J. Math. Tokushima Univ.
Vol. 15 (1981), 87–95

On the Homogeneous Golod Ideal of a Semigraded Local Ring

By

Hiroshi OKUYAMA and Motoyoshi SAKUMA

(Received April 30, 1981)

In his paper [6], M. Steurich introduced the notion of the semigraded local ring as a generalized concept of a power series ring over a field. Corresponding to the non-homogeneous case, we investigate in this paper, how the properties of Golod homomorphisms, due to G. Levin [3], can be transfered to the semigraded case. And, by making use of it, we obtain some change of ring theorems about Poincaré series in our semigraded case.

Throughout the paper, all rings are commutative and Noetherian, and the symbol $(R, \mathfrak{m}, k)$ stands for R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k.

1. Semigraded local rings and modules.

In this section, we briefly recall the definitions of semigraded rings and modules, the reader is referred to [6].

A local ring $(R, \mathfrak{m}, k)$ is called *semigraded* if (i) $R=\prod_{i=0}^{\infty} R_i$ as an abelian group, (ii) $R_iR_j \subseteq R_{i+j}$. The maximal ideal of R_0 is denoted by $\mathfrak{m}_0$. An R-module M is called *semigraded* if M satisfies the conditions: (i) $M=\prod_{i=0}^{\infty} M_i$, (ii) $R_iM_j \subseteq M_{i+j}$. For any semigraded (abbreviated by s.g.) R-modules M and N, R-homomorphism $f\colon M \to N$ is said to be homogeneous of degree $d \geq 0$ if (i) $f(M_i) \subseteq N_{i+d}$ for all i, (ii) $f(\sum_{i=0}^{\infty} x_i)=\sum_{i=0}^{\infty} f(x_i)$ where $x_i \in M_i$.

An R-algebra $W=\bigoplus_{i=0}^{\infty} W_i$ in the sense of Tate [6, § 2], is said to be *homogeneous* if (i) each W_i is a s.g.R-module, $W_i=\prod_{j=0}^{\infty} W_{ij}$, and the multiplication satisfies $W_{ij}W_{kh} \subseteq W_{i+k,j+h}$, (ii) $W_0=R$ and $W_{0j}=R_j$ for all j, (iii) the R-linear differential map d of W satisfies $d(W_{ij}) \subseteq W_{i-1,j}$ for all i and j. For $x \in W_{ij}$, we say x has the degree i (resp. degree j) with respect to the outer (resp. inner) grading of W. Let V and W be homogeneous R-algebras. A *homogeneous* R-algebra map $f\colon V \to W$ is an algebra homomorphism satisfying (i) $d_Wf=fd_V$, (ii) $f(V_i) \subseteq W_i$, (iii) $f\colon V_i \to W_i$ is a homogeneous homomorphism of s.g. R-modules of degree 0.

Let M be a finitely generated s.g.R-module. It is well known that we can construct the minimal resolution (X_*, d_*) of M which enjoys the following properties:

i) each X_i is free and finitely generated s.g. module.

ii) d_i: $X_i \to X_{i-1}$ $(i=1, 2, \dots)$ is homogeneous R-homomorphism of degree 0.

iii) $\operatorname{Ker} d_i \subseteq \mathfrak{m} X_i$ $(i=0, 1, 2, \dots)$.

Thus, $\operatorname{Tor}_i^R(k, M) \cong X_i/\mathfrak{m}X_i$ is a graded k-module. We define the Poincaré series $P_R^M(X, Y)$ of M as the power series in two variables X and Y:

$$P_R^M(X, Y) = \sum_{i,j\geq 0} \dim_k \operatorname{Tor}_{i,j}^R(k, M) X^i Y^j$$

where $\operatorname{Tor}_{i,j}^R(k, M)$ is the j-th homogeneous component of $\operatorname{Tor}_i^R(k, M)$. We note that, for non semigraded case, the Poincaré series of R is given by $P_R^k(X) = P_R^k(X, 1)$.

Let S be a minimal homogeneous generating system of M. We define

$$\chi_M(X) = \sum_{s\in S} X^{\deg s}.$$

The function χ_M has a following property: If F is a free s.g.R-module, then $\chi_M(X)\chi_F(X) = \chi_{M\otimes_R F}(X)$.

Lemma 1. [6, prop. 1. 24]

a) *For any homogeneous free resolution* (F_*, d_*) *of* M *over* R, *we have*

$$P_M^R(X, Y) \leq \sum_{i=0}^{\infty} \chi_{F_i}(Y) X^i \qquad \text{(coefficientwise)}.$$

b) $$P_M^R(X, Y) = \sum_{i=0}^{\infty} \chi_{\operatorname{Tor}_i^R(k,M)}(Y) X^i.$$

2. Semigraded Golod homomorphism.

Let $(R, \mathfrak{m}, k)$ be a s.g. local ring and let W be a homogeneous R-algebra such that $dW \subseteq \mathfrak{m}W$ and its reduced homology algebra $\tilde{H}(W)$ is a vector space over k. Then, there is a spectral sequence of the following form [4]:

$$E_{p,q}^2 = \operatorname{Tor}_p^R(k, H_q(W)) \underset{p}{\Longrightarrow} H_{p+q}(k\otimes W).$$

Lemma 2. *Let* $b_{ij} = \dim_k \operatorname{Tor}_{i,j}^R(k, k)$ *and* $c_{ij} = \dim_k \tilde{H}_{ij}(W)$. *Then,*

i) $b_{ij} \leq \sum_{r=1}^{i} \sum_{k+h=j} b_{i-r,k} c_{r-h,h} + \dim_k (k\otimes W_i)_j \qquad (i\geq 1)$

ii) $b_{0j} \leq \dim_k (k\otimes W_0)_j$

PROOF. Applying a counting argument, we proceed to compare the various vector space dimensions over k. Now,

$$\dim (E_{p,q}^2)_j = \sum_{k+h=j} b_{pk} c_{qh}$$

for $q>0$. Since $E_{i,0}^{r+1}$ is the kernel of d^r: $E_{i,0}^r \to E_{i-r,r-1}^r$, we have

$$\dim (E_{i,0}^r)_j = \dim U_{i,j}^r + \dim (E_{i,0}^{r+1})_j$$

for each j, where U_i^r = image of d^r. Upon iterating, we get

(1) $$d_{ij} = \dim (E_{i,0}^2)_j = \sum_{r=2}^{i} \dim U_{i,j}^r + \dim (E_{i,0}^\infty)_j.$$

Since U_i^r is a subspace of $E_{i-r,r-1}^r$, we have

(2) $$\dim U_{i,j}^r \leq \sum_{k+h=j} b_{i-r,k} c_{r-1,h}.$$

And, since the spectral sequence converges, $E_{p,q}^\infty$ is isomorphic to a subquotient of $H_{p+q}(k \otimes W)$. Hence

(3) $$\dim (E_{i,0}^\infty)_j \leq \dim (k \otimes W_i)_j$$

for all i, j. Therefore, combining the relations (1), (2), (3), we get

$$d_{ij} \leq \sum_{r=2}^{i} \sum_{k+h=j} b_{i-r,k} c_{r-1,h} + \dim (k \otimes W_i)_j$$

for $i \geq 2$.

On the other hand, the long exact sequence associated to the exact sequence $0 \to \tilde{H}_0(W) \to H_0(W) \to k \to 0$ yields

$$b_{ij} \leq d_{ij} + \sum_{k+h=j} b_{i-1,k} c_{0,h}$$

for $i \geq 1$. Hence, we have

$$b_{ij} \leq \sum_{r=1}^{i} \sum_{k+h=j} b_{i-r,k} c_{r-1,h} + \dim (k \otimes W_i)_j$$

for $i \geq 2$.

Since $b_{1j} \leq d_{1j} + \sum_{k+h=j} b_{0k} c_{0h}$ and $d_{1j} = \dim (E_{1,0}^2)_j = \dim (E_{1,0}^\infty)_j \leq \dim (k \otimes W_1)_j$, i) holds for $i = 1$.

Inequality ii) is obvious, since $b_{00} = 1$ and $b_{0j} = 0$ $(j \neq 0)$.

For any bigraded k-vector space $F = \bigoplus_{i,j \geq 0} F_{ij}$, we define the *Hilbert series* of F as the power series $H_F(X, Y) = \sum_{i,j \geq 0} (\dim_k F_{ij}) X^i Y^j$. From lemma 2, we obtain immediately the following

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a s.g.local ring and W be a homogeneous R-algebra such that $dW \subseteq \mathfrak{m}W$ and $\tilde{H}(W)$ is a k-vector space, Then,*

$$P_R^k(X, Y) \leq \frac{H_{W \otimes k}(X, Y)}{1 - X H_{\tilde{H}(W)}(X, Y)}.$$

Let $f: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homogeneous homomorphism of s.g. local rings of degree 0. Suppose $f(\mathfrak{m}) \subset \mathfrak{n}$ and f induces an isomorphism of residue fields $k =$

$R/\mathfrak{m} \simeq S/\mathfrak{n}$. And, suppose further that S is a finite R-module via f. Let X be a homogeneous minimal R-algebra resolution of k. Under these hypothesis Levin's definition of Golod homomorphism can be stated in the following

Definition. $f: R \to S$ is *homogeneous Golod* if the equality holds for the homogeneous R-algebra $W = X \otimes_R S$ in the theorem 1.

A homogeneous ideal $\mathfrak{a}$ in a s.g. local ring R is called *homogeneous Golod* if the canonical map $R \to R/\mathfrak{a}$ is a homogeneous Golod homomorphism. We say R is a *homogeneous Golod ring* if we represent the completion $\hat{R}$ of R, which is again s.g., as the form $\hat{R} \simeq \tilde{R}/\tilde{\mathfrak{a}}$ where $(\tilde{R}, \tilde{\mathfrak{m}})$ is regular and $\tilde{\mathfrak{a}} \subseteq \tilde{\mathfrak{m}}^2$, then $\tilde{\mathfrak{a}}$ is a homogeneous Golod ideal.

Let $x_1, \dots, x_n$ be a minimal homogeneous generators of the maximal ideal $\mathfrak{m}$ of a s.g. local ring R, $\deg x_i = d_i$ $(i = 1, \dots, n)$, and let K be the Koszul complex of R. Then, it is easily seen that $H_{k \otimes K}(X, Y) = \prod_{i=1}^{n} (1 + XY^{d_i})$ and by theorem 1

$$P_R^k(X, Y) \leq \prod_{i=1}^{n} (1 + XY^{d_i})/1 - \sum_{i=1}^{n} \sum_{j=0}^{\infty} c_{ij} X^{i+1} Y^j$$

where $c_{ij} = \dim_k H_{ij}(K)$. And the equality holds if and only if R is a Golod ring, since $P_{\hat{R}}^k(X, Y) = P_R^k(X, Y)$ where $\hat{R}$ is the completion of R.

We say a homogeneous R-algebra W has a *homogeneous* trivial Massey operation, if for each finite sequence of elements $v_1, \dots, v_n$ of $\tilde{H}(W)$ which are homogeneous both with respect to the outer and inner grading (bihomogeneous) of W, there is a bihomogeneous element $\gamma(v_1, \dots, v_n) \in \mathfrak{m}W$ which satisfies the usual properties of trivial Massey operation [4].

The following theorem is the semigraded version of the result of Levin.

Theorem 2. *The following statements are equivalent*:

i) *$f: R \to S$ is a homogeneous Golod homomorphism for s.g. local rings* R and S.

ii) *$X \otimes_R S$ has a homogeneous trivial Massey operation.*

iii) $P_R^k(X, Y) = \dfrac{P_R^k(X, Y)}{1 - X(P_R^S(X, Y) - 1)}$ and $\mathfrak{n}H(X \otimes_R S) = 0$.

3. A change of ring theorem.

In this section, we present some change of ring theorems of Poincaré series for semigraded local rings, which are derived from the non homogeneous case.

Theorem 3. *Let R be a s.g. local ring and let x be a homogeneous non zero* *divisor in* $\mathfrak{m}$ *and* I *a homogeneous ideal. If* $xI \subset \mathfrak{m}^2$, *then* xI *is a homogeneous Golod ideal. In particular, if* $x \in \mathfrak{m}^2$ *and* $\deg x = d$, *then*

$$P^k_{R/(x)}(X, Y) = \frac{P^k_R(X, Y)}{1 - X^2 Y^d}.$$

PROOF. The first part follows from [4, th.2.3]. As for the second, since x is a non zero divisor, the sequence $0 \to (x) \to R \to R/(x) \to 0$ is exact and $P^{R/(x)}_R(X, Y) = 1 + XY^d$. Thus the theorem follows from theorem 2.

Theorem 4. *Let* R *be a s.g. local ring, then there is an integer* N *such that* $\mathfrak{m}^n$ *is a homogeneous Golod ideal for any* $n \geq N$ *and*

$$P^k_{R/\mathfrak{m}^n}(X, Y) = \frac{P^k_R(X, Y)}{1 - X^2 H_F(-X, Y) P^k_R(X, Y)}$$

where $H_F(X, Y)$ *is the Hilbert series of the bigraded* k*-vector space* $F = \bigoplus_{i=0}^{\infty} \mathfrak{m}^{n+i}/\mathfrak{m}^{n+i+1}$.

PROOF. The argument similar to [4, theorem 2.8] shows that $\mathfrak{m}^n$ is a homogeneous Golod ideal for large $n(\geq N)$ and

$$P^k_{R/\mathfrak{m}^n}(X, Y) = \frac{P^k_R(X, Y)}{1 - X(P^{R/\mathfrak{m}^n}_R(X, Y) - 1)}.$$

And, moreover, the canonical map $\tilde{H}(X \otimes R/\mathfrak{m}^n) \to \tilde{H}(X \otimes R/\mathfrak{m}^{n-1})$ is zero for $n \geq N$, where X is a minimal R-algebra resolution of k.

Therefore, the exact sequence $0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to R/\mathfrak{m}^n \to R/\mathfrak{m}^{n-1} \to 0$ yields the exact sequence

$$(*) \qquad 0 \longrightarrow \mathrm{Tor}^R_{p,q}(k, R/\mathfrak{m}^{n-1}) \longrightarrow \mathrm{Tor}^R_{p-1,q}(k, \mathfrak{m}^{n-1}/\mathfrak{m}^n) \longrightarrow \mathrm{Tor}^R_{p-1,q}(k, R/\mathfrak{m}^n) \longrightarrow 0$$

for $p \geq 2$ and $n \geq N$. Put

$$b_{p,q}(i) = \dim_k \mathrm{Tor}^R_{p,q}(k, R/\mathfrak{m}^i), \quad c_{i,j} = \dim_k (\mathfrak{m}^{i-1}/\mathfrak{m}^i)_j, \quad b_{p,q} = b_{p,q}(1).$$

Then, $(*)$ implies that

$$\begin{aligned} b_{p,q}(n) &= \dim_k \mathrm{Tor}^R_{p-1,q}(k, \mathfrak{m}^n/\mathfrak{m}^{n+1}) - b_{p-1,q}(n+1) \\ &= \sum_{k+h=q} c_{n+1,k} b_{p-1,h} - b_{p-1,q}(n+1). \end{aligned}$$

Upon iterating, we have

$$b_{p,q}(n) = \sum_{i=1}^{p-1} \sum_{k+h=q} (-1)^{i-1} c_{n+i,k} b_{p-i,h} + (-1)^{p-1} b_{1,q}(n+p-1).$$

Since $b_{1,q}(n+p-1) = \dim_k \mathrm{Tor}^R_{1,q}(k, R/\mathfrak{m}^{n+p-1}) = \dim_k \mathrm{Tor}^R_{0,q}(k, \mathfrak{m}^{n+p-1}/\mathfrak{m}^{n+p}) =$

$c_{n+p,q}$ and $b_{0,q}=\delta_{0,q}$ (Kronecker's δ), we have

$$b_{p,q}(n)=\sum_{i=1}^{p}\sum_{k+h=q}(-1)^{i-1}c_{n+i,k}b_{p-i,h}.$$

Consequently,

$$\begin{aligned}P_R^{R/\mathfrak{m}^n}(X,\ Y)&=\sum_{p,q\geq 0}b_{p,q}(n)X^pY^q=1+\sum_{p,q\geq 0}b_{p+1,q}(n)X^{p+1}Y^q\\&=1+\Big(\sum_{i,j\geq 0}(-1)^i c_{n+1+i,j}X^{i+1}Y^j\Big)\Big(\sum_{s,t\geq 0}b_{s,t}X^sY^t\Big)\\&=1+XH_F(-X,\ Y)P_R^k(X,\ Y),\end{aligned}$$

which finish our proof.

Theorem 5. *Let R be a s.g. local 0-dimensional Gorenstein ring with $\dim_k\mathfrak{m}/\mathfrak{m}^2>1$ and let σ be a non-zero element of $0:\mathfrak{m}$, $\deg\sigma=d$. Then, $0:\mathfrak{m}$ is a homogeneous Golod ideal and*

$$P_R^k(X,\ Y)=\frac{P_{\bar R}^k(X,\ Y)}{1-X^2Y^dP_{\bar R}^k(X,\ Y)}$$

where $\bar R=R/0:\mathfrak{m}$.

PROOF. By the similar argument in [4], we see the canonical map $R\to\bar R$ is a homogeneous Golod homomorphism and

$$P_{\bar R}^k(X,\ Y)=\frac{P_R^k(X,\ Y)}{1-X(P_R^{\bar R}(X,\ Y)-1)}.$$

By lemma 1, we have

$$P_R^{\bar R}(X,\ Y)=\sum_{p=0}^{\infty}\chi_{\mathrm{Tor}_p^R(k,\bar R)}(Y)X^p.$$

From the exact sequence $0\to 0:\mathfrak{m}\to R\to\bar R\to 0$, we have

$$\mathrm{Tor}_{p,q}^R(k,\ \bar R)\simeq\mathrm{Tor}_{p-1,q+d}^R(k,\ k)\qquad\text{for}\quad p>0.$$

Hence

$$\chi_{\mathrm{Tor}_p^R(k,\bar R)}(Y)=\chi_{\mathrm{Tor}_{p-1}^R(k,k)}(Y)\cdot Y^d,$$

and

$$\begin{aligned}P_R^{\bar R}(X,\ Y)&=1+\sum_{p=0}^{\infty}\chi_{\mathrm{Tor}_p^R(k,k)}(Y)Y^dX^{p+1}\\&=1+XY^dP_R^k(X,\ Y),\end{aligned}$$

which proves our assertion.

Let M be a finitely generated s.g.module over a s.g. local ring R. Then, the trivial extension $R(M)$ of R by M is semigraded: $R(M)_n = R_n \oplus M_n$ $(n \geq 0)$. The canonical map $R \to R(M)$ satisfies the situation described after Theorem 1.

Theorem 6. *The canonical map $R \to R(M)$ is a homogeneous Golod homomorphism and*

$$P^k_{R(M)}(X, Y) = \frac{P^k_R(X, Y)}{1 - XP^M_R(X, Y)}.$$

PROOF. Again, we see $R \to R(M)$ is a homogeneous Golod homomorphism [4, theorem 2.13]. Since the exact sequence of R-modules $0 \to M \to R(M) \to R \to 0$ is regarded as a sequence of $R(M)$-modules, we have $H(X \otimes_R M) \cong \tilde{H}(X \otimes_R R(M))$ as semigraded $R(M)$-modules, where X is a minimal homogeneous R-algebra resolution of k. Hence, $\mathrm{Tor}^R_{ij}(R(M), k) \cong \mathrm{Tor}^R_{ij}(M, k)$ for $i \geq 1$ and $j \geq 0$, and $(R(M) \otimes_R k)_0 \cong k \oplus (M \otimes k)_0$, $(R(M) \otimes_R k)_j \cong (M \otimes k)_j$ for $j > 0$.

Thus, we have

$$P^{R(M)}_R(X, Y) = \sum_{i=0}^{\infty} \chi_{\mathrm{Tor}^R_i(R(M),k)}(Y) X^i = 1 + \sum_{i=0}^{\infty} \chi_{\mathrm{Tor}^R_i(M,k)}(Y) X^i$$
$$= 1 + P^M_R(X, Y),$$

which proves our result in view of Theorem 2.

In the following we state some other examples which are connected with a homogeneous Golod homomorphism.

Example 1. *If $\mathfrak{m}^2 = 0$, then we have*

$$P^k_R(X, Y) = \frac{1}{1 - \chi_{\mathfrak{m}}(Y) X}.$$

PROOF. Since $0 : \mathfrak{m} = \mathfrak{m}$, $\mathfrak{m}$ is a k-vector space and since $\mathrm{Tor}^R_{p+1}(k, k) \cong \mathrm{Tor}^R_p(k, \mathfrak{m})$ $(p \geq 0)$, we have the relation

$$\chi_{\mathrm{Tor}^R_{p+1}(k,k)}(Y) = \chi_{\mathrm{Tor}^R_p(k,\mathfrak{m})}(Y) = \chi_{\mathfrak{m}}(Y) \chi_{\mathrm{Tor}^R_p(k,k)}(Y).$$

Hence

$$P^k_R(X, Y) = 1 + \sum_{p=0}^{\infty} \chi_{\mathrm{Tor}^R_{p+1}(k,k)}(Y) X^{p+1}$$
$$= 1 + \chi_{\mathfrak{m}}(Y) X \sum_{p=0}^{\infty} \chi_{\mathrm{Tor}^R_p(k,k)}(Y) X^p$$
$$= 1 + \chi_{\mathfrak{m}}(Y) X P^k_R(X, Y),$$

which proves our assertion.

Example 2. *If there exists a homogeneous element* $t \in \mathfrak{m}, \notin \mathfrak{m}^2$ *which satisfies the conditions*:

a) $t^2=0$ *and* b) $\mathfrak{m}^n = t\mathfrak{m}^{n-1}$ *for some integer* $n \geq 2$.

Then, $\mathfrak{m}^n$ *is a homogeneous Golod ideal and*

$$P^k_{R/\mathfrak{m}^n}(X, Y) = \frac{P^k_R(X, Y)}{1 - X^2 \chi_{\mathfrak{m}^n}(Y) P^k_R(X, Y)}.$$

In particular, if we can take $n=2$, *we have*

$$P^k_R(X, Y) = \frac{1}{1 - \chi_{\mathfrak{m}}(Y)X + \chi_{\mathfrak{m}^2}(Y)X^2}.$$

PROOF. The assertion that $\mathfrak{m}^n$ is homogeneous Golod follows from the argument in [4, theorem 2.12]. Hence, we only show that

$$P^{R/\mathfrak{m}^n}_R(X, Y) = 1 + X\, \chi_{\mathfrak{m}^n}(Y) P^k_R(X, Y).$$

Since the exact sequence $0 \to \mathfrak{m}^n \to R \to R/\mathfrak{m}^n \to 0$ induces the isomorphisms of graded k-vector spaces

$$\mathrm{Tor}^R_p(k, R/\mathfrak{m}^n) \cong \mathrm{Tor}^R_{p-1}(k, \mathfrak{m}^n) \cong \mathfrak{m}^n \otimes \mathrm{Tor}^R_{p-1}(k, k),$$

we have

$$\begin{aligned} P^{R/\mathfrak{m}^n}_R(X, Y) &= \sum_{p=0}^{\infty} \chi_{\mathrm{Tor}^R_p(k, R/\mathfrak{m}^n)}(Y) X^p \\ &= 1 + X \sum_{p=0}^{\infty} \chi_{\mathfrak{m}^n \otimes \mathrm{Tor}^R_p(k,k)}(Y) X^p \\ &= 1 + X \chi_{\mathfrak{m}^n}(Y) \sum_{p=0}^{\infty} \chi_{\mathrm{Tor}^R_p(k,k)}(Y) X^p \\ &= 1 + X\, \chi_{\mathfrak{m}^n}(Y) P^k_R(X, Y). \end{aligned}$$

The last part follows from the example 1.

Example 3. *If* R *is a Gorenstein local ring of embedding dimension* $n \geq 1$ *satisfying* $\mathfrak{m}^3=0$ *and if* R *is not a complete intersection, then*

$$P^k_R(X, Y) = \frac{1}{1 - \chi_{\mathfrak{m}}(Y)X + X^2 Y^d}$$

where d *is the degree of non-zero homogeneous element which generates* $0: \mathfrak{m}$.

PROOF If $\mathfrak{m}^2=0$, we have $P^k_R(X, Y) = 1/1 - \chi_{\mathfrak{m}}(Y)X = 1/1 - XY^d$ and R is a complete intersection. If $\mathfrak{m}^2 \neq 0$, we have $0: \mathfrak{m} = \mathfrak{m}^2$ and by Theorem 5

$$P_{\bar{R}}^{k}(X, Y)=\frac{P_{R}^{k}(X, Y)}{1-X^{2}Y^{d}P_{\bar{R}}^{k}(X, Y)}$$

where $\bar{R}=R/0:\mathfrak{m}$. Let $\bar{\mathfrak{m}}$ be the maximal ideal of $\bar{R}$. Then, $\bar{\mathfrak{m}}^2=0$ and therefore $P_{\bar{R}}^{k}(X, Y)=1/1-\chi_{\bar{\mathfrak{m}}}(Y)X=1/1-\chi_{\mathfrak{m}}(Y)X$. From these relations we get our result.

Faculty of Education
Tokushima University
and
Faculty of Integrated Arts and Sciences
Hiroshima University

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