioned in Section 3.2, we consider a periodic solution of

$$(5.62)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-x_1^3-kx_2+B\cdot\cos t, \\ \dfrac{dh_1}{dt}=h_2, \\ \dfrac{dh_2}{dt}=-3x_1^2h_1-kh_2 \end{cases}$$

and

$$(5.63)\qquad \begin{cases} x_1(0)+x_1(\pi)=0, \\ x_2(0)+x_2(\pi)=0, \\ h_1(0)-h_1(\pi)=0, \\ h_2(0)-h_2(\pi)=0, \\ h_2(0)-1=0. \end{cases}$$

The results of numerical computations are as follows:

$$(5.64)\qquad \begin{cases} \hat{x}_1(0)=1.87793\ 92526, & \hat{x}_2(0)=0.63506\ 90547, \\ \hat{h}_1(0)=-0.11859\ 06031, & \hat{h}_2(0)=1.0, \\ \hat{B}=2.40337\ 71161. \end{cases}$$

**Example 15.**

Secondly, we consider bifurcations corresponding to Case (II) in Section 3.2. In this case, we consider the equation

$$(5.65)\qquad \frac{d^2x}{dt^2}+2k\frac{dx}{dt}+4x^3=4B\cdot\cos 2t \qquad (k=0.2)$$

instead of the equation (5.56). Thus, we consider a periodic solution of

$$(5.66)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-4x_1^3-2kx_2+4B\cdot\cos 2t, \\ \dfrac{dh_1}{dt}=h_2, \\ \dfrac{dh_2}{dt}=-12x_1^2h_1-2kh_2 \end{cases}$$

and

$$(5.67)\qquad \begin{cases} x_1(0)-x_1(\pi)=0, \\ x_2(0)-x_2(\pi)=0, \\ h_1(0)+h_1(\pi)=0, \\ h_2(0)+h_2(\pi)=0, \\ h_2(0)-1=0. \end{cases}$$

The results of numerical computations are as follows:

$$(5.68)\qquad \begin{cases} \hat{x}_1(0)=2.64693\,5, & \hat{x}_2(0)=4.30580\,6, \\ \hat{h}_1(0)=-0.07177\,8, & \hat{h}_2(0)=1.0, \\ \hat{B}=5.39106\,7. \end{cases}$$

Next, we consider two-point boundary value problems. First, we give examples of turning points and a cusp point.

**Example 16** ([11]).

Let us consider the following two-point boundary value problem:

$$(5.69)\qquad \frac{d^2y}{dt^2}=\delta y\cdot\exp\{\gamma\beta(1-y)/(1+\beta(1-y))\} \qquad (\beta=0.4)$$

and

$$(5.70)\qquad \begin{cases} \dfrac{dy}{dt}(0)=0, \\ y(1)=1. \end{cases}$$

We rewrite (5.69)–(5.70) in the following form:

$$(5.71)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=\delta x_1\cdot\exp\{\gamma\beta(1-x_1)/(1+\beta(1-x_1))\} \end{cases}$$

and

$$(5.72)\qquad \begin{cases} x_1(1)=1, \\ x_2(0)=0. \end{cases}$$

Let $\varphi(t, x(0), B)=(\varphi_1(t, x(0), B), \varphi_2(t, x(0), B))^T$ be a solution of (5.71) with $\varphi(0, x(0), B)=x(0)$, where $x(0)=(x_1(0), x_2(0))^T$ and $B=(\delta, \gamma)^T$. Then we consider the equation

$$F(x(0), B)=L_0\varphi(0, x(0), B)+L_1\varphi(1, x(0), B)-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_1(1, x(0), B)-1 \\ \varphi_2(0, x(0), B) \end{pmatrix}=0, \tag{5.73}$$

where $L_0=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $L_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

This equation (5.73) has turning points and a cusp point.

At first, we put $\gamma=20$ and the parameter $\delta$ is unknown. In this case, the equation (5.73) has two turning points. We compute both of them. The results of numerical computations are as follows:

$$\hat{x}_1(0)=0.22732\ 51767\ 297, \quad \hat{\delta}=0.07793\ 03111\ 191 \tag{5.74}$$

and

$$\hat{x}_1(0)=0.79283\ 87486\ 932, \quad \hat{\delta}=0.13755\ 74408\ 208. \tag{5.75}$$

Next, let $\delta$ and $\gamma$ be (unknown) parameters. In this case, the equation (5.73) has a cusp point. In [11], R. Seydel has suggested the existence of the cusp point, but he has given neither a method for calculating it nor its value.

In our case, as has been mentioned in Section 4, we can compute it. The results of numerical computations are as follows:

$$\begin{cases} \hat{x}_1(0)=0.52648\ 93016, \quad \hat{\delta}=0.22286\ 26066, \\ \hat{\gamma}=14.40322\ 20353. \end{cases} \tag{5.76}$$

Secondly, we give an example of a bifurcation point.

**Example 17** ([14]).

We consider bifurcations of solutions of the two-point boundary value problem

$$\frac{d^2x}{dt^2}+\mu(x+x^2)=0 \tag{5.77}$$

and

$$\begin{cases} \frac{dx}{dt}(0)=0, \\ x(1)=0. \end{cases} \tag{5.78}$$

The equation (5.77) can be rewritten in the form of a first order system as follows:

$$\begin{cases} \frac{dx_1}{dt}=x_2, \\ \frac{dx_2}{dt}=-\mu(x_1+x_1^2). \end{cases} \tag{5.79}$$

Let $\mu$ be a parameter and let us consider the equation (5.79) together with the first variation equation of (5.79) with respect to $x=x(t)$. That is, let us consider the following boundary value problem:

$$(5.80)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\[2mm] \dfrac{dx_2}{dt}=-\mu(x_1+x_1^2), \\[2mm] \dfrac{dh_1}{dt}=h_2, \\[2mm] \dfrac{dh_2}{dt}=-\mu(h_1+2x_1h_1) \end{cases}$$

and

$$(5.81)\qquad \begin{cases} L_0x(0)+L_1x(1)=0, \\ L_0h(0)+L_1h(1)=0, \\ h_1^0-1=0, \end{cases}$$

where $x(t)=(x_1(t),\ x_2(t))^T$, $h(t)=(h_1(t),\ h_2(t))^T$, $x(0)=(x_1(0),\ x_2(0))^T=(x_1^0,\ x_2^0)^T$, $h(0)=(h_1(0),\ h_2(0))^T=(h_1^0,\ h_2^0)^T$ and $L_0=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $L_1=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$.

Let $(\varphi(t,\ \boldsymbol{x}),\ \varphi_1(t,\ \boldsymbol{x}))^T$ be a solution of (5.80) such that $(\varphi(0,\ \boldsymbol{x}),\ \varphi_1(0,\ \boldsymbol{x}))^T=(x(0),\ h(0))^T$ and let us consider the equation

$$(5.82)\qquad F(\boldsymbol{x})=\begin{pmatrix} L_0\varphi(0,\ \boldsymbol{x})+L_1\varphi(1,\ \boldsymbol{x}) \\ L_0\varphi_1(0,\ \boldsymbol{x})+L_1\varphi_1(1,\ \boldsymbol{x}) \\ h_1^0-1 \end{pmatrix}=0,$$

where $\boldsymbol{x}=(x(0),\ h(0),\ \mu)^T$.
Let $F_x(\boldsymbol{x})$ denote the Jacobian matrix of $F(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Then we have

$$(5.83)\qquad F_x(\boldsymbol{x})=\begin{pmatrix} \Psi_1 & 0 & \xi_1 \\ \Psi_2 & \Psi_1 & \xi_2 \\ 00 & 10 & 0 \end{pmatrix},$$

where $\Psi_1=L_0\Phi(0)+L_1\Phi(1)$, $\Psi_2=L_0\Phi_2(0)+L_1\Phi_2(1)$ and

$$\xi_1=L_0\frac{\partial\varphi}{\partial\mu}(0,\ \boldsymbol{x})+L_1\frac{\partial\varphi}{\partial\mu}(1,\ \boldsymbol{x}),\quad \xi_2=L_0\frac{\partial\varphi_1}{\partial\mu}(0,\ \boldsymbol{x})+L_1\frac{\partial\varphi_1}{\partial\mu}(1,\ \boldsymbol{x}).$$

Here $(\Phi(t),\ \Phi_2(t))^T$ is a solution $(4\times 2$ matrix) of the first variation equation of

(5.80) with respect to $(x, h)^T=(\varphi(t, \boldsymbol{x}), \varphi_1(t, \boldsymbol{x}))^T$ (at the given $\mu$) satisfying the initial condition $(\Phi(0), \Phi_2(0))^T=(E_2, 0)^T$, and $\frac{\partial\varphi}{\partial\mu}$ and $\frac{\partial\varphi_1}{\partial\mu}$ are the partial derivatives of $\varphi$ and $\varphi_1$ with respect to $\mu$, respectively.

Here $\mu=\left(\frac{\pi}{2}\right)^2$ is one of bifurcation points of solutions of (5.77)–(5.78). Then $\hat{\boldsymbol{x}}=\left(0, 0, 1, 0, \left(\frac{\pi}{2}\right)^2\right)^T$ is a solution of (5.82) and for this solution $\hat{\boldsymbol{x}}$, we have

(5.84) $$\det F_x(\hat{\boldsymbol{x}})=0$$

because $\frac{\partial\varphi}{\partial\mu}(t, \hat{\boldsymbol{x}})\equiv 0$. In fact, in this case, since

(5.85) $$\hat{\Phi}(t)=\begin{pmatrix} \cos\frac{\pi}{2}t & \frac{2}{\pi}\sin\frac{\pi}{2}t \\ -\frac{\pi}{2}\sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix},$$

we have

(5.86) $$L_0\hat{\Phi}(0)+L_1\hat{\Phi}(1)=\begin{pmatrix} 0 & 1 \\ 0 & \frac{2}{\pi} \end{pmatrix}$$

and

(5.87) $$L_0\frac{\partial\varphi}{\partial\mu}(0, \hat{x})+L_1\frac{\partial\varphi}{\partial\mu}(1, \hat{x})=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which it follows that

(5.88) $$F_x(\hat{\boldsymbol{x}})=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{4}{3} & -\frac{4}{3\pi} & 0 & \frac{2}{\pi} & -\frac{1}{\pi} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By (5.88) we see that

(5.89) $$\operatorname{rank} F_x(\hat{x})=4.$$

Then we apply the method proposed in Section 2.1 to the equation (5.82). That is, introducing a parameter $B$ in (5.82), we consider the system

(5.90)
$$G(\boldsymbol{y})=\begin{pmatrix} F(\boldsymbol{x})-Be_1 \\ F_x(\boldsymbol{x})\boldsymbol{k} \\ k_5-1 \end{pmatrix}=0,$$

where $\boldsymbol{y}=(\boldsymbol{x}, \boldsymbol{k}, B)^T$, $\boldsymbol{x}=(x_1^0, x_2^0, h_1^0, h_2^0, \mu)^T$, $\boldsymbol{k}=(k_1, k_2, k_3, k_4, k_5)^T$ and $e_1=(1, 0, 0, 0, 0)^T$. Here, in the system (5.90), we may adopt the condition $k_1-1=0$ instead of the condition $k_5-1=0$.

Let $G'(\boldsymbol{y})$ be the Jacobian matrix of $G(\boldsymbol{y})$ with respect to $\boldsymbol{y}$. Then we have

(5.91)
$$G'(\boldsymbol{y})=\begin{pmatrix} F_x(\boldsymbol{x}) & 0 & -e_1 \\ F_{xx}(\boldsymbol{x})\boldsymbol{k} & F_x(\boldsymbol{x}) & 0 \\ 00000 & 00001 & 0 \end{pmatrix}.$$

The system (5.90) has a solution $\hat{\boldsymbol{y}}=(\hat{\boldsymbol{x}}, \hat{\boldsymbol{k}}, 0)^T$ $\left(\text{where } \hat{\boldsymbol{x}}=\left(0, 0, 1, 0, \left(\frac{\pi}{2}\right)^2\right)^T \text{ and } \hat{\boldsymbol{k}}=\left(-\frac{3}{4\pi}, 0, 0, 0, 1\right)^T\right)$ and for this solution $\hat{\boldsymbol{y}}$, we have

(5.92)
$$\det G'(\hat{\boldsymbol{y}})\neq 0.$$

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## References

[1] Urabe, M., Galerkin's procedure for nonlinear periodic systems, *Arch. Rational Mech. Anal.*, Vol. **20** (1965), 120–152.
[2] Urabe, M., The Newton method and its application to boundary value problems with nonlinear boundary conditions, *Pro. US-Japan Seminar on Differential and Functional Equations*, Benjamin, New York, (1967), 383–410.
[3] Urabe, M., An existence theorem for multi-point boundary value problems, *Funkcialaj Ekvacioj*, Vol. **9** (1966), 43–60.
[4] Urabe, M., Numerical solution of multi-point boundary value problems in Chebyshev series — Theory of the method, *Numer. Math.*, **9** (1967), 341–366.
[5] Urabe, M., Numerical solution of boundary value problems in Chebyshev series — A method of computation and error estimation, Conference on the numerical solution on differential equations, *Lecture Notes in Mathematics* **109** (1969), 40–86.
[6] Shamanskii, V. E., On the application of Newton's method in a singular case, *USSR.*

*Comp. Math. Math. Phy.*, **7** (1967), 72–85.

[7] Garside, G. R., Jarratt, P. and Mack, C., A new method for solving polynomial equations, *Computer Journal*, Vol. **11** (1968), 87–90.

[8] Ortega, J. M. and Rheinboldt, W. C., Iterative solution of nonlinear equations in several variables, Academic Press, New York, (1970).

[9] Langford, W. F., Numerical solution of bifurcation problems for ordinary differential equations, *Numer. Math.*, **28** (1977), 171–190.

[10] Abbott, J. P., An efficient algorithm for the determination of certain bifurcation points, *J. Comp. and Appl. Math.*, **4** (1978), 19–27.

[11] Seydel, R., Numerical computation of branch points in ordinary differential equations, *Numer. Math.*, **32** (1979), 51–68.

[12] Seydel, R., Numerical computation of branch points in nonlinear equations, *Numr. Math.*, **33** (1979), 339–352.

[13] Moore, G. and Spence, A., The calculation of turning points of nonlinear equations, *SIAM J. Numer. Anal.*, **17** (1980), 567–576.

[14] Decker, D. W. and Kelley, C. T., Newton's method at singular points. I, *SIAM J. Numer. Anal.*, **17** (1980), 66–70.

[15] Decker, D. W. and Kelley, C. T., Newton's method at singular points. II, *SIAM J. Numer. Anal.*, **17** (1980), 465–471.

[16] Griewank, A. and Osborne, M. R., Newton's method for singular problems when the dimension of the null space is $>1$, *SIAM J. Numer. Anal.*, **18** (1981), 145–149.

[17] Pönisch, G. and Schwetlick, H., Computing turning points of curves implicitly defined by nonlinear equations depending on a parameter, *Computing*, **26** (1981), 107–121.

[18] Weber, H. and Werner, W., On the accurate determination of nonisolated solutions of nonlinear equations, *Computing*, **26** (1981), 315–326.

[19] Seydel, R., Numerical computation of periodic orbits that bifurcate from stationary solutions of ordinary differential equations, *Appl. Math. Comp.*, **9** (1981), 257–271.

[20] Ojika, T., Deflation algorithm for the multiple roots of simultaneous nonlinear equations, *Memo. Osaka Kyoiku Univ., Ser. III*, **30** (1982), No. **3**, 197–209.

[21] Watanabe, S., On the deflation algorithm for multiple roots of systems of nonlinear algebraic equations and the order of convergence, *Bull. Yamagata Univ.*, Vol. **10** (1982), No. **3**, 245–263.

[22] Melhem, R. G. and Rheinboldt, W. C., A comparison of methods for determining turning points of nonlinear equations, *Computing*, **29** (1982), 201–226.

[23] Kawakami, H., Bifurcations of periodic responces in forced dynamic nonlinear circuits: Computation of bifurcation values of the system parameters, to appear in *IEEE Trans. Circuits and Systems.*

[24] Yamamoto, N., Galerkin method for autonomous differential equations with unknown parameters, *J. Math. Tokushima Univ.*, **16** (1982), 55–93.

[25] Yamamoto, N., A remark to Galerkin method for nonlinear periodic systems with unknown parameters, *J. Math. Tokushima Univ.*, **16** (1982), 95–126.