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Some Integral Formulas in Complex Projective Spaces

By

Toru ISHIHARA

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§1. Introduction

The integral geometry in the complex projective space $P^n(C)$ was initiated by W. Blaschke [1] and further exploited by L. A. Santalo [4]. In order to get the general kinematic formula for submanifolds in the Euclidean n-space E^n, S. S. Chern [2] gave some formulas about densities of linear subspaces in E^n. In this paper, analogous formulas about densities of linear subspaces in $P^n(C)$ are obtained. Recently, a lot of contribution to the integral geometry in the complex projective space $P^n(C)$ was made by P. A. Griffiths [3] and T. Shifrin [6].

Let $C^{n+1}=\{z=(z^0,\dots, z^n)\}$ be the complex Euclidean $(n+1)$-space with natural inner product $(z, w)=\sum_{k=0}^{n} z^k\bar{w}^k$, $z, w\in C^{n+1}$. The Euclidean metric g on C^{n+1} is given by

$$g(z, w)=Re(z, w), \quad z, w\in C^{n+1}.$$

The unit sphere $S^{2n+1}=\{z\in C^{n+1};\ (z, z)=1\}$ is a principal fibre bundle over $P^n(C)$ with structure group S^1 and projection π. For $z\in C^{n+1}$, put $\pi(z)=[z]\in P^n(C)$. We may regard $z=(z^0,\dots, z^n)$ as the homogeneous coordinate system of the point $[z]$. With the natural identification between vectors tangent to S^{2n+1} and vectors in C^{n+1}, one can show that for $z\in S^{2n+1}$, the tangent space T_zS^{2n+1} at z is given by

$$T_zS^{2n+1}=\{w\in C^{n+1};\ g(z, w)=0\}.$$

If we define T'_z by

$$T'_z=\{w\in C^{n+1};\ g(z, w)=g(iz, w)=0\},$$

then T'_z is a subspace of T_zS^{2n+1} whose orthogonal complement is $\{iz\}$. The projection π induces a linear isomorphism π_* of T'_z onto $T_{[z]}P^n(C)$, and π_* maps $\{iz\}$ into 0 for each $z\in S^{2n+1}$. We define the Fubini-Study metric, $\tilde{g}$ of constant holomorphic sectional curvature 4 by the equation

$$\tilde{g}(X, Y)=g(X', Y'),$$

where X, $Y\in T_{[z]}P^n(C)$ and X', Y' are their respective horizontal lifts at z.

From now on, we will follow the general method in [4]. We consider the group $U(n+1)$, called the unitary group, of all linear transformations $z'=Az$ that leave invariant the sphere S^{2n+1}. Then the $(n+1)\times(n+1)$ complex matrix A satisfies

$$(1.1) \qquad A\bar{A}^t=E, \quad A^{-1}=\bar{A}^t, \; \bar{A}^tA=E,$$

where E is the $(n+1)\times(n+1)$ unit matrix. A linear transformation A acts on $P^n(C)$ such that $A[z]=[Az]$. Then A and $exp(i\alpha)A$ define the same map on $P^n(C)$. Hence if $R\subset U(n+1)$ denotes the groups of matrices $exp(i\alpha)E$, the factor group $H(n+1)=U(n+1)/R$, the Hermitian elliptics group, acts on $P^n(C)$. An element of $H(n+1)$ is represented by $A\in U(n+1)$ with $det\ A=1$.

The Maurer-Cartan forms of $U(n+1)$ are given by

$$(1.2) \qquad \omega_{jk}=\sum_{h=0}^{n}\bar{a}_{hj}da_{hk}=\overline{(a_j,\, da_k)}, \quad \omega_{jk}+\bar{\omega}_{kj}=0,$$

where a_{hk} are the components of the matrix A. The kinematic density of $U(n+1)$ is equal to, ut to a constant facotor,

$$(1.3) \qquad dU(n+1)=\wedge(\omega_{jk}\wedge\bar{\omega}_{jk})\wedge\omega_{hh}, \quad j<k,\ 0\leqq j,\ k,\ h\leqq n.$$

The structure equations are

$$(1.4) \qquad d\omega_{jk}=-\sum_{i=0}^{n}\omega_{ji}\wedge\omega_{ik}.$$

The group $H(n+1)$ has the same invariant forms (1.2) and the same structure equations (1.4). The only difference is that now the relation $\omega_{00}+\omega_{11}+\cdots+\omega_{nn}=0$ holds, as follows by differentiating the relation $det\ A=1$. Hence the kinematic density for $H(n+1)$ is the exterior product (1.3) of all the ω_{ij} except one of the forms ω_{ii}.

§ 2. Densities for linear subspaces

Let L_r^o be a fixed r-plane of $P^n(C)$ and let H_r denote the subgroup of $H(n+1)$ that leaves L_r^o invariant. The invariant density for r-planes is the invariant volume element of the homogeneous space $H(n+1)/H_r$. Let a_k denote the point of C^{n+1} whose coordinates are the columns of matrix A. Conditions (1.1) give $(a_j,\, a_k)=\delta_{jk}$ and from (1.2), it follows that $da_k=\sum_{j=0}^{n}\omega_{jk}a_j$. It follows that, assuming L_r^o defined by the points $a_0,\dots, a_r$, we have $\omega_{jk}=0$ for $0\leqq k\leqq r$ and $r+1\leqq j\leqq n$. Hence it follows that $\bar{\omega}_{jk}=0$. Thus the density for r-planes invariant under $U(n+1)$ is

$$(2.1) \qquad dL_r=\left(\frac{\sqrt{-1}}{2}\right)^{(n-r)(r+1)}\wedge(\omega_{jk}\wedge\bar{\omega}_{jk}), \quad 0\leqq k\leqq r, \quad r+1\leqq j\leqq n.$$

For $r=0$ we get the density for points, that is, the volume element of $P^n(C)$ with respect to the hermitian geometry, which coincides with the volume element deduced from the Fubini-Study metric given in §1. The point a_o moves on the unit sphere S^{2n+1} centered at the origin o. Since $da_0=\sum_{j=0}^{n}\omega_{j0}a_j$, the volume element ds^{2n+1} for S^{2n+1} is given by

$$ds^{2n+1}=\left(\frac{\sqrt{-1}}{2}\right)^n(-\sqrt{-1}\omega_{00})\wedge(\omega_{i0}\wedge\bar{\omega}_{i0}). \tag{2.2}$$

The restriction of the form $\sqrt{-1}\omega_{00}$ to each fibre of the fibre bundle $\pi\colon C^{n+1}\to P^n(C)$ is regarded as the standard volume element of S^1. Hence the total volume $m(P^n(C))$ is given by

$$m(P^n(C))=\frac{1}{2\pi}m(S^{2n+1})=\frac{\pi^n}{n!}. \tag{2.3}$$

Let L_{n-1} be the $(n-1)$-plane in $P^n(C)$ perpendicular to a_0. Calling L_{r-1}^{n-1} to the $(r-1)$-plane $L_r\cap L_{n-1}$, we have that the density for $L_r\cap L_{n-1}$ in L_{n-1} is

$$dL_{r-1}^{n-1}=\frac{(\sqrt{-1})^{r+1}}{2^r}\wedge(\omega_{hi}\wedge\bar{\omega}_{hi}),\quad 1\leqq i\leqq r,\quad r+1\leqq h\leqq n. \tag{2.4}$$

Hence we have

$$dL_r\wedge\frac{(\sqrt{-1})^{r+1}}{2^r}(-\omega_{00})\wedge\omega_{10}\wedge\bar{\omega}_{10}\wedge\cdots\wedge\omega_{r0}\wedge\bar{\omega}_{r0}=dL_{r-1}^{n-1}\wedge ds^{2n+1}. \tag{2.5}$$

If ds^{2r+1} denotes the volume element of the unit $(2r+1)$-sphere in L_r, (2.5) can be written as

$$dL_r\wedge ds^{2r+1}=dL_{r-1}^{n-1}\wedge ds^{2n+1}. \tag{2.6}$$

Exterior multiplication by ds^{2r-1} gives, ut to the sign,

$$\begin{aligned}dL_r\wedge ds^{2r+1}\wedge ds^{2r-1}&=dL_{r-1}^{n-1}\wedge ds^{2r-1}\wedge ds^{2n+1}\\&=dL_{r-1}^{n-1}\wedge ds^{2n-1}\wedge ds^{2n+1}.\end{aligned}$$

As $ds^{2(n-r)+1}=(-\sqrt{-1}\omega_{rr})\wedge dL_0^{n-r}$, Successive exterior multiplication by $ds^{2r-3},\cdots$, ds^3, ds^1 gives

$$dLr\wedge ds^{2r+1}\wedge ds^{2r-1}\wedge\cdots\wedge ds^3\wedge ds^1=ds^{2(n-r)+1}\wedge ds^{2(n-r)+3}\wedge\cdots\wedge ds^{2n+1}. \tag{2.7}$$

Integrating over the unit spheres S^{2n+1}, $S^{2n-1},\ldots$, S^3, S^1, we get (see [4], [5]).

Proposition 1. *The total volume of the r-planes in $P^n(C)$, that is the total volume of the complex Grassmann manifold $G_{r+1,n-r}$ of $(r+1)$-planes in C^{n+1}, is given by*

$$m(G_{r+1,n-r})=\frac{1!\cdots\cdots\cdot r!}{n!\cdots(n-r)!}\pi^{(n-r)(r+1)}.$$

Let q-plane L_q^o be fixed in $P^n(C)$. We seek a density for r-plane $L_r(r>q)$ that contains L_q^o. We assume that $a_0,\dots, a_r$ span L_q^o and that L_q^o and $a_{r+1},\dots, a_q$ span L_r. The group $H_{r[q]}$ of all motions that keep L_r fixed, considered as a subgroup of the group H_q of all motions that that keep L_q^o fixed is defined by

$$\omega_{hi}=0,\quad \bar{\omega}_{hi}=0,\quad q+1\leqq i\leqq r,\quad r+1\leqq h\leqq n,$$

because $e_{q+1},\dots, e_r$ can only vary in L_r^o. Thus the invariant volume element of $H_q/H_{r[q]}$, which is equal to the density for r-planes about L_q^o, reads

$$(2.8)\qquad dL_{r[q]}=\left(\frac{\sqrt{-1}}{2}\right)^{(n-r)(r-q)}\wedge(\omega_{hi}\wedge\bar{\omega}_{hi}),\quad q+1\leqq i\leqq r,\quad r+1\leqq h\leqq n.$$

Let L_{n-q-1} be the $(n-q-1)$-plane perpendicular to L_q. Each $L_{r[q]}$ can be defined by the intersection $L_{r[q]}\cap L_{n-q-1}$, which is an $(r-q-1)$-plane, and consequently the density of all $L_{r[q]}$ is equal to the density of all L_{r-q-1} in L_{n-q-1}, that is,

$$(2.9)\qquad dL_{r[q]}=dL_{r-q-1}^{n-q-1}.$$

§3. Relations between densities of linear subspaces

Let L_r and L_q be a moving r-plane and a fixed q-plane respectively. Assume $q+r>n$ so that $L_q\cap L_r$ is, in general, an $(r+q-n)$-plane, which we denote by L_{r+q-n}. We can suppose that

(a) $a_0,\dots, a_{r+q-n}$ span L_{r+q-n},

(b) $a_{r+q-n+1},\dots, a_r$ lie on L_r.

Take points $b_{r+q-n+1},\dots, b_n$ such that

(c) $a_0,\dots, a_{r+q-n}, b_{r+q-n+1},\dots, b_n$ form an unitary base of C^{n+1},

(d) $a_0,\dots, a_{r+q-n}, b_{r+q-n+1},\dots, b_r$ span L_q.

Then the equation (2.1) can be written as

$$(3.1)\qquad dL_r=\left(\frac{\sqrt{-1}}{2}\right)^{(n-r)(r+1)}\wedge(\omega_{hi}\wedge\bar{\omega}_{hi})\wedge(\omega_{h\alpha}\wedge\bar{\omega}_{h\alpha}),$$

$$0\leqq i\leqq r+q-n,\quad r+1\leqq h\leqq n,\quad r+q-n+1\leqq\alpha\leqq r.$$

According to (2.8), we get

$$(3.2)\qquad dL_{r[r+q-n]}=\left(\frac{\sqrt{-1}}{2}\right)^{(n-r)(n-q)}\wedge(\omega_{h\alpha}\wedge\bar{\omega}_{h\alpha}),$$

$$r+1\leqq h\leqq n,\quad r+q-n+1\leqq\alpha\leqq r.$$

The density for $(r+q-n)$-planes in L_q is given by

(3.3) $$dL^q_{r+q-n}=\left(\frac{\sqrt{-1}}{2}\right)^{(n-r)(r+q-n+1)}\wedge((da_i,\ b_h)\wedge\overline{(da_i,\ b_h)}),$$
$$0\leqq i\leqq r+q-n,\quad r+1\leqq h\leqq n.$$

Put

$$a_h=\sum u_{h\alpha}b_\alpha+\sum u_{hk}b_k,\quad r+q-n+1\leqq\alpha\leqq r,\quad r+1\leqq h,\quad k\leqq n.$$

From $(b_\alpha,\ a_i)=0$, it follows that $(b_\alpha,\ da_i)=-(db_\alpha,\ a_i)=0$. Hence we have

$$\omega_{hi}=(da_i,\ a_h)=\sum u_{hk}(da_i,\ b_k).$$

From (3.1), (3.2), (3.3) and (3.4), we get the desired formula

(3.5) $$dL_r=|\varDelta|^{2(r+q-n+1)}dL_{r[r+q-n]}\wedge dL^q_{r+q-n},$$

where

(3.6) $$\varDelta=det\ (a_h,\ b_k).$$

Since $\varDelta$ depends only on $L_{r[r+q-n]}$, integrating (3.5) over all L_r, we obtain

(3.7) $$\int|\varDelta|^{2(r+q-n+1)}dL_{r[r+q-n]}=\frac{m(G_{r+1,n-r})}{m(G_{r+q-n+1,n-r})}$$

From (2.9), if follows that

$$dL_{r[r+q-n]}=dL^{2n-q-r-1}_{n-q-1}.$$

Hence making the change of notations $2n-r-q-1=N$, $r+q-n+1=\nu$, $n-q-1=\rho$, we get from (3.7).

Proposition 2. *Let L^o_ρ be a fixed ρ-plane in $P^N(C)$. We have*

$$\int_{G_{N-\rho,\rho+1}}|\langle L_\rho,\ L^o_\rho\rangle|^{2\nu}dL^N_\rho=\frac{m(G_{N-\rho,\nu+\rho+1})}{m(G_{N-\rho,\nu})}$$

Notice that if L^o_ρ is spanned by $a^o_0,\ldots,a^o_\rho$, and L_ρ is spanned by $a_0,\ldots,a_\rho$, then $\varDelta=det(a^o_i,\ a_j)$, $0\leqq i,\ j\leqq\rho$.

Department of Mathematics
Faculty of Education
Tokushima University

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