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On Spectra of the Laplacian on Vector Bundles

By

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§1. Introduction

There have been gotten many interesting results on spectra of the Laplace-Beltrami operator on Riemannian manifolds. These results answer what extent the spectrum of this operator determines the geometric or topological structures of the manifold. See [2], [7] for example.

The present article is concerned with the spectra of more general operators.

The motivation of our study is the following. Let M be a C^∞ manifold. A second order elliptic operator D on M is expressed in a local coordinate neighbourhood as

$$D=a^{jk}(x)\frac{\partial^2}{\partial x^j\partial x^k}+b^j(x)\frac{\partial}{\partial x^j}+c(x)\,,$$

with respect to the coordinates (x^j), where a^{jk}, b^j and c are C^∞ complex-valued functions. Here and hereafter we adopt the convention of summing over repeated indices unless indicated otherwise. If $a=(a^{jk})$ is real and positive definite, we can regard $g=a^{-1}=(a_{jk})$ as a Riemannian structure on M. Moreover, considering the trivial complex line bundle over M, we can see that coefficients b^j and c are concerned with intrinsic objects of the line bundle. That is, b^j and c determine a linear connection and an endomorphism of the line bundle, respectively.

Let E be a vector bundle over a Riemannian manifold (M, g), and give a linear connection $\tilde{d}$ on E. Then, noting the above discussion, we can define a natural elliptic operator of second order called the Laplacian on E. We study in this paper how the spectrum of the Laplacian on E is related to the geometric or topological structures of $(M, g; E, \tilde{d})$.

The paper is organized as follows. In §2 we introduce the Laplacian on vector bundles, and study in §3 the basic properties of its sepctrum, in particular concerning the zero eigenvalue. In §4 we consider the relationship between the sepctrum and the gauge transformation on the vector bundle. Section 5 treats the spectra of the line bundles over 2-sphere and the trivial line bundle over flat torus, which will be guiding examples in our studies. In §6 we give estimations of the non-zero first eigenvalue by the curvatures. Finally in §7 the Minakshisundaram's expansion is considered.

§2. Laplacian on vector bundles

Let (M, g) be a compact n-dimensional C^∞ Riemannian manifold without boundary, and E be a C^∞ complex vector bundle over M with rank r. We assume that E has a C^∞ Hermitian structure $\langle\ ,\ \rangle$. Let $A^p(M)$, $(p=0,\dots, n)$, denote the set of C^∞ p-forms on M, and $A^p(M, E)$ the set of E-valued C^∞ p-forms on M, that is, $A^p(M, E)=C^\infty(E)\otimes_\Gamma A^p(M)$, where $C^\infty(E)$ is the set of C^∞ sections of E. We have $A^0(M, E)=C^\infty(E)$.

Let $\tilde{d}$: $A^0(M, E)\to A^1(M, E)$ be a linear connection on E compatible with the Hermitian structure (cf. [11]). For the linear connection $\tilde{d}$, we define $\tilde{d}_p$: $A^p(M, E)\to A^{p+1}(M, E)$ by $\tilde{d}_p(s\otimes\theta)=\tilde{d}s\wedge\theta+s\otimes d\theta$, where $s\in C^\infty(E)$ and $\theta\in A^p(M)$. We introduce inner products in $A^p(M, E)$ by

$$\langle s\otimes\theta,\ s'\otimes\theta'\rangle_x=\langle s, s'\rangle_x\langle\theta, \theta'\rangle_{g,x},\ x\in M,$$

and

$$(\cdot\,,\cdot)=\int_M\langle\cdot\,,\cdot\rangle_x dV_g(x),$$

where $\langle\ ,\ \rangle_g$ is the natural inner product of $A^p(M)$ and dV_g is the volume element of M induced by the Riemannian metric g. Let $\tilde{\delta}_p$: $A^{p+1}(M, E)\to A^p(M, E)$ be the adjoint operator of $\tilde{d}_p$ with respect to the inner product $(\ ,\ )$, and set

$$L_p=\tilde{\delta}_p\tilde{d}_p+\tilde{d}_{p-1}\tilde{\delta}_{p-1}, \tag{2.1}$$

which is called the *Laplacian* acting on $A^p(M, E)$. The Laplacian is a positive, formally self-adjoint elliptic operator of second order.

We study the spectrum of the Laplacian $L=L_0$ acting on $C^\infty(E)$.

We write the operator L by the local coordinates of M and the local frame of E. Let $(x^1,\dots, x^n)$ be local coordinates of M, and $(e_1,\dots, e_r)$ be a local frame of E, i.e., a system of linear independent local sections of E. Then, every section $s\in C^\infty(E)$ is written as $s=s^\alpha e_\alpha=\boldsymbol{e}\cdot\boldsymbol{s}$, where we set

$$\boldsymbol{s}=\begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix},\quad \boldsymbol{e}=(e_1,\dots, e_r)\,.$$

For a linear connection $\tilde{d}$, we have

$$\tilde{d}e_\alpha=\omega_\alpha^\beta e_\beta,$$

where $\omega=(\omega_\alpha^\beta)$ is an $r\times r$ matrix of 1-forms, called the *connection matrix* of $\tilde{d}$ with respect to the frame $\{e_\alpha\}$.

Lemma 2.1. *Set $H=(H_{\alpha\beta})=(\langle e_\alpha, e_\beta\rangle)$. Then, the connection $\tilde{d}$ is compatible with the Hermitian structure on E if and only if $dH=H\bar{\omega}+{}^t\omega H$.*

PROOF. See [11, p. 78].

Set $\omega=\omega_j dx^j$, where ω_j is an $r\times r$ matrix.

Proposition 2.2. *The Laplacian L is expressed as*

$$Ls=-g^{jk}\nabla_j\nabla_k\boldsymbol{s}-2g^{jk}\omega_j\nabla_k\boldsymbol{s}-g^{jk}(\nabla_j\omega_k+\omega_j\omega_k)\boldsymbol{s}, \tag{2.2}$$

where ∇ is the Levi-Civita connection induced by g.

PROOF. For $s=\boldsymbol{e}\cdot\boldsymbol{s}\in C^\infty(E)$ and $t=\boldsymbol{e}\cdot\boldsymbol{t}\in A^1(M, E)$, we can easily get

$$(\tilde{d}\boldsymbol{s})_j=\nabla_j\boldsymbol{s}+\omega_j\boldsymbol{s}, \tag{2.3}$$

$$\tilde{\delta}\boldsymbol{t}=-g^{jk}(\nabla_k\boldsymbol{t}_j+\omega_k\boldsymbol{t}_j). \tag{2.4}$$

These formulas lead to (2.2). Q.E.D.

Remark. As to (2.4) we show more general formula in §6 (Lemma 6.1).

Example. Suppose E is the trivial line bundle, i.e., $E=M\times\boldsymbol{C}$ with the natural inner product in $\boldsymbol{C}$. Then, for the global section $e=1$, the connection matrix is given by a 1-form $\omega=i\alpha=ia_j dx^j$ on M, where a_j's are real (Lemma 2.1). Moreover, the Laplacian is expressed as

$$Ls=-g^{jk}\nabla_j\nabla_k s-2ia^j\nabla_j s+(a_j a^j-i\nabla_j a^j)s, \tag{2.5}$$

which acts on complex-valued functions on M.

Next, we represent the Laplacian by covariant differentiation of E-valued tensors. Let $T^p_q(M, E)$ denote the set of E-valued tensor fields of type (p, q) on M. An element T of $T^p_q(M, E)$ is written as $T=s\otimes\xi$, where $s\in C^\infty(E)$ and ξ is a usual tensor field on M of type (p, q). We define *covariant differentiation* $\tilde{\nabla}: T^p_q(M, E)\to T^p_{q+1}(M, E)$ by

$$\tilde{\nabla}(s\otimes\xi)=\tilde{d}s\otimes\xi+s\otimes\nabla\xi.$$

Let $\{e_\alpha\}$ be a local frame, and $T\in T^p_q(M, E)$ be given by

$$T=\boldsymbol{e}\cdot\boldsymbol{T}^{k\cdots m}_{h\cdots j}\,dx^h\otimes\cdots\otimes dx^j\otimes\frac{\partial}{\partial x^k}\otimes\cdots\otimes\frac{\partial}{\partial x^m},$$

where $\boldsymbol{T}^{k\cdots m}_{h\cdots j}$ are r-column vectors. Then, we have

$$\tilde{\nabla}_t\boldsymbol{T}^{k\cdots m}_{h\cdots j}=\nabla_t\boldsymbol{T}^{k\cdots m}_{h\cdots j}+\omega_t\boldsymbol{T}^{k\cdots m}_{h\cdots j}. \tag{2.6}$$

and the Laplacian L is given by

$$Ls = -g^{jk}\tilde{\nabla}_j\tilde{\nabla}_k s. \tag{2.7}$$

In the remainder of this section, we discuss about the curvature of the linear connection on E. For a connection matrix ω of a linear connection on E, we define

$$\Omega = d\omega + \omega \wedge \omega,$$

which is an $r \times r$ matrix of 2-forms. The matrix Ω is called the *curvature matrix* associated with the connection matrix ω. It is easy to see that Ω is an element of $A^2(M, \mathrm{End}(E))$, where $\mathrm{End}(E)$ denotes the bundle of endomorphisms of E.

A linear connection $\tilde{d}$ on E is called *flat* if the curvature of $\tilde{d}$ vanishes.

The following properties of the curvature are basic.

Lemma 2.3. (1) *If the connection is compatible with the Hermitian structure, then the curvature matrix is skew-Hermitian for an orthonormal frame of the bundle.*

(2) *For* $s \in A^p(M, E)$, $\tilde{d}^2 s = \Omega s$.

(3) *(Bianchi identity)* $d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0$.

(4) *(Ricci formula) Set* $\Omega = \frac{1}{2}\Omega_{jk} dx^j \wedge dx^k$, *and*

$$\begin{aligned}\tilde{\nabla}_a\tilde{\nabla}_b T^{k\cdots m}_{h\cdots j} - \tilde{\nabla}_b\tilde{\nabla}_a T^{k\cdots m}_{h\cdots j} = {} & R^k_{cab} T^{c\cdots m}_{h\cdots j} + \cdots + R^m_{cab} T^{k\cdots c}_{h\cdots j} \\ & - R^c_{hab} T^{k\cdots m}_{c\cdots j} - \cdots - R^c_{jab} T^{k\cdots m}_{h\cdots c} + \Omega_{ab} T^{k\cdots m}_{h\cdots j},\end{aligned} \tag{2.8}$$

where R^a_{bcd} *is the Riemannian curvature tensor associated with the metric* g.

§3. Basic properties of spectrum

The Laplacian L is a positive, formally self-adjoint elliptic operator of second order. Therefore, L has an infinite sequence

$$(0 \le)\quad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \uparrow \infty$$

of non-negative eigenvalues, each eigenvalue being repeated as many times as its multiplicity indicates. We denote this set of eignevalues by $\mathrm{Sp}(M, g; E, \tilde{d})$.

Different from the case of the Laplace-Beltrami operator on (M, g) acting on functions, the Laplacian L generally has not zero eigenvalue. The section s satisfying $Ls = 0$ is called *harmonic*. Obviously, the condition $Ls = 0$ is equivalent to $\tilde{d}s = 0$. Thus a harmonic section is a *parallel* section.

We have the following propositions converning the zero eigenvalue.

Proposition 3.1. (1) *If L has zero eigenvalue with multiplicity $k(\le r)$, then $E = E' \oplus T_k$ (Whitney sum), where T_k is a trivial bundle of rank k.*

(2) *If L has zero eigenvalue with multiplicity r, then E is a trivial bundle and*

the curvature Ω of the connection $\tilde{d}$ vanishes.

PROOF. (1) Let $f_1,\dots,f_k$ be k independent parallel sections of E. Set $T_k=\{(x, v);\ x\in M,\ v=\sum_{\alpha=1}^{k} v^\alpha f_\alpha(x),\ (v^\alpha\colon \text{const.})\}\subset E$, and T_k is a trivial subbundle of E. By setting $E'=T_k^\perp$ (the orthogonal complement of T_k), we have $E=E'\oplus T_k$.

(2) We have only to show $\Omega=0$. Let $f_1,\dots,f_r$ be r independent parallel sections of E. Then, we have $\Omega f_\alpha=\tilde{d}^2 f_\alpha=0$ for $\alpha=1,\dots, r$, which leads to $\Omega=0$.

Q. E. D.

Let $\mathrm{Sp}(M, g)$ denote the spectrum of the Laplace-Beltrami operator $\Delta=-g^{jk}\nabla_j\nabla_k$ acting on functions on M.

Proposition 3.2. *If L has zero eigenvalue, then*

$$\mathrm{Sp}(M, g; E, \tilde{d})\supset \mathrm{Sp}(M, g).$$

In particular, L has zero eigenvalue with multiplicty r, then

$$\mathrm{Sp}(M, g; E, \tilde{d})=r\cdot \mathrm{Sp}(M, g),$$

where $r\cdot\mathrm{Sp}(M, g)=\mathrm{Sp}(M, g)\cup\cdots\cup\mathrm{Sp}(M, g)$, *($r$ times).*

PROOF. Since $0\in \mathrm{Sp}(M, g; E, \tilde{d})$, there is $f\in C^\infty(E)$ such that

$$\tilde{\nabla}_j f=\nabla_j f+\omega_j f=0. \tag{3.1}$$

It is obvious that f nowhere vanishes. Suppose $\lambda\in\mathrm{Sp}(M, g)$ and $\Delta\phi=\lambda\phi$. Set $s=\phi f\in C^\infty(E)$, and we have

$$\begin{aligned} Ls&=(\Delta\phi)f-\phi\nabla^j\nabla_j f-2(\nabla_j\phi)(\nabla^j f)-2\omega^j(\nabla_j\phi)f\\ &\quad-2\omega^j\phi\nabla_j f-\omega^j\omega_j\phi f-(\nabla^j\omega_j)\phi f\\ &=(\Delta\phi)f=\lambda\phi f=\lambda s,\end{aligned}$$

using (3.1). Thus $\lambda\in\mathrm{Sp}(M, g; E, \tilde{d})$. Next, suppose $f_1,\dots,f_r$ are independent and each f_α satisfies (3.1). Let $\lambda_m\in\mathrm{Sp}(M, g)$ and $\Delta\phi_m=\lambda_m\phi_m$, $(m=1, 2,\dots)$. Set $s_{m\alpha}=\phi_m f_\alpha\in C^\infty(E)$, $(\alpha=1,\dots, r)$. Then, $s_{m\alpha}$'s are independent and $Ls_{m\alpha}=\lambda_m s_{m\alpha}$ holds. We have only to show that $\{s_{m\alpha};\ m=1, 2,\dots, \alpha=1,\dots, r\}$ is a basis for $L^2(E)$ (the Hilbert space of L^2 sections of E). We regard $\{f_\alpha;\ \alpha=1,\dots, r\}$ as a global frame of E. For each $s\in L^2(E)$, we have $s=\sum_{\alpha=1}^{r} s^\alpha f_\alpha$, where s^α is a L^2 function on M. Since the eigenfunctions ϕ_m; $m=1, 2,\dots$, of Δ form a basis of $L^2(M)$, we have $s^\alpha=\sum_{m=1}^{\infty} a^{m\alpha}\phi_m$, $(a^{m\alpha}\colon \text{const.})$. Therefore, we get

$$s=\sum_\alpha s^\alpha f_\alpha=\sum_\alpha\Big(\sum_m a^{m\alpha}\phi_m\Big)f_\alpha=\sum_{\alpha,m} a^{m\alpha}\phi_m f_\alpha=\sum_{\alpha,m} a^{m\alpha}s_{m\alpha}.$$

Q. E. D.

§4. Gauge transformations and spectra

A *gauge transformation* on a vector bundle E with the Hermitian structure is a diffeomorphism $\psi: E \to E$ which maps each fibre E_x isometrically and linearly onto itself. For a linear connection $\tilde{d}$ on E, we define a new connection $\psi^*\tilde{d} = \psi^{-1} \circ \tilde{d} \circ \psi$ (ψ being regarded as a map: $A^p(M, E) \to A^p(M, E)$) by a gauge transformation ψ on E. Obviously, $\psi^*\tilde{d}$ is compatible with the Hermitian structure. Take a local frame $\{e_\alpha\}$ of E. Then, for a gauge transformation ψ, we have

$$\psi(e_\alpha) = \Psi_\alpha^\beta e_\beta,$$

where $\Psi = (\Psi_\alpha^\beta)$ is an $r \times r$ non-singular matrix satisfying $H = {}^t\Psi H \bar{\Psi}$. Let ω be the connection matrix of $\tilde{d}$ with respect to $\{e_\alpha\}$. Then, the connection matrix $\psi^*\omega$ of $\psi^*\tilde{d}$ is given by

$$\psi^*\omega = \Psi^{-1} d\Psi + \Psi^{-1}\omega\Psi. \tag{4.1}$$

Two connections $\tilde{d}$ and $\tilde{d}'$ on E are called *gauge equivalent* to each other (denoted by $\tilde{d} \underset{(g)}{\sim} \tilde{d}'$) if there is a gauge transformation ψ on E such that $\tilde{d}' = \psi^*\tilde{d}$.

Proposition 4.1. *Suppose $\tilde{d}$ and $\tilde{d}'$ are two connections on E. If $\tilde{d} \underset{(g)}{\sim} \tilde{d}'$ holds, then* $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g; E, \tilde{d}')$.

PROOF. Let L and L' be the Laplacians defined from $\tilde{d}$ and $\tilde{d}'$, respectively. Let ψ be a gauge transformation such that $\tilde{d}' = \psi^*\tilde{d}$. Suppose $Ls = \lambda s$. Set $s' = \psi^{-1}(s) \in C^\infty(E)$, and we get $L's' = \lambda s'$ by straightforward calculation. This proves the proposition. Q. E. D.

Remark. The converse of Proposition 4.1 is false. In fact, we can easily give a counterexample (see §5).

We define a more natural equivalence relation between connections.

Let E be a vector bundle with the Hermitian structure over a Riemannian manifold (M, g). A diffeomorphism $\psi: E \to E$ is called a *weak gauge transformation* on E if (1) ψ maps each fibre E_x isometrically and linearly onto one of the fibre $E_{x'}$, and (2) $\bar{\psi}: (M, g) \to (M, g)$ defined by $\bar{\psi}(x) = x'$ is an isometry.

Two connections $\tilde{d}$ and $\tilde{d}'$ on E are called *weakly gauge equivalent* to each other (denoted by $\tilde{d}_{(\widetilde{wg})}\tilde{d}'$) if there is a weak gauge transformation ψ on E such that $\tilde{d}' = \psi^*\tilde{d} = \psi^{-1} \circ \tilde{d} \circ \psi$.

Similarly to Proposition 4.1, we get

Proposition 4.2. *If $\tilde{d}_{(\widetilde{wg})}\tilde{d}'$ holds, then* $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g; E, \tilde{d}')$.

PROOF. Since $\bar{\psi}$ is an isometry of (M, g), the proof is on the same line as that of Proposition 4.1. Q.E.D.

The following problem is fundamental.

Problem. *When* $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g; E, \tilde{d}')$ *holds, is* $\tilde{d}'$ *weakly gauge equivalent to* $\tilde{d}$?

This type question for the Laplace-Beltrami operator on (M, g) was answered negatively by Milnor [8]. Concerning the above problem, however, we don't know whether there is a counterexample or not.

In the remainder of this section we discuss the case of line bundles.

First, we give a definition. The injection $\varepsilon\colon \boldsymbol{Z} \to \boldsymbol{R}$ induces a homomorphism $\varepsilon\colon H^k(M; \boldsymbol{Z}) \to H^k(M; \boldsymbol{R})$ between cohomology groups. A class $\gamma \in H^k(M; \boldsymbol{R})$ is called *integral* in case γ lies in the image $\tilde{H}^k(M; \boldsymbol{Z})$ of the map ε. A real closed k-form θ is called *integral* if $[\theta] \in \tilde{H}^k(M; \boldsymbol{Z})$. It is easily shown a real k-form θ is integral if and only if $\int_c \theta$ is an integer for every k-cycle c of M.

Let E be a line bundle over (M, g), and $\tilde{d}_1$ and $\tilde{d}_2$ be two connections on E. Let ω_1 and ω_2 be the connection 1-forms of $\tilde{d}_1$ and $\tilde{d}_2$, respectively, with respect to a local section e of E. For a change of frame $e \mapsto e' = fe$, f being a non-vanishing function, we have $\omega_i' = \omega_i + f^{-1}df$, $(i = 1, 2)$. Hence, $\omega_1' - \omega_2' = \omega_1 - \omega_2$. Thus, $\omega_1 - \omega_2$ is a global 1-form on M. Moreover, we have the following.

Proposition 4.3. $\tilde{d}_1 \underset{(g)}{\sim} \tilde{d}_2$ *holds if and only if* $(\omega_1 - \omega_2)/2\pi i$ *is an integral 1-form.*

Proof. Suppose $\tilde{d}_2 = \psi^* \tilde{d}_1$ by a gauge transformation ψ on E. In this case ψ is regarded as a non-vanishing complex-valued function on M, and we have $\omega_2 = \psi^{-1} d\psi + \omega_1$ from (4.1). Let c be any closed cuve on M. Then, we obtain

$$\int_c (\omega_2 - \omega_1) = \int_c \psi^{-1} d\psi = 2\pi k i, \quad k\colon \text{integer}.$$

Conversely, suppose $(\omega_2 - \omega_1)/2\pi i$ is integral. Fix $p \in M$, and set $\psi(x) = \exp\left\{\int_{\gamma(x)} (\omega_2 - \omega_1)\right\}$, where $\gamma(x)$ is a curve from p to $x \in M$. Let $\gamma'(x)$ be another curve from p to x. Then, $\gamma'(x) = \gamma(x) + c$, c being a closed curve, and we have

$$\begin{aligned}
\exp\left\{\int_{\gamma'(x)} (\omega_2 - \omega_1)\right\} &= \exp\left\{\int_{\gamma(x)} (\omega_2 - \omega_1) + \int_c (\omega_2 - \omega_1)\right\} \\
&= \exp\left\{\int_{\gamma(x)} (\omega_2 - \omega_1)\right\} \exp\left\{\int_c (\omega_2 - \omega_1)\right\} \\
&= \exp\left\{\int_{\gamma(x)} (\omega_2 - \omega_1)\right\}.
\end{aligned}$$

Thus, $\psi(x)$ is well defined, and $\omega_2 - \omega_1 = \psi^{-1} d\psi$ holds, which fact shows that $\tilde{d}_2 =$

$\psi^* \tilde{d}_1$. Q.E.D.

Noting Propositions 3.1, 3.2 and 4.3, we obtain the following.

Theorem 4.4. *Let E be a line bundle over (M, g), and $\tilde{d}$ a linear connection on E. Then, the following conditions are equivalent to each other.*

(1) $0 \in \mathrm{Sp}(M, g; E, \tilde{d})$,

(2) $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g)$,

(3) *E is a trivial bundle and $\tilde{d} \underset{(g)}{\sim} 0$, where 0 denotes the connection whose connection form is identically zero.*

(4) *E is a trivial bundle and $\omega/2\pi i$ is an integral 1-form on M, where ω is the connection form of $\tilde{d}$. Moreover, if M is simply connected, the above conditions are equivalent to the following.*

(5) *The curvature Ω of $\tilde{d}$ vanishes.*

PROOF. About the equivalence of the first four conditions we have only to prove that the condition (1) derive (4). If $0 \in \mathrm{Sp}(M, g; E, \tilde{d})$ holds, there is a non-vanishing $f \in C^\infty(E)$ such that $df + f\omega = 0$. Hence, $\omega = -f^{-1}df$, which shows that $\omega/2\pi i$ is integral. Next, under the assumption that M is simply connected, we show that (5) derives (3). This is directly obtained from the following propositon.

Proposition 4.5. (Kostant [5, p. 135]). *Assume M is simply connected and assume $\Omega/2\pi i$ is an integral 2-form on M. Then, up to gauge equivalence there is a unique line bundle with connection $\tilde{d}$ such that the curvature of $\tilde{d}$ is equal to Ω.*

§ 5. Examples

5.1. Spectra of line bundles over S^2

Consider (S^2, g_0), the 2-dimensional sphere in $\boldsymbol{R}^3$:

$$S^2 = \{(x^1, x^2, x^3) \in \boldsymbol{R}^3;\ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\},$$

with the canonical metric induced by the Euclidean metric of $\boldsymbol{R}^3$. Let (θ, φ) be the polar coordinates of S^2 given by

$$x^1 = \sin\theta\cos\varphi, \quad x^2 = \sin\theta\sin\varphi, \quad x^3 = \cos\theta, \quad (0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi \leqq 2\pi).$$

The volume element Θ induced by the metric g_0 is given by $\Theta = \sin\theta d\theta \wedge d\varphi$. For each integer m, set

$$\Omega_m = \frac{i}{2} m\Theta = \frac{i}{2} m \sin\theta d\theta \wedge d\varphi. \tag{5.1}$$

Then, the 2-form $\Omega_m/2\pi i$ is closed and integral. Since $\tilde{H}^2(S^2; \boldsymbol{Z}) = H^2(S^2; \boldsymbol{Z}) = \boldsymbol{Z}$ holds, each m associates a line bundle E_m over S^2 and by Kostant's theorem

(Proposition 4.5) there exists a unique connection $\tilde{d}_m$ whose curvature form is equal to Ω_m. In particular, $(E_0, \tilde{d}_0)$ is a trivial bundle with flat connection. The Laplacian on the line bundle $(E_m, \tilde{d}_m)$ is (locally) given by

$$L_m = \Delta_2 + im \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{m^2}{4}\frac{\cos^2\theta}{\sin^2\theta}, \tag{5.2}$$

where

$$\Delta_2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Laplace-Beltrami operator on (S^2, g_0).

The above line bundles with connection $(E_m, \tilde{d}_m)$ are induced from the principal S^1-bundle over S^2 called the Hopf fibre bundle. Consider Lie groups

$$SU(2) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} ;\ |z|^2 + |w|^2 = 1,\ z, w \in \boldsymbol{C} \right\} \cong S^3 \subset \boldsymbol{R}^4$$

and

$$S^1 = \{\varepsilon(t) = e^{it};\ 0 \leqq t \leqq 2\pi\}.$$

S^1 acts on $SU(2)$ on the right:

$$h\varepsilon(t) = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix},$$

and this action derives a principal S^1-bundle:

$$P\colon SU(2) = S^3 \xrightarrow{\pi} SU(2)/S^1 = S^2$$

over S^2. If we choose the coordinates (θ, φ, ψ) of $SU(2)$ as

$$h = \begin{bmatrix} \cos\frac{\theta}{2}\, e^{i(\varphi+\psi)/2} & i\sin\frac{\theta}{2}\, e^{i(\varphi-\psi)/2} \\ i\sin\frac{\theta}{2}\, e^{-i(\varphi-\psi)/2} & \cos\frac{\theta}{2}\, e^{-i(\varphi+\psi)/2} \end{bmatrix} \in SU(2),$$

$$(0 \leqq \theta \leqq \pi,\quad 0 \leqq \varphi \leqq 2\pi,\quad 0 \leqq \psi \leqq 4\pi),$$

then the projection $\pi(h) = (x^1, x^2, x^3) \in S^2$ is given by

$$x^1 = \sin\theta\cos\varphi, \quad x^2 = \sin\theta\sin\varphi, \quad x^3 = \cos\theta.$$

Thus, the fibre over $(\theta_0, \varphi_0) \in S^2$ is the submanifold $\{(\theta_0, \varphi_0, \psi) \in S^3;\ 0 \leqq \psi \leqq 4\pi\}$, and its tangent space is generated by $\partial/\partial\psi$. We define a vector $X \in T_h(S^3)$, $h \in S^3$

to be horizontal if X is orthogonal to $(\partial/\partial\psi)_h$ with respect to the canonical metric g_0 on S^3 induced by the Euclidean metric of $\boldsymbol{R}^4$. This notion of horizontal vectors on S^3 define a connection on the principal S^1-bundle P. Next, for each integer m, let ρ_m be a representation of S^1 on $\boldsymbol{C}$ defined by

$$\rho_m(\varepsilon(t))z=\varepsilon(t)^{-m}z, \quad z\in\boldsymbol{C}.$$

Let E'_m denote the associated line bundle with P by the representation, ρ_m, that is, the quatient manifold of $S^3\times\boldsymbol{C}$ by the equivalence relation $(h, z)\sim(h\varepsilon, \varepsilon^m z)$, $\varepsilon\in S^1$. For each $h\in S^3$, define $q_h\colon \boldsymbol{C}\to\tilde{\pi}^{-1}(\pi(h))$ by $z\mapsto[(h, z)]$, where $\tilde{\pi}\colon E'_m\to S^2$ is the projection. Let $C^\infty_m(S^3)$ be the set consisting of every C^∞ function f on S^3 such that

$$f(h\varepsilon)=\varepsilon^m f(h) \tag{5.3}$$

for every $h\in S^3$ and $\varepsilon\in S^1$, which is called an equivariant function with respect to ρ_m. For $s\in C^\infty(E'_m)$, define a C^∞ function $q^\#_m s$ on S^3 by $(q^\#_m s)(h)=q_h^{-1}(s(\pi(h)))$, $h\in S^3$. Then $q^\#_m s$ belongs to $C^\infty_m(S^3)$ and $q^\#_m$ gives a one-one correspondence between $C^\infty(E'_m)$ and $C^\infty_m(S^3)$. A linear connection $\tilde{d}'_m$ on E'_m associated with the connection on P is defined as the covariant derivative:

$$\nabla'_X s=(q^\#_m)^{-1}X^\# q^\#_m s,\ s\in C^\infty(E'_m),\ X\in T(S^2),$$

where $X^\#$ is the horizontal lift of X. We find by straightforward calculations, that the curvature form of $\tilde{d}'_m$ is equal to Ω_m. Therefore, by virtue of Kostant's theorem, we have $(E'_m, \tilde{d}'_m)=(E_m, \tilde{d}_m)$.

Now, define an operator $L^\#_m$ on $C^\infty_m(S^3)$ by

$$L^\#_m=q^\#_m L_m(q^\#_m)^{-1}.$$

Then, $\mathrm{Sp}\,(S^2, g_0; E_m, \tilde{d}_m)$ consists of eigenvalues of $L^\#_m$, and each eigensection s associates to $q^\#_m s$. Using coordinates (θ, φ, ψ) of S^3, we get

$$L^\#_m=-\frac{\partial^2}{\partial\theta^2}-\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2\cos\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi\partial\psi}-\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2},$$

and therefore

$$L^\#_m=\frac{1}{4}\Delta_3+\frac{\partial^2}{\partial\psi^2},$$

where Δ_3 is the Laplace-Beltrami operator on (S^3, g_0). Noting (5.3), we have $\partial f/\partial\psi=(i/2)mf$ for $f\in C^\infty_m(S^3)$. Therefore,

$$L^\#_m f=\frac{1}{4}\Delta_3 f-\frac{m^2}{4}f \tag{5.4}$$

holds. Thus, if $L^\#_m f=\lambda f$, then $(1/4)\Delta_3 f=\{\lambda+(m^2/4)\}f$, that is, f is an eigenfunction of Δ_3. It is well known that eigenvalues of Δ_3 are

$$\mu_n = n(n+2),\ n=0,\ 1,\ 2,\dots$$

and the eigenfunctions corresponding to μ_n are

$$Y^n_{jk} = e^{i(j\varphi+k\psi)/2}\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\beta P^{(\alpha,\beta)}_\gamma(\cos\theta),$$

$$|j|,\ |k| = n,\ n-2,\ n-4,\dots \geqq 0 \quad \text{(see Fig. 1)},$$

$$\alpha = |k-j|/2,\ \beta = |k+j|/2,\ \gamma = (n-\alpha-\beta)/2,$$

where $P^{(\alpha,\beta)}_\gamma(x)$ is the Jacobi polynomial (cf. [3, Ch. X]).

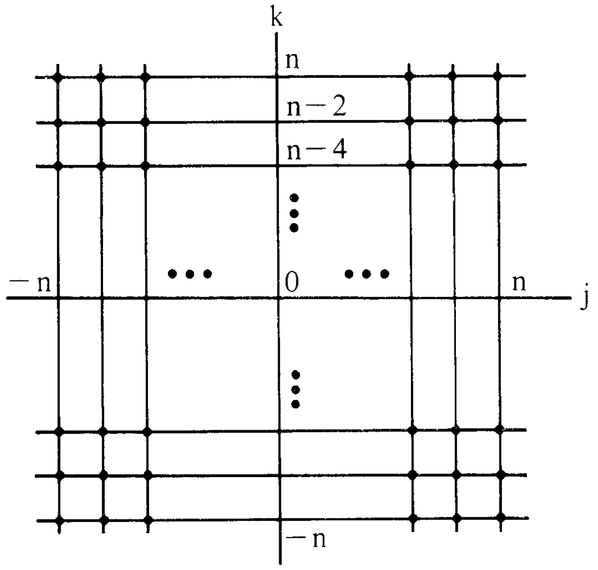

Fig. 1

Since $\partial Y^n_{jk}/\partial\psi = (i/2)kY^n_{jk}$, Y^n_{jk} belongs to $C^\infty_m(S^3)$ if and only if $k=m$. From (5.4), we have

$$L^\#_m Y^n_{jm} = \left\{\frac{n}{2}\left(\frac{n}{2}+1\right) - \frac{m^2}{4}\right\} Y^n_{jm},$$

$$n = |m|,\ |m|+2,\ |m|+4,\dots$$

$$j = n,\ n-2,\dots,\ -n+2,\ -n.$$

Noting that $\iiint Y^n_{jk} Y^n_{lm} d\theta d\varphi d\psi = 0$ if $k \neq m$, we see that $\{(q^\#_m)^{-1} Y^n_{jm}\}$ forms a complete system of eigensections of L_m.

As a consequence, we obtain the following.

Theorem 5.1. *Let (S^2, g_0) be the 2-sphere with canonical metric, and $(E_m, \tilde{d}_m)$, $m \in \boldsymbol{Z}$ be a line bundle with connection whose curvature form is* $\Omega_m =$

$(i/2)m\Theta$, Θ *being the volume element on* S^2 *induced by* g_0. *Then,*

$$\mathrm{Sp}(S^2, g_0; E_m, \tilde{d}_m)=\left\{l(l+1)-\frac{m^2}{4};\ l=\frac{|m|}{2}, \frac{|m|}{2}+1, \frac{|m|}{2}+2,\dots\right\},$$

and the eigensections associated with $l(l+1)-m^2/4$ *are given by* $(q_m^\#)^{-1}Y_{jm}^{2l}$, $j=2l, 2l-2,\dots, -2l+2, -2l$.

Remark. W. Greub and H.-R. Petry [12] obtained the same result under the discussions about the quantum-mechanical motion of a charged particle in a magnetic monopole field.

5.2. Spectrum of line bundle over flat torus

Let $(M, g)=\boldsymbol{R}^n/\Gamma$ be a flat torus, where Γ is a lattice, i.e., a discrete abelian subgroup of the group of Euclidean motions on $\boldsymbol{R}^n$. Let E be a trivial line bundle over (M, g), and $\tilde{d}$ be a flat connection on E. Then, $\tilde{d}$ is defined by a closed 1-form $\omega=i\alpha$ (α: real) on M (see §2, Example), and the Laplacian is given by

$$L=-\sum_{k=1}^{n}\frac{\partial^2}{\partial x^{k^2}}-2i\sum_{k=1}^{n}a_k\frac{\partial}{\partial x^k}+\sum_{k=1}^{n}\left(a_k a_k-i\frac{\partial a_k}{\partial x^k}\right), \tag{5.5}$$

where (x^k) are the coordinates induced from $\boldsymbol{R}^n$, and $\alpha=a_k dx^k$.

First, we consider the case where $a_k=$const. $(k=1,\dots, n)$. Let Γ^* denote the dual lattice, consisting of all $x\in\boldsymbol{R}^n$ such that $(x\,|\,y)=\sum_{k=1}^n x^k y^k$ is an integer for all $y\in\Gamma$. For $\xi\in\Gamma^*$, we set

$$s_\xi(x)=\exp(2\pi i(x\,|\,\xi)).$$

Then $s_\xi(x)$ is a function on M and we have

$$\begin{aligned} Ls_\xi(x)&=4\pi^2(\xi\,|\,\xi)s_\xi(x)+4\pi(a\,|\,\xi)s_\xi(x)+(a\,|\,a)s_\xi(x)\\ &=|2\pi\xi+a|^2 s_\xi(x), \end{aligned}$$

where $|x|^2=(x|x)$, $(x\in\boldsymbol{R}^n)$, and $a=(a_1,\dots, a_n)\in\boldsymbol{R}^n$. Thus, s_ξ is an eigenfunction of L with the eigenvalue $|2\pi\xi+a|^2$. It is well known that $\{s_\xi;\ \xi\in\Gamma^*\}$ is the complete set of eigenfunctions of the Laplace-Beltrami operator on (M, g). Therefore, it is also complete set of eigenfunctions of L.

Next, we consider the case of a general closed 1-form α. Note that the set of harmonic 1-forms on $(M, g)=\boldsymbol{R}^n/\Gamma$ consists of $\alpha^*=a_k dx^k$ with a_k's being constants. By the Hodge-de Rham theorem, there is a unique harmonic 1-form $\alpha^*=a_k dx^k$ which is cohomologous to α. That is,

$$\alpha-\alpha^*=d\psi \tag{5.6}$$

holds for some function ψ on M. Let $\tilde{d}^*$ be the connection on E induced by the

1-form $\omega^* = i\alpha^*$. Form Propositions 4.1 find 4.3, we have $\tilde{d} \underset{(g)}{\sim} \tilde{d}^*$ and accordingly, $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g; E, \tilde{d}^*) = \{|2\pi\xi + a|^2;\ \xi \in \Gamma^*\}$. We find the eigenfunction associated with the eigenvalue $|2\pi\xi + a|^2$. By setting $\Psi = \exp(i\psi)$, we have $\omega = \Psi^{-1} d\Psi + \omega^*$ from (5.6). Therefore, as shown in the proof of Proposition 4.1, $s(x) = \Psi^{-1} s_\xi(x) = \exp\{2\pi i(x \mid \xi) - i\psi(x)\}$ is an eigenfunction of L for the eigenvalue $|2\pi\xi + a|^2$.

Thus, we have obtained the following theorem.

Theorem 5.2. *Let $(M, g) = \boldsymbol{R}^n/\Gamma$ be a flat torus and E be a trivial line bundle over (M, g). Suppose $\tilde{d}$ is a flat linear connection on E defined by a closed 1-form $\omega = i\alpha$ (α: real) on M, and suppose $\alpha - a_k dx^k = d\psi$, where $a_k = $ const. and ψ is a real function on M. Then,*

$$\mathrm{Sp}(M, g; E, \tilde{d}) = \{|2\pi\xi + a|^2;\ \xi \in \Gamma^*\},$$

where $a = (a_1, \dots, a_n) \in \boldsymbol{R}^n$, and the eigenfunction associated with $|2\pi\xi + a|^2$ is given by

$$\exp\{2\pi i(x \mid \xi) - i\psi(x)\}.$$

Given a vector $a = (a_1, \dots, a_n) \in \boldsymbol{R}^n$, we denote by $\tilde{d}_a$ the connection on the trivial line bundle which is defined by the 1-form $\omega = i a_k dx^k$.

By virtue of Theorem 5.2, we have the following theorem concerning the first eigenvalue.

Theorem 5.3. *Let $(M, g) = \boldsymbol{R}^n/\Gamma$ be a flat torus and E be a trivial line bundle with a flat connection $\tilde{d}$. Suppose λ_1 be the first eigenvalue of the Laplacian on E, and $V = \mathrm{vol}(M, g)$ is fixed. Then, the following holds.*

(1) *For any large $N \in \boldsymbol{R}$, there are Γ and $\tilde{d}$ such that $\lambda_1 > N$.*

(2) *Let $m(\lambda_1)$ denote the multiplicity of λ_1. Then, $m(\lambda_1) \leq 2^n$. Moreover, if $m(\lambda_1) = 2^n$, then Γ is a rectangular lattice and*

$$\lambda_1 \geq n\pi^2 (V^{-2/n}), \tag{5.7}$$

where the equality holds if and only if Γ is cubic and $\tilde{d} \underset{(g)}{\sim} \tilde{d}_a$, $a = (\pi V^{-1/n}, \dots, \pi V^{-1/n})$.

Remark. A lattice Γ is called *rectangular* if there is a generator $\{\gamma_1, \dots, \gamma_n\}$ of Γ such that $(\gamma_i \mid \gamma_j) = 0$ if $i \neq j$. Moreover, Γ is *cubic* if it is rectangular and $|\gamma_1| = \cdots = |\gamma_n|$.

PROOF. (1) Let D_Γ denote the fundamental domain of the lattice Γ. When $\mathrm{vol}(M, g) = [\mathrm{vol}(D_{\Gamma^*})]^{-1}$ is fixed, there is Γ such that one of the side lines of $D_{2\pi\Gamma^*}$ has an any long length. By setting a to be the half of the maximal side line and $\tilde{d} \underset{(g)}{\sim} \tilde{d}_a$, λ_1 takes an any large value.

(2) Note that Γ is rectangular if and only if Γ^* is so. It is obvious that $m(\lambda_1)$ takes the maximal value 2^n if and only if Γ is rectangular and $\tilde{d} \underset{(g)}{\sim} \tilde{d}_a$, a being the center of $D_{2\pi\Gamma^*}$. Moreover, $\lambda_1 = |a|^2$ if $m(\lambda_1) = 2^n$. When Γ^* is rectangular and $\mathrm{vol}(D_{2\pi\Gamma^*})$ is fixed, the length of the diagonal lines of $D_{2\pi\Gamma^*}$ takes the minimal value $2\pi n^{1/2}[\mathrm{vol}(D_{\Gamma^*})]^{1/n}$ if and only if Γ^* is cubic, i.e., Γ is cubic. Therefore, we have (5.7). Q.E.D.

We conclude this section by giving a counterexample for the converse of Proposition 4.1. Let $\Gamma = \{(p, q) \in \boldsymbol{R}^2;\ p, q \in \boldsymbol{Z}\}$ and $(M, g) = \boldsymbol{R}^2/\Gamma$. Obviously, $\Gamma^* = \Gamma$ holds. Set $a = (a_1, a_2)$ and $a' = (a_1, -a_2)$, where $0 < a_2 < \pi$ is satisfied. Then, for $\xi \in \Gamma$, we have $|2\pi\xi + a'| = |2\pi\bar{\xi} + a|$, where $\bar{\xi} = (\xi_1, -\xi_2) \in \Gamma$ for $\xi = (\xi_1, \xi_2)$. Therefore, $\mathrm{Sp}(M, g; E, \tilde{d}_a) = \mathrm{Sp}(M, g; E, \tilde{d}_{a'})$ holds. But, for $\alpha = a_1 dx^1 + a_2 dx^2$, $\alpha' = a_1 dx^1 - a_2 dx^2$ and a closed curve $c = c(t) = (0, t) \in \boldsymbol{R}^2$, $(0 \leq t \leq 1)$, we have

$$\frac{1}{2\pi}\int_c (\alpha - \alpha') = \frac{a_2}{\pi},$$

which is not an integer. Hence, from Proposition 4.3, $\tilde{d}_a$ is not gauge equivalent to $\tilde{d}_{a'}$.

Remark. In the above example, $\tilde{d}_a$ is weakly gauge equivalent to $\tilde{d}_{a'}$. In fact, the map $\psi\colon \boldsymbol{R}^2 \times \boldsymbol{C} \to \boldsymbol{R}^2 \times \boldsymbol{C}$, $(x, s) \mapsto (-x, s)$ induces a weak gauge transformation on $E = (\boldsymbol{R}^2/\Gamma) \times \boldsymbol{C}$, and obviously $\tilde{d}_{a'} = \psi^* \tilde{d}_a$ holds. Thus, we propose the following conjecture. Let $\tilde{d}$ and $\tilde{d}'$ be two flat connections on the trivial line bundle E over a flat torus (M, g). Then, $\mathrm{Sp}(M, g; E, \tilde{d}) = \mathrm{Sp}(M, g; E, \tilde{d}')$ holds if and only if $\tilde{d} \underset{(wg)}{\sim} \tilde{d}'$.

§6. Estimations of first eigenvalue

In this section, we estimate the non-zero first eigenvalue of the Laplacian by the curvatures. These are generalizations of Lichnerowicz's result (cf. [6], [9]).

Let (,) be the natural inner product of $T_q(M, E)$, that is, for a frame $\boldsymbol{e} = \{e_\alpha\}$ of E,

$$(T, T') = \int_M {}^t\boldsymbol{T}_{k\cdots m} H \boldsymbol{T}'^{k\cdots m} dV_g, \tag{6.1}$$

where $T^{(\prime)} = \boldsymbol{e} \cdot \boldsymbol{T}^{(\prime)}{}_{k\cdots m} dx^k \otimes \cdots \otimes dx^m \in T_q(M, E)$, and $H = (H_{\alpha\beta}) = (\langle e_\alpha, e_\beta \rangle)$.

Lemma 6.1. *The adjoint operator $\tilde{\nabla}^*\colon T_{q+1}(M, E) \to T_q(M, E)$ of $\tilde{\nabla}$ with respect to (,) is given by*

$$(\tilde{\nabla}^* T)_{j\cdots k} = -\tilde{\nabla}^m T_{mj\cdots k}. \tag{6.2}$$

PROOF. We have from (6.1),

$$(\tilde{\nabla}S, T) = \int_M \tilde{\nabla}_m {}^t\boldsymbol{S}_{j\cdots k} H \boldsymbol{T}^{mj\cdots k} dV_g$$

$$= \int_M [\nabla_m {}^t\boldsymbol{S}_{j\cdots k} H \overline{\boldsymbol{T}}^{mj\cdots k} + {}^t\boldsymbol{S}_{j\cdots k} {}^t\omega_m H \overline{\boldsymbol{T}}^{mj\cdots k}] dV_g$$

$$= \int_M [-{}^t\boldsymbol{S}_{j\cdots k} (\nabla_m H) \overline{\boldsymbol{T}}^{mj\cdots k} - {}^t\boldsymbol{S}_{j\cdots k} H \nabla_m \overline{\boldsymbol{T}}^{mj\cdots k}$$

$$+ {}^t\boldsymbol{S}_{j\cdots k} {}^t\omega_m H \overline{\boldsymbol{T}}^{mj\cdots k}] dV_g.$$

Since the connection is compatible with the Hermitian structure, we have $\nabla_m H = H\overline{\omega}_m + {}^t\omega_m H$ (Lemma 2.1). Therefore, we obtain

$$(\tilde{\nabla} S, T) = -\int_M [{}^t\boldsymbol{S}_{j\cdots k} H \overline{\omega}_m \overline{\boldsymbol{T}}^{mj\cdots k} + {}^t\boldsymbol{S}_{j\cdots k} H \nabla_m \overline{\boldsymbol{T}}^{mj\cdots k}] dV_g$$

$$= -\int_M {}^t\boldsymbol{S}_{j\cdots k} H \tilde{\nabla}_m \overline{\boldsymbol{T}}^{mj\cdots k} dV_g,$$

which means (6.2). Q.E.D.

We define *covariant differentiation* $\tilde{\nabla}: T^p_q(M, \operatorname{End}(E)) \to T^p_{q+1}(M, \operatorname{End}(E))$ by

$$\tilde{\nabla}_t K^{i\cdots j}_{k\cdots m} = \nabla_t K^{i\cdots j}_{k\cdots m} + \omega_t K^{i\cdots j}_{k\cdots m} - K^{i\cdots j}_{k\cdots m} \omega_t,$$

where $K^{i\cdots j}_{k\cdots m}$ is an $r \times r$ matrix. Then, for $K \in T^p_q(M, \operatorname{End}(E))$ and $s \in T^{p'}_{q'}(M, E)$,

$$\tilde{\nabla}_t(Ks) = (\tilde{\nabla}_t K)s + K(\tilde{\nabla}_t s)$$

holds good.

Remarks. 1. The Riemannian metric g is regarded as an element $g \otimes I$ of $T_2(M, \operatorname{End}(E))$, and $\tilde{\nabla} g = 0$ holds.

2. The Bianchi identity in Lemma 2.3 is expressed as $\tilde{\nabla}_j \Omega_{km} + \tilde{\nabla}_k \Omega_{mj} + \tilde{\nabla}_m \Omega_{jk} = 0$.

Definition. A linear connection $\tilde{d}$ on E is called a *harmonic* (or *Yang-Mills*) *connection* if $(\delta\Omega)_j = \tilde{\nabla}^k \Omega_{jk} = 0$ holds for the curvature Ω of $\tilde{d}$. See [13], for details.

Remark. Suppose E is a line bundle. Then, the curvature Ω of the connection $\tilde{d}$ is a closed 2-form on M such that $\Omega/2\pi i$ is integral. Moreover, $\tilde{d}$ is hormonic if and only if Ω is a harmonic 2-form. The connections $\tilde{d}_m$ studied in §5.1, i.e., $(S^2, g_0, E_m, \tilde{d}_m)$, are harmonic.

Now, let L be the Laplacian on E, and assume $Ls = \lambda s$ holds for $s \in C^\infty(E)$ and $\lambda > 0$. Then, we have the following formulas.

(6.3) $$\lambda(s, s) = (Ls, s) = -(\tilde{\nabla}^2 s, gs) = (\tilde{\nabla} s, \tilde{\nabla} s),$$

(6.4) $$(\tilde{\nabla}^* \tilde{\nabla}^2 s)_j = -\tilde{\nabla}^k \tilde{\nabla}_k \tilde{\nabla}_j s = \lambda \tilde{\nabla}_j s - g^{mk} R_{kj} \tilde{\nabla}_m s - 2g^{mk} \Omega_{kj} \tilde{\nabla}_m s - (\tilde{\nabla}^m \Omega_{mj}) s,$$

where R_{jk} is the Ricci tensor of (M, g).

In fact, (6.3) is immediate from the definition of L. As to (6.4), using the Ricci formula (2.6), we have

$$
\begin{aligned}
-\tilde{\nabla}^k \tilde{\nabla}_k \tilde{\nabla}_j s &= -\tilde{\nabla}^k \tilde{\nabla}_j \tilde{\nabla}_k s - \tilde{\nabla}^k(\Omega_{kj} s) \\
&= -\tilde{\nabla}_j \tilde{\nabla}^k \tilde{\nabla}_k s + g^{km} R^i_{kmj} \tilde{\nabla}_i s - g^{km} \Omega_{mj} \tilde{\nabla}_k s \\
&\qquad - \Omega_{kj} \tilde{\nabla}^k s - (\tilde{\nabla}^k \Omega_{kj}) s \\
&= \lambda \tilde{\nabla}_j s - g^{mk} R_{kj} \tilde{\nabla}_m s - 2 g^{mk} \Omega_{kj} \tilde{\nabla}_m s - (\tilde{\nabla}^k \Omega_{kj}) s.
\end{aligned}
$$

We define two endomorphisms ρ and Θ of the bundle $E \otimes T^*(M)$ by $\rho: v_j \mapsto g^{mk} R_{kj} v_m$ and $\Theta: v_j \mapsto g^{mk} \Omega_{kj} v_m$, respectively. It is easily shown that ρ and Θ are Hermitian operators.

We set

$$
u = \tilde{\nabla}^2 s + \frac{\lambda}{n} g s,
$$

and assume that the connection is harmonic. Then, from (6.3) and (6.4), we have

$$
\begin{aligned}
(u, u) &= (\tilde{\nabla}^2 s, \tilde{\nabla}^2 s) + \frac{2\lambda}{n} (g s, \tilde{\nabla}^2 s) + \left(\frac{\lambda}{n}\right)^2 (g s, g s) \\
&= (\tilde{\nabla} s, \tilde{\nabla}^* \tilde{\nabla}^2 s) - \frac{2\lambda}{n} (\tilde{\nabla} s, \tilde{\nabla} s) + \frac{\lambda}{n} (\tilde{\nabla} s, \tilde{\nabla} s) \\
&= \left(\lambda - \frac{\lambda}{n}\right) (\tilde{\nabla} s, \tilde{\nabla} s) - (\tilde{\nabla} s, \rho \tilde{\nabla} s) - (\tilde{\nabla} s, 2\Theta \tilde{\nabla} s).
\end{aligned}
$$

Since $(u, u) \geq 0$ and $(\tilde{\nabla} s, \tilde{\nabla} s) > 0$, we get the following.

Proposition 6.2. *Let E be a vector bundle over (M, g), and $\tilde{d}$ be a harmonic connection on E. Assume $Ls = \lambda s$ holds for $s \in C^\infty(E)$ and $\lambda > 0$. Then,*

$$
\lambda \geq \frac{n}{n-1} \frac{(\tilde{\nabla} s, (\rho + 2\Theta) \tilde{\nabla} s)}{(\tilde{\nabla} s, \tilde{\nabla} s)}. \tag{6.5}
$$

Let Θ_x^+ (resp. Θ_x^-) denote the maximum (resp. minimum) of the eigenvalues of the operator $\Theta \mid E_x \otimes T_x^*(M)$, $(x \in M)$.

Theorem 6.3. *Let E be a vector bundle over (M, g), and $\tilde{d}$ be a harmonic connection on E. Assume that $R_{jk} \geq k g_{jk}$, and $\Theta_x^- \geq c$ hold for every $x \in M$, where k and c are constants subject to $k + 2c > 0$. Then, the non-zero first eigenvalue λ of the Laplacian L satisfies*

$$\lambda \geq \frac{n}{n-1}(k+2c). \tag{6.6}$$

Remark. The above theorem does not assert that the first eigenvalue λ_1 satisfies

$$\lambda_1 \geq \frac{n}{n-1}(k+2c). \tag{6.7}$$

If we assume E is a line bundle and $\Omega \neq 0$ besides the assumptions of the theorem, then (6.7) is satisfied.

Next, we give another estimation without the assumption that $\tilde{d}$ is harmonic. Set

$$w = \tilde{\nabla}^2 s + \frac{\lambda}{n} g s - \Omega s = u - \Omega s.$$

Then we have

$$\begin{aligned}(w, w) &= (u - \Omega s, u - \Omega s) \\ &= (u, u) - (\Omega s, u) - (u, \Omega s) + (\Omega s, \Omega s).\end{aligned}$$

It is easy to see that $(\Omega s, u)$ is real and

$$\begin{aligned}(\Omega s, u) &= (u, \Omega s) = (\tilde{\nabla}^2 s, \Omega s) \\ &= -\frac{1}{2}(\tilde{\nabla} s, \Theta \tilde{\nabla} s) + \frac{1}{2}(\tilde{\nabla} s, (\tilde{\delta}\Omega) s).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(w, w) &= \frac{(n-1)}{n}(\tilde{\nabla} s, \tilde{\nabla} s)\lambda \\ &\quad - (\tilde{\nabla} s, (\rho + \Theta)\tilde{\nabla} s) + (\Omega s, \Omega s).\end{aligned} \tag{6.8}$$

Choose a local coordinate system (x^i) of (M, g) orthonormal at $x \in M$, and a local frame $\boldsymbol{e}$ of E orthonormal at E_x. An element α of $E_x \otimes T_x^*(M)$ is written as $\alpha = \boldsymbol{e} \cdot (\boldsymbol{a}_j + i\boldsymbol{b}_j) dx^j$, where $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ are real r-column vectors. We set

$${}^t\boldsymbol{a} = ({}^t\boldsymbol{a}_1 \cdots {}^t\boldsymbol{a}_n {}^t\boldsymbol{b}_1 \cdots {}^t\boldsymbol{b}_n),$$

and $\boldsymbol{a}$ is a $2nr$-column vector. Moreover, for $\Omega_{jk} = A_{jk} + iB_{jk}$ (A_{jk}, B_{jk}: real $(r \times r)$ matrices), we set

$$C=\begin{pmatrix} 0 & -A_{12}\cdots -A_{1n} & 0 & B_{12}\cdots\cdots B_{1n} \\ -A_{21} & 0\ \cdots\ -A_{2n} & B_{21} & 0\ \cdots\cdots\ B_{2n} \\ \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots\ \ddots\ \vdots \\ -A_{n1} & -A_{n2}\cdots\cdots 0 & B_{n1} & B_{n2}\cdots\cdots 0 \\ 0 & B_{21}\cdots\cdots B_{n1} & 0 & -A_{12}\cdots -A_{1n} \\ B_{12} & 0\ \cdots\cdots\ B_{n2} & -A_{21} & 0\ \cdots\ -A_{2n} \\ \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots\ \ddots\ \vdots \\ B_{1n} & B_{2n}\cdots\cdots 0 & -A_{n1} & -A_{n2}\cdots\cdots 0 \end{pmatrix}$$

Lemma 6.4. *The matrix C is a real symmetric $(2nr\times 2nr)$ matrix and*

(6.9) $$\langle\alpha,\ \Theta_x\alpha\rangle={}^t\boldsymbol{a}C\boldsymbol{a}$$

is satisfied.

PROOF. Since ${}^t\bar{\Omega}_{jk}=-\Omega_{jk}$ holds, we have

$${}^tA_{jk}=-A_{jk},\quad {}^tB_{jk}=B_{jk}.$$

Hence, C is a symmetric matrix, and (6.9) is derived by direct computations. Q.E.D.

Next, set

$$Z_1=\sum_{j,k}(-A_{jk}A_{jk}+B_{jk}B_{jk}),$$

$$Z_2=\sum_{j,k}(B_{jk}A_{jk}+A_{jk}B_{jk}),$$

and

$$Z=\begin{bmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{bmatrix}$$

Then, similarly to Lemma 6.4, we have

Lemma 6.5. *The matrix Z is a real symmetric $(2r\times 2r)$ matrix, and for $s=\boldsymbol{e}\cdot(\boldsymbol{s}_1+i\boldsymbol{s}_2)\in E_x$,*

(6.10) $$\langle\Omega s,\ \Omega s\rangle_x=\frac{1}{4}\,{}^t\boldsymbol{s}Z\boldsymbol{s}$$

holds, where ${}^t\boldsymbol{s}=({}^t\boldsymbol{s}_1,\ {}^t\boldsymbol{s}_2)$.

Lemma 6.6. $\mathrm{Tr}(C^2)=\mathrm{Tr}(Z)$.

PROOF. By straightforward calculatons, we have

$$C^2=\begin{pmatrix} C_{11}\cdots\cdots C_{1n} & D_{11}\cdots D_{1n} \\ \vdots \qquad \vdots & \vdots \quad \vdots \\ C_{n1}\cdots\cdots C_{nn} & D_{n1}\cdots D_{nn} \\ -D_{11}\cdots -D_{1n} & C_{11}\cdots C_{1n} \\ \vdots \qquad \vdots & \vdots \quad \vdots \\ -D_{n1}\cdots -D_{nn} & C_{n1}\cdots C_{nn} \end{pmatrix}$$

and $Z_1=\sum_{k=1}^n C_{kk}$, $Z_2=\sum_{k=1}^n D_{kk}$. Therefore,

$$\begin{aligned}\operatorname{Tr}(C^2)&=2\sum_{k=1}^n \operatorname{Tr}(C_{kk})=2\operatorname{Tr}(\sum C_{kk})\\ &=2\operatorname{Tr}(Z_1)=\operatorname{Tr}(Z).\end{aligned}$$

Q.E.D.

The following lemma by Tanno [10] is useful.

Lemma 6.7 (Tanno [10]). *Let H be a symmetric linear operator of a vector space W with inner product. Then, for every integer $k\geq 1$ and every $w\in W$, we have*

$$\langle Hw, w\rangle^2\leq[\operatorname{Tr}(H^{2k})]^{1/k}\langle w, w\rangle^2.$$

Set

$$\begin{aligned}|\Omega|^2&=-\operatorname{Tr}(\sum_{j,k}\Omega_{jk}\Omega_{jk})=\operatorname{Tr}(\sum_{j,k}\Omega_{jk}{}^t\bar{\Omega}_{jk})\\ &=\sum_{\alpha,\beta,j,k}\Omega^\alpha_{\beta jk}\bar{\Omega}^\alpha_{\beta jk}=\frac{1}{2}\operatorname{Tr}(Z).\end{aligned}$$

Lemma 6.8. (A) *If $|\Theta^\pm_x|\leq c$, then $|\Omega|^2\leq nrc^2$ and $\langle\Omega s, \Omega s\rangle_x\leq nrc^2\langle s, s\rangle_x/4$ hold for every $s\in E_x$.*

(B) *If $\langle\Omega s, \Omega s\rangle_x\leq d^2\langle s, s\rangle_x$ for every $s\in E_x$, then $|\Theta^\pm_x|\leq 2\sqrt{2r}d$ and $|\Omega|^2\leq 4rd^2$ hold.*

(C) *If $|\Omega|^2\leq d^2$, then $|\Theta^\pm_x|\leq\sqrt{2}d$ and $\langle\Omega s, \Omega s\rangle_x\leq d^2\langle s, s\rangle_x/4$ hold for every $s\in E_x$.*

Proof. (A) The condition $|\Theta^\pm_x|\leq c$ leads to $\operatorname{Tr}(C^2)=\operatorname{Tr}(Z)\leq 2nrc^2$. It is easy to see that the matrix Z has eigenvalues $d_1,\dots, d_r$, each multiplicity being two. Thus,

$$2(d_1+\cdots+d_r)\leq 2nrc^2.$$

Hence $\max\{d_x\}\leq nrc^2$, which completes the proof.

(B) The condition $\langle\Omega s, \Omega s\rangle\leq d^2\langle s, s\rangle$ means that the eigenvalues of Z are lower than or equal to $4d^2$. This leads to the conclusions using Lemma 6.7.

(C) If $|\Omega|^2\leq d^2$, we have $|\Theta^\pm_x|\leq\sqrt{2}d$ by Lemma 6.7, and the eigenvalues of Z are lower than or equal to d^2. Thus the proof is completed.

Using (6.8) and the above lemma, we have the following theorem.

Theorem 6.9. *Let E be a Hermitian vector bundle over (M, g) and $\tilde{d}$ be a connection on E. Assume that $R_{jk}\geq kg_{jk}$, k being some positive constant. Then we have the following estimations for the eigenvalue λ of the Laplacian.*

(A) *If $|\Theta_x^{\pm}|\leq c$ holds for every $x\in M$, then*

$$\lambda\leq\frac{n}{2(n-1)}\{(k-c)-\sqrt{(k-c)^2-(n-1)c^2}\}$$

or

$$\lambda\geq\frac{n}{2(n-1)}\{(k-c)+\sqrt{(k-c)^2-(n-1)c^2}\}.$$

(B) *If $\langle\Omega s, \Omega s\rangle_x\leq d^2\langle s, s\rangle_x$ for every $x\in M$ and every $s\in E_x$, then*

$$\lambda\leq\frac{n}{2(n-1)}\left\{(k-2\sqrt{2r}d)-\sqrt{(k-2\sqrt{2r}d)^2-4\left(1-\frac{1}{n}\right)d^2}\right\}$$

or

$$\lambda\geq\frac{n}{2(n-1)}\left\{(k-2\sqrt{2r}d)+\sqrt{(k-2\sqrt{2r}d)^2-4\left(1-\frac{1}{n}\right)d^2}\right\}.$$

(C) *If $|\Omega|_x^2\leq d^2$ for every $x\in M$, then*

$$\lambda\leq\frac{n}{2(n-1)}\left\{(k-\sqrt{2}d)-\sqrt{(k-\sqrt{2}d)^2-\left(1-\frac{1}{n}\right)d^2}\right\}$$

or

$$\lambda\geq\frac{n}{2(n-1)}\left\{(k-\sqrt{2}d)+\sqrt{(k-\sqrt{2}d)^2-\left(1-\frac{1}{n}\right)d^2}\right\}.$$

PROOF. (A) From (6.8) we have

$$\begin{aligned}0&\leq(w, w)\\&\leq\frac{(n-1)}{n}\lambda(\tilde{\nabla}s, \tilde{\nabla}s)-(k-c)(\tilde{\nabla}s, \tilde{\nabla}s)+\frac{nrc^2}{4}(s, s)\\&=\frac{(n-1)}{n}\lambda(\tilde{\nabla}s, \tilde{\nabla}s)-(k-c)(\tilde{\nabla}s, \tilde{\nabla}s)+\frac{nrc^2}{4\lambda}(\tilde{\nabla}s, \tilde{\nabla}s).\end{aligned}$$

Since $(\tilde{\nabla}s, \tilde{\nabla}s)>0$ and $\lambda>0$, we get

$$\frac{(n-1)}{n}\lambda^2-(k-c)\lambda+\frac{nrc^2}{4}\geq 0.$$

From this inequality we obtain the estimation.

The other results are similarly obtained. Q. E. D.

Corollary 6.10. *Besides the assumptions of Theorem* 6.9, *we assume E is a trivial line bundle. Then,*

(A)
$$\lambda_1 \leq \frac{n}{2(n-1)}\{(k-c)-\sqrt{(k-c)^2-(n-1)c^2}\},$$
$$\lambda_2 \geq \frac{n}{2(n-1)}\{(k-c)+\sqrt{(k-c)^2-(n-1)c^2}\}.$$

In the cases (B) *and* (C) *the similar estimations hold good.*

PROOF. Let ω be the connection 1-form of $\tilde{d}$, and $\omega_t = t\omega$, $(0 \leq t \leq 1)$. Let $\tilde{d}_t$ be the connection defined by the 1-form ω_t, and L_t be the Laplacian. Obviously, $C_t = tC$ holds, where C_t is the matrix similarly defined from $\tilde{d}_t$ as in Lemma 6.4. Therefore, the eigenvalue $\lambda(t)$ of L_t satisfies

$$\lambda(t) \leq \frac{n}{2(n-1)}\{(k-tc)-\sqrt{(k-tc)^2-(n-1)(tc)^2}\},$$

or

$$\lambda(t) \geq \frac{n}{2(n-1)}\{(k-tc)+\sqrt{(k-tc)^2-(n-1)(tc)^2}\}.$$

Since the coefficients of L_t are analytic with respect to t, $\lambda_k(t)$ varies analytically (cf. [9, Lemma 3.15]). On the other hand, we have $L_0 = \Delta$ and $\lambda_1(0) = 0$, $\lambda_2(0) \geq nk/(n-1)$. Hence we get the corollary. Q. E. D.

§7. Minakshisundaram's expansion

In [4] P. B. Gilkey studied the asymptotic expansion of Minakshisundaram's type for the general self-adjoint elliptic operator of second order on the vector bundle over M. That is,

$$\sum_{k=1}^{\infty} \exp(-t\lambda_k) \underset{t \downarrow 0}{\sim} \left(\frac{1}{4\pi t}\right)^{n/2} (a_0 + a_1 t + a_2 t^2 + \cdots), \tag{7.1}$$

where $n = \dim M$. In the case of the Laplacian on the vector bundle E over (M, g), one has the following formula for the coefficients a_0, a_1 and a_2.

Proposition 7.1. (Gilkey). *Let $r = \operatorname{rank} E$, and we have*

$$a_0 = r \operatorname{vol}(M, g), \tag{7.2}$$

$$a_1 = \frac{r}{6}\int_M \tau dV_g, \tag{7.3}$$

(7.4) $$a_2=\frac{r}{360}\int_M(2|R|^2-2|\rho|^2+5\tau^2)dV_g-\frac{1}{12}\int_M|\Omega|^2dV_g,$$

where τ *is the scalar curvature,* $|\rho|^2=R_{jk}R^{jk}$, $|R|^2=R_{ijkm}R^{ijkm}$ *and* $|\Omega|^2=-\mathrm{Tr}\cdot(\Omega_{jk}\Omega^{jk})$.

Using the above proposition, we get the following.

Theorem 7.2. *Let* $(E, \tilde{d})$ *and* $(E', \tilde{d}')$ *be two vector bundles with connection over a Riemannian manifold* (M, g). *Suppose* $\mathrm{Sp}(M, g; E, \tilde{d})=\mathrm{Sp}(M, g; E', \tilde{d}')$. *Then,* $\mathrm{rank}\, E=\mathrm{rank}\, E'$, *and if* $\tilde{d}$ *is a flat connection, so is* $\tilde{d}'$.

PROOF. $a_0=a_0'$ leads to $\mathrm{rank}\, E=\mathrm{rank}\, E'$. From $a_2=a_2'$, we have $\int_M|\Omega|^2dV_g=\int_M|\Omega'|^2dV_g$. Hence we have only to show $|\Omega|^2\geqq 0$ for every connection. Take a orthonormal basis of (M, g) and a orthonormal frame of E. Then, since ${}^t\bar{\Omega}=-\Omega$ holds, we have

$$|\Omega|^2=-\mathrm{Tr}(\Omega_{jk}\Omega^{jk})=\mathrm{Tr}({}^t\bar{\Omega}_{jk}\Omega^{jk})$$
$$=\sum_{\alpha,\beta,j,k}\bar{\Omega}^{\beta}_{\alpha jk}\Omega^{\beta}_{\alpha jk}=\sum_{\alpha,\beta,j,k}|\Omega^{\beta}_{\alpha jk}|^2\geqq 0. \qquad \text{Q.E.D.}$$

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