J. Math. Tokushima Univ.
Vol. 15 (1981), 53–85

Galerkin Method for Autonomous Differential Equations[1]

By

Yoshitane SHINOHARA

(Received May 15, 1981)

Abstract

As for the periodic differential equations, M. Urabe [8] developed Galerkin method for numerical analysis of periodic solution.

But, in the autonomous cases, the period of periodic solution is also unknown. Hence, how to deal with the unknown period is a problem.

In the previous papers [4], [5], the author has proposed a Galerkin method for calculating the periodic solution and its period simultaneously to autonomous cases by making use of a boundary value problem.

It is clear that, when $\boldsymbol{x}(t)$ is a solution of autonomous differential equation $\boldsymbol{x}(t+\alpha)$ is also a solution for an arbitrary constant α. The fact tells us the Galerkin approximation to $\boldsymbol{x}(t)$ is not uniquely determined by the periodic boundary condition alone. Hence, in order to determine the Galerkin approximation uniquely, the author considered an additional linear functional and gave a rule how to choose the linear functional.

In the present paper we shall give a mathematical foundation to the Galerkin method for autonomous differential equations, similar to the one for periodic cases given by M. Urabe [8], and summarize our results obtained in the previous papers [4], [5], [12].

It is worth stressing that, in autonomous cases, the quantity $\mathscr{L}(m)$ appeared in the inequalities (5.30) and (5.36) may vanish just as in periodic cases if we choose as $l(\boldsymbol{u})=\int_0^{2\pi}\boldsymbol{x}(t)\cdot\cos pt\,dt$ $(p\leq m)$ the additional linear functional.

§1. Galerkin method for autonomous cases

In the present paper we shall consider a general d-dimensional autonomous differential equation

$$\frac{d\boldsymbol{x}}{d\tau}=\boldsymbol{X}(\boldsymbol{x}), \tag{1.1}$$

---

1) This paper was prepared during the stay in University of Pittsburg in order to study with Professor Werner C. Rheinboldt, supported by Japanese Government.

where $\boldsymbol{X}(\boldsymbol{x}) \in C_x^1[\Delta]$, Δ being a domain in the $\boldsymbol{x}$-space.

In order to compute a Galerkin approximation of ω-periodic solution $\boldsymbol{x}(\tau)$ of (1.1) we transform τ to t by

$$\tau = \frac{\omega t}{2\pi}, \tag{1.2}$$

then the equation (1.1) is rewritten in the form

$$\frac{d\boldsymbol{x}}{dt} = \frac{\omega}{2\pi} \boldsymbol{X}(\boldsymbol{x}). \tag{1.3}$$

The problem then reduced to the one of finding a 2π-periodic solution of (1.3). But, in our case, ω is also unknown. Hence, we consider the differential system

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \dfrac{\omega}{2\pi} \boldsymbol{X}(\boldsymbol{x}), \\ \dfrac{d\omega}{dt} = 0, \end{cases} \tag{1.4}$$

where $\boldsymbol{x}$ and ω are unknown functions. The periodic boundary condition for (1.4) is then as follows:

$$\boldsymbol{x}(0) = \boldsymbol{x}(2\pi). \tag{1.5}$$

As can be seen, when $\boldsymbol{x}(t)$ is a solution of the autonomous system (1.3) $\boldsymbol{x}(t+\alpha)$ is also a solution for an arbitrary constant α. The fact tells us that no 2π-periodic solution to (1.3) is uniquely determined by the condition (1.5) alone. Hence we will consider one more condition, say,

$$l(\boldsymbol{u}) = \beta, \tag{1.6}$$

where $\boldsymbol{u} = \mathrm{col}(\boldsymbol{x}(t), \omega)$ and l is a linear functional satisfying the isolatedness condition $l[\mathrm{col}(\boldsymbol{X}(\boldsymbol{x}), 0)] \neq 0$ (See Theorem 5) and β is a constant number.

We shall write the set of boundary conditions (1.5) and (1.6) in the form

$$\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}, \tag{1.7}$$

where

$$\boldsymbol{f}(\boldsymbol{u}) = \mathrm{col}\,[\boldsymbol{x}(0) - \boldsymbol{x}(2\pi),\ l(\boldsymbol{u}) - \beta]. \tag{1.8}$$

Then the boundary value problem (1.4)–(1.6) can be rewritten as follows:

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} = \boldsymbol{V}(\boldsymbol{u}), \\ \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}, \end{cases} \tag{1.9}$$

where $\boldsymbol{V}(\boldsymbol{u})=\text{col}\,[\omega\boldsymbol{X}(\boldsymbol{x})/2\pi,\ 0]$.

In order to get a 2π-periodic approximate solution to (1.9), we consider a trigonometric polynomial of the form

$$(1.10)\qquad \boldsymbol{x}_m(t)=\boldsymbol{\alpha}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{\alpha}_{2n-1}\sin nt+\boldsymbol{\alpha}_{2n}\cos nt)$$

and the $(d+1)$-dimensional vector $\boldsymbol{u}_m(t)=(\boldsymbol{x}_m(t),\ \omega_0)$, where $\boldsymbol{\alpha}_i$ $(i=0, 1, 2,\dots, 2m)$ are d-dimensional vectors and ω_0 is a real number.

By Galerkin method we shall determine the unknown coefficients $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\dots$, $\boldsymbol{\alpha}_{2m-1}$, $\boldsymbol{\alpha}_{2m}$ and ω_0 so that

$$(1.11)\qquad \frac{d\boldsymbol{x}_m(t)}{dt}=P_m\boldsymbol{X}[\boldsymbol{u}_m(t)]$$

and

$$(1.12)\qquad \boldsymbol{f}(\boldsymbol{u}_m(t))=\boldsymbol{0}$$

may be valid, where $\boldsymbol{X}[\boldsymbol{u}_m(t)]=\omega_0\boldsymbol{X}(\boldsymbol{x}_m(t))/2\pi$ and P_m denotes a truncation of the Fourier series of the 2π-periodic operant function discarding all harmonic terms of the order higher than m. The equalities (1.11) and (1.12) are clearly equivalent to the system of $d(2m+1)+d+1$ equations

$$(1.13)\qquad \begin{cases} F_0(\boldsymbol{\alpha})\equiv \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]ds=\boldsymbol{0}, \\ F_{2n-1}(\boldsymbol{\alpha})\equiv \dfrac{1}{\sqrt{2}\,\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\sin ns\,ds+n\boldsymbol{\alpha}_{2n}=\boldsymbol{0}, \\ F_{2n}(\boldsymbol{\alpha})\equiv \dfrac{1}{\sqrt{2}\,\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\cos ns\,ds-n\boldsymbol{\alpha}_{2n-1}=\boldsymbol{0}, \\ F_f(\boldsymbol{\alpha})\equiv\boldsymbol{f}[\boldsymbol{u}_m(t)]=\boldsymbol{0}, \\ (n=1, 2,\dots, m), \end{cases}$$

where $\boldsymbol{\alpha}=\text{col}\,[\boldsymbol{\alpha}_0,\ \boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\dots,\ \boldsymbol{\alpha}_{2m-1},\ \boldsymbol{\alpha}_{2m},\ \omega_0]$ is a $(d(2m+1)+1)$-dimensional vector.

In (1.13) the system $F_f(\boldsymbol{\alpha})=\boldsymbol{0}$ essentially consists of a single equation $l(\boldsymbol{u}_m(t))-\beta=0$. Hence the number of the equations (1.13) and the one of the unknown coefficients are both $d(2m+1)+1$.

The system of equations (1.13) is called the *determining equations* of Galerkin approximation to autonomous cases.

For brevity let us write the determining equations (1.13) consisting of $d(2m+1)+1$ equations in vector form as follows:

$$(1.14)\qquad \boldsymbol{F}_m(\boldsymbol{\alpha})=\boldsymbol{0}.$$

In what follows we consider the product spaces $B \equiv \Delta \times R^1$ and $\Omega \equiv I \times B$, where $I \equiv [0, 2\pi]$ and R^1 is the real space.

Put

$$C^1[I] = \{\boldsymbol{x}(t) \equiv \text{col}\,[x_1(t), \dots, x_d(t)] \mid x_i(t)\,(i=1, \dots, d) \text{ are } C^1\text{-class on } I\},$$

$$C[I] = \{\boldsymbol{x}(t) \equiv \text{col}\,[x_1(t), \dots, x_d(t)] \mid x_i(t)\,(i=1, \dots, d) \text{ are continuous on } I\},$$

$$S^1 = \{\boldsymbol{u}(t) \equiv [x_1(t), \dots, x_d(t), \omega] \mid (t, \boldsymbol{u}(t)) \in \Omega \text{ for all } t \in I,\ \boldsymbol{u}(t) \in C^1[I] \times R^1\}$$

and

$$S = \{\boldsymbol{u}(t) \equiv \text{col}\,[x_1(t), \dots, x_d(t), \omega] \mid (t, \boldsymbol{u}(t)) \in \Omega \text{ for all } t \in I,\ \boldsymbol{u}(t) \in C[I] \times R^1\}.$$

In the present paper we shall denote the Euclidean norm by $\|\cdots\|$ and define the norms in the product spaces $C[I] \times R^1$ and $W = C[I] \times R^{d+1}$ by the formulas

$$\|\boldsymbol{u}(t)\|_n = \max_{t \varepsilon I} \|\boldsymbol{u}(t)\|$$

and

$$\|\|\boldsymbol{w}\|\| = \|\boldsymbol{x}(t)\|_n + \|\boldsymbol{v}\|$$

respectively, where $\boldsymbol{w} = (\boldsymbol{x}(t), \boldsymbol{v})$. Then the product spaces $C[I] \times R^1$ and W are evidently Banach spaces with respect to the above norms, respectively.

If $\boldsymbol{g}(t)$ is a vector-valued trigonometric polynomial of the form

$$\boldsymbol{g}(t) = \boldsymbol{c}_0 + \sqrt{2} \sum_{n=1}^{m} (\boldsymbol{c}_{2n} \cos nt + \boldsymbol{c}_{2n-1} \sin nt),$$

then by the definition

$$\|\boldsymbol{g}\|_q \equiv \left[\frac{1}{2\pi} \int_0^{2\pi} \|\boldsymbol{g}(t)\|^2 dt\right]^{\frac{1}{2}} \tag{1.15}$$

we easily have

$$\|\boldsymbol{g}\|_q = \|\boldsymbol{\gamma}\|, \tag{1.16}$$

$$\|\boldsymbol{g}\|_n \leqq \sqrt{2m+1}\, \|\boldsymbol{\gamma}\|, \tag{1.17}$$

where $\boldsymbol{\gamma} = \text{col}\,(\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_{2m-1}, \boldsymbol{c}_{2m})$.

Now, the boundary value problem (1.9) is reduced to the one of finding $\boldsymbol{u} \in C^1[I] \times R^1$ satisfying the equation

$$\boldsymbol{F}(\boldsymbol{u}) \equiv \left[\frac{d\boldsymbol{x}}{dt} - \frac{\omega}{2\pi} \boldsymbol{X}(\boldsymbol{x}),\ \boldsymbol{f}(\boldsymbol{u})\right] = \boldsymbol{0}. \tag{1.18}$$

In (1.18) we assume that the function $\boldsymbol{F}(\boldsymbol{u})$ with domain $S^1 \subset C^1[I] \times R^1$ and range

W is continuously weak Fréchet differentiable.

§ 2. General solution to linear boundary value problem

Consider a linear boundary value problem

$$\frac{d\boldsymbol{y}}{dt}=A(t)\boldsymbol{y}+\boldsymbol{g}(t), \tag{2.1}$$

$$L\boldsymbol{y}=\boldsymbol{v}, \tag{2.2}$$

where $\boldsymbol{y}$ is a $(d+1)$-dimensional vector-value function, $A(t)$ is a $(d+1)\times(d+1)$ continuous matrix whose $(d+1)th$ low is zero vector and L is a linear operator mapping $C^1[I]\times R^1$ into R^{d+1}.

By $\Phi(t)$ let us denote the fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{y}}{dt}=A(t)\boldsymbol{y}$$

with the initial condition $\Phi(0)=E$ (*unit matrix*).

By $L[\Phi(t)]$ we denote the matrix whose column vectors are $L[\Phi_i(t)]$ $(i=1, 2,\dots, d+1)$, where $\Phi_i(t)$ are column vectors of $\Phi(t)$.

For the linear boundary value problem (2.1), (2.2), we have

Proposition 1. *If the matrix $G\equiv L[\Phi(t)]$ is nonsingular, that is,*

$$\det G=\det L[\Phi(t)]\neq 0, \tag{2.3}$$

then for any continuous $(d+1)$-dimensional vector-valued function $\boldsymbol{g}(t)$ whose $(d+1)th$ element is zero and for any $(d+1)$-dimensional vector $\boldsymbol{v}$ the linear boundary value problem (2.1), (2.2) *possesses a unique solution $\boldsymbol{y}(t)$, which is given by*

$$\boldsymbol{y}(t)=H^1\boldsymbol{g}+H^2\boldsymbol{v},$$

where

$$H^1\boldsymbol{g}=\Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{g}(s)ds-\Phi(t)G^{-1}L[\Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{g}(s)ds],$$

$$H^2\boldsymbol{v}=\Phi(t)G^{-1}\boldsymbol{v}.$$

Here H^1 is a linear operator mapping $C[I]\times R^1$ into $C^1[I]\times R^1$ and H^2 is a linear operator mapping R^{d+1} into $C^1[I]\times R^1$.

PROOF. The general solution of (2.1) is given by

$$\boldsymbol{y}(t)=\Phi(t)\boldsymbol{c}+\Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{g}(s)ds, \tag{2.4}$$

where $\boldsymbol{c}$ is an arbitrary constant vector. To determine $\boldsymbol{c}$ so that (2.4) may satisfy (2.2), we substitute (2.4) into (2.2). Then we have

$$L[\Phi]\boldsymbol{c}+L[\Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{g}(s)ds]=\boldsymbol{v}.$$

From the assumption we have

$$\boldsymbol{c}=G^{-1}\boldsymbol{v}-G^{-1}L[\Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{g}(s)ds].$$

If we substitute this into (2.4), we end the proof. Q. E. D.

Consider an additive operator T mapping $C^1[I]\times R^1$ into W as follows:

$$T\boldsymbol{y}=\left[\frac{d\boldsymbol{y}}{dt}-A(t)\boldsymbol{y},\ L\boldsymbol{y}\right]. \tag{2.5}$$

Then Proposition 1 tells us that if the matrix G is non-singular the operator T has the linear inverse operator T^{-1} which can be written as follows:

$$T^{-1}\boldsymbol{w}=H^1\boldsymbol{g}+H^2\boldsymbol{v} \tag{2.6}$$

for any $\boldsymbol{w}=(\boldsymbol{g},\ \boldsymbol{v})\in W$. Furthermore we have

$$\|T^{-1}\|_n\leq\max(\|H^1\|_n,\ \|H^2\|_n), \tag{2.7}$$

because of

$$\begin{aligned}\|T^{-1}\boldsymbol{w}\|_n&\leq\|H^1\|_n\cdot\|\boldsymbol{g}\|_n+\|H^2\|_n\cdot\|\boldsymbol{v}\|\\&\leq\max(\|H^1\|_n,\ \|H^2\|_n)(\|\boldsymbol{g}\|_n+\|\boldsymbol{v}\|)\\&\leq\max(\|H^1\|_n,\ \|H^2\|_n)\cdot|||\boldsymbol{w}|||\end{aligned}$$

for any $\boldsymbol{w}=(\boldsymbol{g},\ \boldsymbol{v})\in W$.

Let $\hat{\boldsymbol{u}}(t)$ be a solution of the boundary value problem (1.9) and suppose that $\boldsymbol{V}(\boldsymbol{u})$ in the right-hand side of (1.9) is continuously differentiable.

Let $\Xi(\boldsymbol{u})$ denote the Jacobian matrix of $\boldsymbol{V}(\boldsymbol{u})$ with respect to $\boldsymbol{u}$. Then the solution $\hat{\boldsymbol{u}}(t)$ will be called to be *isolated* if the matrix $\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Phi(t)]$ is nonsingular, where $\Phi(t)$ is the fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{y}}{dt}=\Xi[\hat{\boldsymbol{u}}(t)]\boldsymbol{y}$$

with the initial condition $\Phi(0)=E$ (unit matrix) and $\boldsymbol{f}'(\hat{\boldsymbol{u}})$ denotes the weak Fréchet derivative of $\boldsymbol{f}(\boldsymbol{u})$.

It will be shown that in a sufficiently small neighborhood of a isolated solution there is no solution to (1.9). (*See* Proposition 3).

In what follows, if the condition (2.3) is satisfied, the operator T will be called to be *regular*.

§3. Truncation of Fourier series

As can be seen from (1.11), Galerkin method is based on the truncation of Fourier series. Concerning the truncation of Fourier series we have the following lemma due to Cessari [1].

Lemma 1. *Let $g(t)$ be a continuously differentiable 2π-periodic vector-valued function. Then we have*

$$\|g - P_m g\|_n \leq \sigma(m)\|\dot{g}\|_q, \tag{3.1}$$

$$\|g - P_m g\|_q \leq \sigma_1(m)\|\dot{g}\|_q, \tag{3.2}$$

where $\cdot = d/dt$ *and*

$$\sigma(m) = \sqrt{2}\,[\sum_{n=m+1}^{\infty} n^{-2}]^{\frac{1}{2}}, \tag{3.3}$$

$$\sigma_1(m) = (m+1)^{-1}. \tag{3.4}$$

For $\sigma(m)$, *it holds that*

$$\frac{\sqrt{2}}{\sqrt{m+1}} < \sigma(m) < \frac{\sqrt{2}}{\sqrt{m}}. \tag{3.5}$$

For the proof, see [8].

In what follows, let us assume that $V(u)$ in the right-hand side of (1.9) and its Jacobian matrix $\Xi(u)$ are continuously differentiable in a closed bounded region D of the $(d+1)$-dimension Euclidean space R^{d+1}.

By this assumption it is clear that there are non-negative constants K and K_1 such that

$$\begin{cases} \|V(u)\| \leq K, \\ \|\Xi(u)\| \leq K_1 \end{cases} \tag{3.6}$$

for all $u \in D$.

Let $\hat{u}(t)$ be an arbitrary 2π-periodic solution of (1.9) such that $\hat{u}(t) \in D$ for all $t \in R^1$. Then applying Lemma 1 to $\hat{u}(t)$, we readily have

$$(3.7)\qquad \begin{cases} \text{(i)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_n \leq K\sigma(m), \\ \text{(ii)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_q \leq K\sigma_1(m), \\ \text{(iii)} & \|\dot{\hat{\boldsymbol{u}}}-\dot{\hat{\boldsymbol{u}}}_m\|_n \leq KK_1\sigma(m), \end{cases}$$

where $\hat{\boldsymbol{u}}_m(t)=P_m\hat{\boldsymbol{u}}(t)$.

Suppose that $\hat{\boldsymbol{u}}(t)\in\mathring{D}$ for all $t\in R^1$, where $\mathring{D}$ denotes the interior of D. Since (i) of (3.7) implies

$$(3.8)\qquad \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_n \longrightarrow 0 \quad as \quad m\longrightarrow\infty,$$

it is clear that there is a positive integer m_0 such that for all $m\geq m_0$, $\hat{\boldsymbol{u}}_m(t)\in D$ for all $t\in R^1$.

For any $m\geq m_0$, it is readily seen from (iii) of (3.7) that there is a non-negative constant K_2 such that

$$(3.9)\qquad \left\|\frac{d}{dt}\,\Xi[\hat{\boldsymbol{u}}_m(t)]\right\|\leq K_2.$$

Suppose that $\hat{\boldsymbol{u}}(t)$ is isolated. Then by the definition the operator T is regular. Since $\Xi(\boldsymbol{u})$ is uniformly continuous in D, by (3.8) it is seen that there is a positive integer $m_1\geq m_0$ such that, for any $m\geq m_1$, the operator $\hat{T}_m$ defined by

$$\hat{T}_m\boldsymbol{y}=\left[\frac{d\boldsymbol{y}}{dt}-\Xi[\hat{\boldsymbol{u}}_m(t)]\boldsymbol{y},\ L\boldsymbol{y}\right]$$

is always regular and moreover the mapping $\hat{T}_m^{-1}$ corresponding to $\hat{T}_m$ is equibounded, that is, there is a positive constant M independent of m such that

$$(3.10)\qquad \|\hat{T}_m^{-1}\|_q,\quad \|\hat{T}_m^{-1}\|_n\leq M.$$

§4. Jacobian matrices related with determining equations

In the present section we shall consider some basic properties of the Jacobian matrix $J_m(\boldsymbol{\alpha})$ of the determining equation (1.14) with respect to $\boldsymbol{\alpha}$. The basic properties of $J_m(\boldsymbol{\alpha})$ are obtained from the analysis of the linear algebraic equation

$$(4.1)\qquad J_m(\boldsymbol{\alpha})\boldsymbol{\xi}=\boldsymbol{\gamma},$$

where $\boldsymbol{\alpha}=\mathrm{col}(\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2,\dots, \boldsymbol{a}_{2m-1}, \boldsymbol{a}_{2m}, \omega_0)$, $\boldsymbol{\xi}=\mathrm{col}(\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2,\dots, \boldsymbol{c}_{2m-1},\boldsymbol{c}_{2m}, \mu)$ and $\boldsymbol{\gamma}=\mathrm{col}(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2,\dots, \boldsymbol{p}_{2m-1}, \boldsymbol{p}_{2m}, \boldsymbol{v})$.

Put

$$
(4.2) \qquad \begin{cases} \boldsymbol{x}_m(t)=\boldsymbol{a}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}_{2n}\cos nt+\boldsymbol{a}_{2n-1}\sin nt), \\ \boldsymbol{z}(t)=\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n}\cos nt+\boldsymbol{c}_{2n-1}\sin nt), \\ \boldsymbol{\phi}(t)=\boldsymbol{p}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{p}_{2n}\cos nt+\boldsymbol{p}_{2n-1}\sin nt), \end{cases}
$$

then from (1.18) the equation (4.1) means

$$
\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{\lambda}[\boldsymbol{F}(\boldsymbol{u}_m+\lambda\boldsymbol{h})-\boldsymbol{F}(\boldsymbol{u}_m)] \\
&=\left[\frac{d\boldsymbol{z}}{dt}-X_u(\boldsymbol{u}_m)\boldsymbol{h}, \boldsymbol{f}'(\boldsymbol{u}_m)\boldsymbol{h}\right] \\
&=(\boldsymbol{\phi}, \boldsymbol{v}),
\end{aligned}
$$

where $\boldsymbol{h}=(\boldsymbol{z}(t), \mu)$ and $X_u(\boldsymbol{u})$ denotes the Jacobian matrix of $\boldsymbol{X}(\boldsymbol{u})$ with respect to $\boldsymbol{u}$. Hence we have the following boundary value problem corresponding to (4.1):

$$
(4.3) \qquad \begin{cases} \dfrac{d\boldsymbol{z}(t)}{dt}-P_m\{X_u(\boldsymbol{u}_m)\boldsymbol{h}\}=\boldsymbol{\phi}(t), \\ \boldsymbol{f}'(\boldsymbol{u}_m)\boldsymbol{h}=\boldsymbol{v}. \end{cases}
$$

Consider the following $(d+1)\times(d+1)$ matrix

$$
\Xi(\boldsymbol{u})=\begin{pmatrix} X_u(\boldsymbol{u}) \\ 0\cdots 0 \end{pmatrix}.
$$

Concerning the boundary value problem (4.3), we get

Lemma 2. *Suppose that* (1.9) *possesses an isolated solution* $\hat{\boldsymbol{u}}(t)$ *such that* $\hat{\boldsymbol{u}}(t)\in\mathring{D}$ *for all* $t\in R^1$. *Taking* $m_2\geq m_1$ *sufficiently large, consider the boundary value problem*

$$
(4.4) \qquad \begin{cases} \dfrac{d\boldsymbol{h}(t)}{dt}=P_m\{\Xi(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\}+\boldsymbol{\psi}(t), \\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m)\boldsymbol{h}=\boldsymbol{v} \end{cases}
$$

for $m\geq m_2$, *where* $\hat{\boldsymbol{u}}_m(t)=P_m\hat{\boldsymbol{u}}(t)$, $\boldsymbol{\psi}(t)$ *is an arbitrary* $(d+1)$*-dimensional* 2π*-periodic continuous vector-valued function such as* $\boldsymbol{\psi}(t)\equiv(\boldsymbol{\phi}(t), 0)$ *and* $\boldsymbol{v}$ *is an arbitrary real vector of the form* $\boldsymbol{v}=(\boldsymbol{0}, v)$.

Then for any 2π*-periodic solution* $\boldsymbol{h}(t)$ *of* (4.4) (*if any exists*), *it holds that*

$$
(4.5) \qquad \|\boldsymbol{h}\|_q\leq\frac{M(1+K_1)(\|\boldsymbol{\psi}\|_q+\|\boldsymbol{v}\|)}{1-M(K_2+K_1^2)\sigma_1(m)}
$$

and

$$\|\boldsymbol{\xi}\| \leq \frac{M(1+K_1)}{1-M(K_2+K_1^2)\sigma_1(m)} \|\boldsymbol{\gamma}\|. \tag{4.6}$$

PROOF. For brevity, put

$$A_m(t) \equiv \Xi(\hat{\boldsymbol{u}}_m).$$

Then for any 2π-periodic solution $\boldsymbol{h}(t)$ of (4.4), we have

$$\frac{d\boldsymbol{h}(t)}{dt} = A_m(t)\boldsymbol{h}(t) + \boldsymbol{\psi}(t) + \boldsymbol{\eta}(t),$$

where

$$\boldsymbol{\eta}(t) = -(I-P_m)[A_m(t)\boldsymbol{h}(t)].$$

Here I denotes the identity operator.
Put

$$\boldsymbol{b}(t) \equiv A_m(t)\boldsymbol{h}(t),$$

then by (4.4) we have

$$\dot{\boldsymbol{b}}(t) = \dot{A}_m(t)\boldsymbol{h}(t) + A_m(t)[P_m\{\Xi(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\} + \boldsymbol{\psi}(t)],$$

from which by (3.6) and (3.9) follows

$$\|\dot{\boldsymbol{b}}\|_q \leq K_2\|\boldsymbol{h}\|_q + K_1[\|P_m\{\Xi(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\}\|_q + \|\boldsymbol{\psi}\|_q]. \tag{4.7}$$

But by Bessel's inequality,

$$\|P_m\{\Xi(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\}\|_q \leq \|\Xi(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\|_q \leq K_1\|\boldsymbol{h}\|_q. \tag{4.8}$$

Hence from (4.7) and (4.8) we have

$$\|\dot{\boldsymbol{b}}\|_q \leq (K_2+K_1^2)\|\boldsymbol{h}\|_q + K_1\|\boldsymbol{\psi}\|_q. \tag{4.9}$$

Applying Lemma 1 to $\boldsymbol{\eta}(t)$ and using (4.9), we have

$$\|\boldsymbol{\eta}\|_q \leq \sigma_1(m)[(K_2+K_1^2)\|\boldsymbol{h}\|_q + K_1\|\boldsymbol{\psi}\|_q]. \tag{4.10}$$

On the other hand, from Proposition 1 with $L\boldsymbol{h} = \boldsymbol{f}'(\hat{\boldsymbol{u}}_m)\boldsymbol{h}$ and from (3.10) we have

$$\|\boldsymbol{h}\|_q \leq M(\|\boldsymbol{\psi}\|_q + \|\boldsymbol{\eta}\|_q + \|\boldsymbol{v}\|) \tag{4.11}$$

for $m \geq m_2 \geq m_1$. If we substitute (4.10) into (4.11) we get

$$\|\boldsymbol{h}\|_q \leq M\|\boldsymbol{\psi}\|_q + \sigma_1(m)M[(K_2+K_1^2)\|\boldsymbol{h}\|_q + K_1\|\boldsymbol{\psi}\|_q] + \|\boldsymbol{v}\|M.$$

Now $1-M(K_2+K_1^2)\sigma_1(m)>0$ for $m\geq m_2$, since m_2 supposed to be sufficiently large. Thus from (4.12) we get (4.5). (4.6) is clear from (4.5). Q. E. D.

In Lemma 2, let the Fourier series of $\hat{\boldsymbol{x}}(t)$ be

$$\hat{\boldsymbol{x}}(t)=\hat{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{\infty}(\hat{\boldsymbol{a}}_{2n}\cos nt+\hat{\boldsymbol{a}}_{2n-1}\sin nt).$$

Then clearly

$$\hat{\boldsymbol{x}}_m(t)=\hat{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{m}(\hat{\boldsymbol{a}}_{2n}\cos nt+\hat{\boldsymbol{a}}_{2n-1}\sin nt).$$

Put

$$\hat{\boldsymbol{\alpha}}\equiv\mathrm{col}(\hat{\boldsymbol{a}}_0,\ \hat{\boldsymbol{a}}_1,\ \hat{\boldsymbol{a}}_2,\dots,\ \hat{\boldsymbol{a}}_{2m-1},\ \hat{\boldsymbol{a}}_{2m},\ \hat{\omega}),$$

then from Lemma 2 the following corollaries follow readily.

Corollary 2.1. *For any* $m\geq m_2$,

(4.13) $$\det J_m(\hat{\boldsymbol{\alpha}})\neq 0.$$

PROOF. Suppose the contrary. Then there is a non-trivial vector $\boldsymbol{\xi}$ satisfying

$$J_m(\hat{\boldsymbol{\alpha}})\boldsymbol{\xi}=\mathbf{0}.$$

But this is the equation (4.1) with $\boldsymbol{\gamma}\equiv\mathbf{0}$. Hence, by (4.6) the assumption implies a contradiction. Q. E. D.

Corollary 2.2. *For any* $m\geq m_2$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ *exists and*

(4.14) $$\|J_m^{-1}(\hat{\boldsymbol{\alpha}})\|\leq\frac{M(1+K_1)}{1-M(K_2+K_1^2)\sigma_1(m)}.$$

PROOF. By Corollary 2.1 the existence of $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ is clear. Then the equation

$$J_m(\hat{\boldsymbol{\alpha}})\boldsymbol{\xi}=\boldsymbol{\gamma}$$

possesses a unique solution

$$\boldsymbol{\xi}=J_m^{-1}(\hat{\boldsymbol{\alpha}})\boldsymbol{\gamma}$$

for an arbitrary vector $\boldsymbol{\gamma}$ provided $m\geq m_2$. Taking account of (4.6), we have the estimate (4.14). Q. E. D.

Now put

(4.15) $$\begin{cases}\boldsymbol{\alpha}'=\mathrm{col}(\boldsymbol{a}_0',\ \boldsymbol{a}_1',\ \boldsymbol{a}_2',\dots,\ \boldsymbol{a}_{2m-1}',\ \boldsymbol{a}_{2m}',\ \omega')\\ \boldsymbol{\alpha}''=\mathrm{col}(\boldsymbol{a}_0'',\ \boldsymbol{a}_1'',\ \boldsymbol{a}_2'',\dots,\ \boldsymbol{a}_{2m-1}'',\ \boldsymbol{a}_{2m}'',\ \omega'').\end{cases}$$

Let

(4.16) $$\begin{cases} \boldsymbol{x}'_m(t)=\boldsymbol{a}'_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}'_{2n}\cos nt+\boldsymbol{a}'_{2n-1}\sin nt), \\ \boldsymbol{x}''_m(t)=\boldsymbol{a}''_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}''_{2n}\cos nt+\boldsymbol{a}''_{2n-1}\sin nt) \end{cases}$$

be arbitrary d-dimensional trigonometric polynomials such that

(4.17) $$\theta\boldsymbol{u}'(t)+(1-\theta)\boldsymbol{u}''(t)\in D$$

for all $t\in R^1$ and all θ satisfying $0\leq\theta\leq 1$, where $\boldsymbol{u}'(t)=(\boldsymbol{x}'_m(t), \omega')$ and $\boldsymbol{u}''(t)=(\boldsymbol{x}''_m(t), \omega'')$. Then we have

Lemma 3.

(4.18) $$\|J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')\|\leq K_3\sqrt{2m+1}\|\boldsymbol{\alpha}'-\boldsymbol{\alpha}''\|.$$

Here K_3 is a positive constant such that

(4.19) $$\left\{\sum_{i,j,k=1}^{d+1}\left[\frac{\partial^2 V_i(\boldsymbol{u})}{\partial u_j\partial u_k}\right]^2\right\}^{\frac{1}{2}}\leq K_3,$$

where $V_i(\boldsymbol{u})$ and u_i $(i=1, 2,\dots, d+1)$ are components of vectors $\boldsymbol{V}(\boldsymbol{u})$ and $\boldsymbol{u}$, respectively.

PROOF. Take an arbitrary vector $\boldsymbol{\xi}=\mathrm{col}(\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2,\dots, \boldsymbol{c}_{2m-1}, \boldsymbol{c}_{2m}, \mu)$ and consider

(4.20) $$\boldsymbol{z}(t)\equiv\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n}\cos nt+\boldsymbol{c}_{2n-1}\sin nt).$$

Put

(4.21) $$\boldsymbol{\gamma}'\equiv J_m(\boldsymbol{\alpha}')\boldsymbol{\xi}, \quad \boldsymbol{\gamma}''\equiv J_m(\boldsymbol{\alpha}'')\boldsymbol{\xi}$$

and let

$$\boldsymbol{\gamma}'=\mathrm{col}(\boldsymbol{p}'_0, \boldsymbol{p}'_1, \boldsymbol{p}'_2,\dots, \boldsymbol{p}'_{2m-1}, \boldsymbol{p}'_{2m}, \boldsymbol{v}'),$$
$$\boldsymbol{\gamma}''=\mathrm{col}(\boldsymbol{p}''_0, \boldsymbol{p}''_1, \boldsymbol{p}''_2,\dots, \boldsymbol{p}''_{2m-1}, \boldsymbol{p}''_{2m}, \boldsymbol{v}'').$$

Consider

$$\boldsymbol{\phi}'(t)=\boldsymbol{p}'_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{p}'_{2n}\cos nt+\boldsymbol{p}'_{2n-1}\sin nt),$$
$$\boldsymbol{\phi}''(t)=\boldsymbol{p}''_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{p}''_{2n}\cos nt+\boldsymbol{p}''_{2n-1}\sin nt),$$

and

$$\boldsymbol{\psi}'(t)=\mathrm{col}\,(\boldsymbol{\phi}'(t),\,\mathbf{0}),\quad \boldsymbol{\psi}''(t)=\mathrm{col}\,(\boldsymbol{\phi}''(t),\,\mathbf{0}).$$

Then comparing (4.21) with (4.1), by (4.4) we have

$$\frac{d\boldsymbol{h}(\boldsymbol{t})}{dt}=P_m\{\Xi(\boldsymbol{u}'_m)\boldsymbol{h}(t)\}+\boldsymbol{\psi}'(t),\quad \boldsymbol{f}'(\boldsymbol{u}'_m)\boldsymbol{h}=\boldsymbol{v}',$$

$$\frac{d\boldsymbol{h}(t)}{dt}=P_m\{\Xi(\boldsymbol{u}''_m)\boldsymbol{h}(t)\}+\boldsymbol{\psi}''(t),\quad \boldsymbol{f}'(\boldsymbol{u}''_m)\boldsymbol{h}=\boldsymbol{v}'',$$

from which readily follows

$$\boldsymbol{\psi}'(t)-\boldsymbol{\psi}''(t)=P_m\{[\Xi(\boldsymbol{u}'_m)-\Xi(\boldsymbol{u}''_m)]\boldsymbol{h}(t)\}. \tag{4.22}$$

However on account of (4.17), by (4.19) we have

$$\|\Xi(\boldsymbol{u}')-\Xi(\boldsymbol{u}'')\|\leq K_3\|\boldsymbol{u}'-\boldsymbol{u}''\|.$$

Then by Bessel's inequality it follows from (4.22) that

$$\|\boldsymbol{\psi}'-\boldsymbol{\psi}''\|_q\leq K_3\|\boldsymbol{u}'_m-\boldsymbol{u}''_m\|_n\cdot\|\boldsymbol{h}\|_q.$$

Since $\|\boldsymbol{h}\|_q=\|\boldsymbol{\xi}\|$ and $\|\boldsymbol{\gamma}'-\boldsymbol{\gamma}''\|=\{\|\boldsymbol{\psi}'-\boldsymbol{\psi}''\|_q^2+\|[\boldsymbol{f}'(\boldsymbol{u}'_m)-\boldsymbol{f}'(\boldsymbol{u}''_m)]\boldsymbol{h}\|^2\}^{\frac{1}{2}}=\|\boldsymbol{\psi}'-\boldsymbol{\psi}''\|_q$ because of the linearity of $\boldsymbol{f}(\boldsymbol{u})$, we have

$$\|\boldsymbol{\gamma}'-\boldsymbol{\gamma}''\|\leq K_3\|\boldsymbol{u}'_m-\boldsymbol{u}''_m\|_n\cdot\|\boldsymbol{\xi}\|.$$

Since $\boldsymbol{\gamma}'-\boldsymbol{\gamma}''=[J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')]\boldsymbol{\xi}$ by (4.21) and $\boldsymbol{\xi}$ is an arbitrary vector, we thus have

$$\|J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')\|\leq K_3\|\boldsymbol{u}'_m-\boldsymbol{u}''_m\|_n,$$

from which (4.18) follows readily by (1.17). Q. E. D.

§5. Existence and convergence of Galerkin approximations

Our proof of the existence and the convergence of Galerkin approximations is based on the following proposition due to Urabe.

Proposition 2. *Let*

$$\boldsymbol{F}(\boldsymbol{\alpha})=\mathbf{0} \tag{5.1}$$

be a given real system of equations where $\boldsymbol{\alpha}$ *and* $\boldsymbol{F}(\boldsymbol{\alpha})$ *are vectors of the same dimension and* $\boldsymbol{F}(\boldsymbol{\alpha})$ *is continuously differentiable with respect to* $\boldsymbol{\alpha}$ *in a region* Ω *of the* $\boldsymbol{\alpha}$*-space. Assume that* (5.1) *possesses an approximate solution* $\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}$ *for which the Jacobian matrix* $J(\boldsymbol{\alpha})$ *of* $\boldsymbol{F}(\boldsymbol{\alpha})$ *with respect to* $\boldsymbol{\alpha}$ *is non-singular at* $\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}$ *and there are a positive constant* δ *and a non-negative constant* $\kappa<1$ *such that*

$$
(5.2) \qquad \begin{cases} \text{(i)} & \Omega_\delta = \{\boldsymbol{\alpha} \mid \|\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\| \leq \delta\} \subset \Omega, \\ \text{(ii)} & \|J(\boldsymbol{\alpha}) - J(\hat{\boldsymbol{\alpha}})\| \leq \dfrac{\kappa}{M'} \qquad \textit{for any} \quad \boldsymbol{\alpha} \in \Omega_\delta, \\ \text{(iii)} & M'r/(1-\kappa) \leq \delta, \end{cases}
$$

where r *and* $M'(>0)$ *are numbers such that*

$$
(5.3) \qquad \|\boldsymbol{F}(\hat{\boldsymbol{\alpha}})\| \leq r, \quad \|J^{-1}(\hat{\boldsymbol{\alpha}})\| \leq M'.
$$

Then the system of equations (5.1) *possesses one and only one solution* $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$ *in* Ω_δ *and*

$$
(5.4) \qquad \det J(\bar{\boldsymbol{\alpha}}) \neq 0
$$

and

$$
(5.5) \qquad \|\bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}\| \leq \frac{M'r}{1-\kappa}.
$$

For the proof, see [8].

We can now state a theorem which asserts the existence and the convergence of an infinite sequence of Galerkin approximations corresponding to a 2π-periodic isolated solution.

Theorem 1. *Let* (1.9) *be a given boundary value problem. We assume that both* $\boldsymbol{V}(\boldsymbol{u})$ *and its Jacobian matrix* $\Xi(\boldsymbol{u})$ *are continuously differentiable in* D, *where* D *is a closed bounded region of the* $(d+1)$*-dimension Euclidean space.*

Suppose that (1.9) *possesses a* 2π*-periodic isolated solution* $\hat{\boldsymbol{u}}(t)$ *such that* $\hat{\boldsymbol{u}}(t) \in \mathring{D}$ *for all* $t \in R^1$, *where* $\mathring{D}$ *denotes the interior of* D.

Then there is a positive integer $\bar{m}$ *such that for any* $m \geq \bar{m}$ *there is a Galerkin approximation* $\bar{\boldsymbol{u}}_m(t)$ *of the order* m *converging to* $\hat{\boldsymbol{u}}(t)$ *uniformly together with its derivatives as* $m \to \infty$.

PROOF. Put $\hat{\boldsymbol{u}}_m(t) \equiv P_m \hat{\boldsymbol{u}}(t)$. Then we have

$$
(5.6) \qquad \frac{d\hat{\boldsymbol{u}}_m(t)}{dt} = P_m \frac{d\hat{\boldsymbol{u}}(t)}{dt} = P_m \boldsymbol{V}[\hat{\boldsymbol{u}}(t)].
$$

Now $\hat{\boldsymbol{u}}(t)$ is 2π-periodic and $\hat{\boldsymbol{u}}(t) \in \mathring{D}$ for all $t \in R^1$. Therefore there is a positive number δ_0 such that

$$
(5.7) \qquad U = \{\boldsymbol{u} \mid \|\boldsymbol{u} - \hat{\boldsymbol{u}}(t)\| \leq \delta_0 \quad \textit{for some } t \in R^1\} \subset D.
$$

Then, by (3.8) $\hat{\boldsymbol{u}}_m(t) \in U$ for all $t \in R^1$ and for all $m \geq m_3 \geq m_2$, provided m_3 is sufficiently large.

For $m \geq m_3$, equation (5.6) can be written as follows:

(5.8) $$\frac{d\hat{\boldsymbol{u}}_m(t)}{dt} = P_m\boldsymbol{V}[\hat{\boldsymbol{u}}_m(t)]+\boldsymbol{R}_m(t),$$

where

(5.9) $$\boldsymbol{R}_m(t)= P_m\{\boldsymbol{V}[\hat{\boldsymbol{u}}(t)]-\boldsymbol{V}[\hat{\boldsymbol{u}}_m(t)]\}.$$

Now, if m_3 is sufficiently large, then for any $m \geq m_3$ we have

$$\boldsymbol{V}[\hat{\boldsymbol{u}}(t)]-\boldsymbol{V}[\hat{\boldsymbol{u}}_m(t)]=-\int_0^1 \Xi\{\hat{\boldsymbol{u}}(t)+\theta[\hat{\boldsymbol{u}}_m(t)-(\hat{\boldsymbol{u}}t)]\}[\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)]d\theta.$$

Hence by (3.6) we have

$$\|\boldsymbol{V}[\hat{\boldsymbol{u}}(t)]-\boldsymbol{V}[\hat{\boldsymbol{u}}_m(t)]\| \leq K_1\|\hat{\boldsymbol{u}}(t)-\hat{\boldsymbol{u}}_m(t)\|,$$

from which by (ii) of (3.7) follows

$$\|\boldsymbol{V}[\hat{\boldsymbol{u}}(t)]-\boldsymbol{V}[\hat{\boldsymbol{u}}_m(t)]\|_q \leq KK_1\sigma_1(m).$$

Then from (5.9), by Bessel's inequality, we have

(5.10) $$\|\boldsymbol{R}_m\|_q \leq K\,K_1\sigma_1(m).$$

Let the Fourier series of $\hat{\boldsymbol{x}}(t)$ in $\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t),\,\hat{\omega})$ be

(5.11) $$\hat{\boldsymbol{x}}(t)=\hat{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{\infty}(\hat{\boldsymbol{a}}_{2n}\cos nt+\hat{\boldsymbol{a}}_{2n-1}\sin nt).$$

and put

(5.12) $$\boldsymbol{R}_m(t)=\boldsymbol{r}_0+\sqrt{2}\sum_{n=1}^{\infty}(\boldsymbol{r}_{2n}\cos nt+\boldsymbol{r}_{2n-1}\sin nt).$$

Then comparing (5.8) with (1.11), by (1.13) we get

(5.13) $$\begin{cases} F_0(\hat{\boldsymbol{a}})=-\boldsymbol{r}_0, \\ F_{2n-1}(\hat{\boldsymbol{a}})=-\boldsymbol{r}_{2n-1}, \\ F_{2n}(\hat{\boldsymbol{a}})=-\boldsymbol{r}_{2n}, \\ F_f(\hat{\boldsymbol{a}})=v(m), \end{cases}$$

where $\hat{\boldsymbol{a}}=\mathrm{col}(\hat{\boldsymbol{a}}_0,\,\hat{\boldsymbol{a}}_1,\,\hat{\boldsymbol{a}}_2,\ldots,\,\hat{\boldsymbol{a}}_{2m-1},\,\hat{\boldsymbol{a}}_{2m},\,\hat{\omega})$ and $v(m)=l(\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t))$.

By the use of the notation in section 4, we can write (5.13) in vector form as follows:

(5.14) $$\boldsymbol{F}_m(\hat{\boldsymbol{a}})=-\boldsymbol{\rho}_m,$$

where $\boldsymbol{\rho}_m=\mathrm{col}(\boldsymbol{r}_0,\,\boldsymbol{r}_1,\,\boldsymbol{r}_2,\ldots,\,\boldsymbol{r}_{2m-1},\,\boldsymbol{r}_{2m},\,-v(m))$. By (1.16), from (5.10) we have

(5.15) $$\|\boldsymbol{\rho}_m\| \leq K\, K_1\sigma_1(m) + \mathscr{L}(m),$$

where $\mathscr{L}(m) = \|l\|_n \cdot \|\boldsymbol{u}_m(t) - \boldsymbol{u}(t)\|_n$.

Take m_3 so large that the inequality $\delta_0 - K\sigma(m) > 0$ may be valid for any $m \geq m_3$, and consider the region

(5.16) $$U_m = \{\boldsymbol{u} \mid \|\boldsymbol{u} - \hat{\boldsymbol{u}}_m(t)\| \leq \delta_0 - K\sigma(m) \quad \textit{for some} \quad t \in R^1\}$$

for $m \geq m_3$. Then by (i) of (3.7) it is clear that

(5.17) $$U_m \subset U \subset D$$

for any $m \geq m_3$. Consider the set

(5.18) $$\Omega_m = \left\{\boldsymbol{\alpha} \mid \|\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\| < \frac{\delta_0 - K\sigma(m)}{\sqrt{2m+1}}\right\},$$

then for any $\boldsymbol{\alpha} = \mathrm{col}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, \omega_0) \in \Omega_m$, by (1.17) the corresponding trigonometric polynomials

(5.19) $$\begin{cases} \boldsymbol{x}_m(t) = \boldsymbol{\alpha}_0 + \sqrt{2}\sum_{n=1}^{m} (\boldsymbol{\alpha}_{2n}\cos nt + \boldsymbol{\alpha}_{2n-1}\sin nt), \\ \omega_m(t) = \omega_0 \end{cases}$$

satisfies the inequality

$$\|\boldsymbol{u}_m(t) - \hat{\boldsymbol{u}}_m(t)\| \leq \delta_0 - K\sigma(m),$$

therefore $\boldsymbol{u}_m(t) \in U_m$ for all $t \in R^1$. By (5.17) this implies that $\boldsymbol{F}_m(\boldsymbol{\alpha})$ is well defined for any $\boldsymbol{\alpha} \in \Omega_m$ provided $m \geq m_3$.

Now by the definition a Galerkin approximation of order m consists of a trigonometric polynomials of the form (5.19) whose Fourier coefficients $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, \omega_0$ satisfy the equation

(5.20) $$\boldsymbol{F}_m(\boldsymbol{\alpha}) = \boldsymbol{0},$$

where $\boldsymbol{\alpha} = \mathrm{col}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, \omega_0)$. Since $\hat{\boldsymbol{\alpha}}$ is an approximate solution of (5.20) for large m as can be seen from (5.14) and (5.15), we shall apply Proposition 2 to (5.20) in order to prove the existence of an exact solution, that is, the existence of a Galerkin approximation of the order m.

For any $m \geq m_3$, $m \geq m_2$ since $m_3 \geq m_2$. Therefore by Corollary 2.2 for any $m \geq m_3$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ exists and

(5.21) $$\|J_m^{-1}(\hat{\boldsymbol{\alpha}})\| \leq M',$$

where

(5.22) $$M'=\frac{M[1+K_1\sigma_1(m_3)]}{1-M(K_2+K_1^2)\sigma_1(m_3)}.$$

Further by Lemma 3, we have

(5.23) $$\|J_m(\boldsymbol{\alpha})-J_m(\hat{\boldsymbol{\alpha}})\|\leq K_3\sqrt{2m+1}\,\|\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}\|$$

for any $\boldsymbol{\alpha}\in\Omega_m$ provided $m\geq m_3$.

Take an arbitrary positive number $\kappa<1$, and put

(5.24) $$\delta_1\equiv\min\left[\frac{\kappa}{K_3M'},\quad \delta_0-K\sigma(m_3)\right].$$

Let us take $\overline{m}\geq m_3$ so that, for any $m\geq\overline{m}$,

$$\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}<\frac{\delta_1}{\sqrt{2m+1}}.$$

This is possible because of

$$\frac{\sqrt{2m+1}}{m+1}\longrightarrow 0\quad\text{and}\quad\sqrt{2m+1}\cdot\mathscr{L}(m)\longrightarrow 0\quad(\text{as } m\longrightarrow\infty).$$

Hence we can take a positive number δ_m so as to have

(5.25) $$\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}\leq\delta_m\leq\frac{\delta_1}{\sqrt{2m+1}}.$$

Let

(5.26) $$\Omega_{\delta_m}=\{\boldsymbol{\alpha}\mid\|\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}\|\leq\delta_m\}.$$

Then for any $\boldsymbol{\alpha}\in\Omega_{\delta_m}$ and any $m\geq\overline{m}$, we have

$$\begin{aligned}\|\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}\|&\leq\frac{\delta_1}{\sqrt{2m+1}}\\&\leq\frac{\delta_0-K\sigma(m_3)}{\sqrt{2m+1}}\\&\leq\frac{\delta_0-K\sigma(m)}{\sqrt{2m+1}},\end{aligned}$$

which implies

(5.27) $$\Omega_{\delta_m}\subset\Omega_m$$

for $m\geq\overline{m}$. Then for any $\boldsymbol{\alpha}\in\Omega_{\delta_m}$ and $m\geq\overline{m}$, by (5.23) we have

$$\|J_m(\boldsymbol{\alpha})-J_m(\hat{\boldsymbol{\alpha}})\|\leq K_3\sqrt{2m+1}\,\delta_m,$$

which by (5.24) and (5.25) implies

(5.28) $$\|J_m(\boldsymbol{\alpha})-J_m(\hat{\boldsymbol{\alpha}})\|\leq\frac{\kappa}{M'}$$

for $\boldsymbol{\alpha}\in\Omega_{\delta_m}$ and $m\geq\overline{m}$. Further by (5.15) and (5.25) we have

(5.29) $$\frac{M'\|\boldsymbol{\rho}_m\|}{1-\kappa}\leq\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}\leq\delta_m$$

for $m\geq\overline{m}$.

The expressions (5.27)–(5.29) show that the conditions of Proposition 2 are all fulfilled. Thus by the proposition we see that equation (5.20) possesses one and only one solution $\bar{\boldsymbol{\alpha}}=\mathrm{col}\,(\bar{\boldsymbol{\alpha}}_0, \bar{\boldsymbol{\alpha}}_1, \bar{\boldsymbol{\alpha}}_2,\dots, \bar{\boldsymbol{\alpha}}_{2m-1}, \bar{\boldsymbol{\alpha}}_{2m}, \omega)$ in Ω_{δ_m} for any $m\geq\overline{m}$. This proves the existence of a Galerkin approximation $\bar{\boldsymbol{u}}_m(t)$ of the order m for any $m\geq\overline{m}$.

Next, we shall show the uniform convegence of the Galerkin approximation $\bar{\boldsymbol{u}}_m(t)$ obtained. By our definition, we have

$$\bar{\boldsymbol{x}}_m(t)=\bar{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{m}(\bar{\boldsymbol{a}}_{2n}\cos nt+\bar{\boldsymbol{a}}_{2n-1}\sin nt),$$

$$\bar{\omega}_m(t)=\bar{\omega}_0$$

and $\bar{\boldsymbol{\alpha}}=\mathrm{col}\,(\bar{\boldsymbol{a}}_0, \bar{\boldsymbol{a}}_1, \bar{\boldsymbol{a}}_2,\dots, \bar{\boldsymbol{a}}_{2m-1}, \bar{\boldsymbol{a}}_{2m}, \bar{\omega}_0)$ is a solution of (5.20) in Ω_{δ_m}. By Proposition 2, we have

$$\|\bar{\boldsymbol{\alpha}}-\hat{\boldsymbol{\alpha}}\|\leq\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}.$$

Then by (1.17) we have

$$\|\bar{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}_m\|_n\leq\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}\cdot\sqrt{2m+1}.$$

On the other hand, by (i) of (3.7) we have

$$\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_n\leq K\sigma(m).$$

Thus we have

(5.30) $$\|\bar{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_n\leq\frac{M'[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}\cdot\sqrt{2m+1}+K\sigma(m),$$

which proves that $\bar{\boldsymbol{u}}_m(t)$ converges to $\hat{\boldsymbol{u}}(t)$ uniformly as $m\to\infty$.

Now, since $\bar{\boldsymbol{u}}_m(t)\,(m\geq\overline{m})$ is a Galerkin approximation corresponding to $\bar{\boldsymbol{\alpha}}\in\Omega_{\delta_m}$ it satisfies the equation

(5.31) $$\frac{d\bar{\boldsymbol{u}}_m(t)}{dt}=P_m\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)].$$

This equation can be written as follows:

$$\frac{d\bar{\boldsymbol{u}}_m(t)}{dt}=\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)]+\boldsymbol{\eta}_m(t), \tag{5.32}$$

where

$$\boldsymbol{\eta}_m(t)=-(I-P_m)\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)]. \tag{5.33}$$

By (5.31)

$$\frac{d}{dt}\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)]=\Xi[\bar{\boldsymbol{u}}_m(t)]\,P_m\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)],$$

therefore by (3.6) we have

$$\left\|\frac{d}{dt}\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)]\right\|_q\le K_1K.$$

Then by Lemma 1, from (5.33) we have

$$\|\boldsymbol{\eta}_m\|_n\le KK_1\sigma(m). \tag{5.34}$$

Since

$$\frac{d\hat{\boldsymbol{u}}(t)}{dt}=\boldsymbol{V}[\hat{\boldsymbol{u}}(t)],$$

from (5.32) we have

$$\frac{d\bar{\boldsymbol{u}}_m(t)}{dt}-\frac{d\hat{\boldsymbol{u}}(t)}{dt}=\{\boldsymbol{V}[\bar{\boldsymbol{u}}_m(t)]-\boldsymbol{V}[\hat{\boldsymbol{u}}(t)]\}+\boldsymbol{\eta}_m(t),$$

which, for sufficiently large m, can be written as follows;

$$\frac{d\bar{\boldsymbol{u}}_m(t)}{dt}-\frac{d\hat{\boldsymbol{u}}(t)}{dt}=\int_0^1\Xi[\hat{\boldsymbol{u}}(t)+\theta\{\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\}]\{\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\}\,d\theta+\boldsymbol{\eta}_m(t). \tag{5.35}$$

Then by (3.6), (5.30) and (5.35) we have

$$\|\dot{\bar{\boldsymbol{u}}}_m-\dot{\hat{\boldsymbol{u}}}\|_n\le\sqrt{2m+1}\,\frac{K_1\cdot M'\cdot[K\cdot K_1\cdot\sigma_1(m)+\mathscr{L}(m)]}{1-\kappa}+2K\cdot K_1\cdot\sigma(m) \tag{5.36}$$

for sufficiently large m. From inequality (5.36) it readily follows that $\dot{\bar{\boldsymbol{u}}}_m(t)$ converges to $\dot{\hat{\boldsymbol{u}}}(t)$ uniformly as $m\to\infty$. Q. E. D.

§ 6. Numerical computation of Galerkin approximations

In order to obtain the Galerkin approximations on a computer, it suffies to solve numerically the determining equation.

In practical computations, it is convenient to discrete (1.13) as follows:

$$(6.1)\qquad \begin{cases} G_0(\boldsymbol{\alpha}) \equiv \dfrac{1}{2N} \displaystyle\sum_{i=1}^{2N} \boldsymbol{X}[\boldsymbol{u}_m(t_i)] = \mathbf{0}, \\ G_{2n-1}(\boldsymbol{\alpha}) \equiv \dfrac{1}{\sqrt{2N}} \displaystyle\sum_{i=1}^{2N} \boldsymbol{X}[\boldsymbol{u}_m(t_i)] \sin nt_i + n\boldsymbol{\alpha}_{2n} = \mathbf{0}, \\ G_{2n}(\boldsymbol{\alpha}) \equiv \dfrac{1}{\sqrt{2N}} \displaystyle\sum_{i=1}^{2N} \boldsymbol{X}[\boldsymbol{u}_m(t_i)] \cos nt_i - n\boldsymbol{\alpha}_{2n-1} = \mathbf{0} \\ \qquad (n=1, 2,\dots, m), \\ G_{2m+1}(\boldsymbol{\alpha}) \equiv l(\boldsymbol{u}_m(t)) - \beta = 0, \end{cases}$$

where $t_i=(2i-1)\pi/2N$ and N is a positive integer greater than or equal to $m+1$. Now put

$$(6.2)\qquad \boldsymbol{G}(\boldsymbol{\alpha}) = \mathrm{col}\,[G_0(\boldsymbol{\alpha}), G_1(\boldsymbol{\alpha}), G_2(\boldsymbol{\alpha}),\dots, G_{2m}(\boldsymbol{\alpha}), G_{2m+1}(\boldsymbol{\alpha})],$$

then the determining equation (6.1) can be written briefly as

$$(6.3)\qquad \boldsymbol{G}(\boldsymbol{\alpha}) = \mathbf{0}.$$

If the function $\boldsymbol{X}(\boldsymbol{u})$ is nonlinear in $\boldsymbol{u}$, (6.3) is also a nonlinear equation in $\boldsymbol{\alpha}$. Hence, for numerical solution of (6.3) the Newton method will be efficient.

Starting from a certain approximation $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$, we compute the sequence $\{\boldsymbol{\alpha}_p\}$ successively by the iterative process

$$(6.4)\qquad \begin{cases} J(\boldsymbol{\alpha}_p)\boldsymbol{h}_p + \boldsymbol{G}(\boldsymbol{\alpha}_p) = \mathbf{0}, \\ \boldsymbol{\alpha}_{p+1} = \boldsymbol{\alpha}_p + \boldsymbol{h}_p \qquad (p=0, 1, 2,\dots), \end{cases}$$

where $J(\boldsymbol{\alpha})$ is the Jacobian matrix of $\boldsymbol{G}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$.

Note that in order to practice the iterative process (6.4) on a computer, it suffices to evaluate $\boldsymbol{G}(\boldsymbol{\alpha})$ and $J(\boldsymbol{\alpha})$ for a known $\boldsymbol{\alpha}$. For the details, see [5], [12].

In order to practice the Newton method, it is, however, necessary to find a starting value $\boldsymbol{\alpha}_0$. When a Galerkin approximation is known, in other words a solution of the determining equation is known for a system slightly different from the given system (1.1), it can be used as starting value for the given system. In special, this method can be used effectively for systems depending continuously on some parameters.

On the other hand, the following method seems to be also useful one of the methods which can be used for considerable wide class of differential systems. That is, we solve the determining equation of Galerkin approximations of very low order, say, of order 1 or 2. The determining equation under consideration can be solved sometimes analytically or graphically. When it is difficult to solve the determining equations under question analytically or graphically, the FORTRAN program developed by the author [3] can be used effectively. For the details, see [3].

§7. A posteriori error estimation of Galerkin approximations

When a Galerkin approximation has been obtained on a computer, it is important in practical applications to know whether a corresponding exact peridodic solution to the given differential equation exists or not, and further to know an error bound of the Galerkin approximation obtained if an exact periodic solutions exists.

In the present section a method of assuring the existence of an exact periodic solution and of obtaining error bounds of the approximate solution $\bar{\boldsymbol{u}}(t)$ and the approximate period $\bar{\omega}$ will be considered.

In (2.5) we take $A(t)$ and L such that

$$A(t)=\Xi[\bar{\boldsymbol{u}}(t)], \quad L=\boldsymbol{f}'(\bar{\boldsymbol{u}}(t))$$

respectively. Then we have

Theorem 2 ([5]). *Assume that the boundary value problem* (1.9) *possesses an approximate solution* $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ *in* S^1 *such that the matrix*

$$G\equiv\boldsymbol{f}'(\bar{\boldsymbol{u}})[\Phi[(t)]$$

is nonsingular, where $\Phi(t)$ *is the fundamental matrix of the following linear system*

$$\frac{dy}{dt}=\Xi[\bar{\boldsymbol{u}}(t)]y$$

satisfying the initial condition $\Phi(0)=E$ *(unit matrix).*

Let μ *and* r *be the positive numbers such that*

$$\mu=\max(\|H^1\|_n, \|H^2\|_n)\geq\|T^{-1}\|_n, \tag{7.1}$$

$$r\geq\left\|\frac{d\bar{\boldsymbol{u}}}{dt}-\boldsymbol{V}(\bar{\boldsymbol{u}})\right\|_n+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|. \tag{7.2}$$

If there exist a positive number δ *and a non-negative number* $\kappa<1$ *such that*

$$D'_\delta=\{\boldsymbol{u}\mid \|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_n\leq\delta,\ \boldsymbol{u}\in C[I]\times R^1\}\subset S, \tag{7.3}$$

$$\|\Xi(\boldsymbol{u})-\Xi(\bar{\boldsymbol{u}})\|_n+\|\boldsymbol{f}'(\boldsymbol{u})-\boldsymbol{f}'(\bar{\boldsymbol{u}})\|\leq\frac{\kappa}{\mu} \quad \text{on} \quad D'_\delta, \tag{7.4}$$

$$\frac{\mu r}{1-\kappa}\leq\delta, \tag{7.5}$$

then the boundary value problem (1.9) *has one and only one solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *in*

(7.6) $$D_\delta=\{\boldsymbol{u}\mid \|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_n\leq\delta,\quad \boldsymbol{u}\in C^1[I]\times R^1\}$$

and for this exact solution $\hat{\boldsymbol{u}}(t)$ *we have an error estimation as follows:*

(7.7) $$\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_n\leq\frac{\mu r}{1-\kappa}.$$

PROOF. Let us put

(7.8) $$\boldsymbol{F}(\boldsymbol{u})=\left[\frac{d\boldsymbol{u}}{dt}-\boldsymbol{V}(\boldsymbol{u}),\ \boldsymbol{f}(\boldsymbol{u})\right],$$

then $\boldsymbol{F}(\boldsymbol{u})$ maps $S^1\subset C^1[I]\times R^1$ into W and the boundary value problem (1.9) is rewritten to the equation

(7.9) $$\boldsymbol{F}(\boldsymbol{u})=\boldsymbol{0}.$$

The weak Fréchet derivative $\boldsymbol{F}'(\boldsymbol{u})$ can be written as follows:

(7.10) $$\boldsymbol{F}'(\boldsymbol{u})\boldsymbol{y}=\left[\frac{d\boldsymbol{y}}{dt}-\Xi(\boldsymbol{u})\boldsymbol{y},\ \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{y}\right],$$

where $\boldsymbol{y}$ is an arbitrary element belonging to $C^1[I]\times R^1$. Then, by (2.5), (7.4) and (7.10) we have

(7.11) $$|||\boldsymbol{F}'(\boldsymbol{u})-T|||\leq\frac{\kappa}{\mu}\quad on\quad D_\delta\subset D'_\delta.$$

For the approximate solution $\bar{\boldsymbol{u}}(t)\in S^1$ we have from (7.2) that

(7.12) $$|||\boldsymbol{F}(\bar{\boldsymbol{u}})|||=\left\|\frac{d\bar{\boldsymbol{u}}}{dt}-\boldsymbol{V}(\bar{\boldsymbol{u}})\right\|_n+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|\leq r.$$

From (7.3) and (7.6), we have $D_\delta\subset D'_\delta$ and $D'_\delta\subset S$. Hence we have

(7.13) $$D_\delta\subset S\cap\{C^1[I]\times R^1\}=S^1.$$

By Proposition 1, the operator T has a linear inverse operator T^{-1} satisfying (7.1). The facts tell us that the Newton interative process

(7.14) $$\begin{cases}\boldsymbol{u}_{p+1}=\boldsymbol{u}_p-T^{-1}\boldsymbol{F}(\boldsymbol{u}_p),\\ \boldsymbol{u}_0=\bar{\boldsymbol{u}}\quad(p=0,1,2,\ldots)\end{cases}$$

is well defined in D_δ.

In fact we shall prove by the induction that

(7.15) $$\|\boldsymbol{u}_{p+1}-\boldsymbol{u}_p\|_n\leq\kappa^p\|\boldsymbol{u}_1-\boldsymbol{u}_0\|_n,$$

(7.16) $$\boldsymbol{u}_{p+1}\in D_\delta$$

for $p=0, 1, 2, \ldots$.

For $p=0$, the inequality (7.15) is evident. For $\boldsymbol{u}_1$, we have

$$\boldsymbol{u}_1 \in C^1[I] \times R^1,$$

since $\boldsymbol{u}_1 = \boldsymbol{u}_0 - T^{-1}\boldsymbol{F}(\boldsymbol{u}_0)$. Moreover, from (2.7), (7.1) and (7.2) we have

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n = \|T^{-1}\boldsymbol{F}(\boldsymbol{u}_0)\|_n \leq \mu \cdot r \leq \delta(1-\kappa) \leq \delta. \tag{7.17}$$

Hence we have $\boldsymbol{u}_1 \in D_\delta$, which tells us that (7.16) is valid for $p=0$.

Now let us assume that the iterative process (7.14) yielded the sequence $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{p-1}, \boldsymbol{u}_p\}$ and we had (7.15) and (7.16) up to $p-1$. Then, since $\boldsymbol{u}_p \in D_\delta$, we really have $\boldsymbol{u}_{p+1}$ and from (7.14) we have

$$\boldsymbol{u}_{p+1} - \boldsymbol{u}_p = \boldsymbol{u}_p - \boldsymbol{u}_{p-1} - T^{-1}[\boldsymbol{F}(\boldsymbol{u}_p) - \boldsymbol{F}(\boldsymbol{u}_{p-1})] \; (p \geq 1). \tag{7.18}$$

Now, by our assumption we have $\boldsymbol{u}_p, \boldsymbol{u}_{p-1} \in D_\delta$. Therefore

$$\boldsymbol{u}_{p-1} + \theta(\boldsymbol{u}_p - \boldsymbol{u}_{p-1}) \in D_\delta \quad \textit{for all} \quad \theta \quad \textit{such that} \quad 0 \leq \theta \leq 1.$$

Then, by the mean value theorem we have

$$\boldsymbol{u}_{p+1} - \boldsymbol{u}_p = T^{-1}\int_0^1 \{T - \boldsymbol{F}'[\boldsymbol{u}_{p-1} + \theta(\boldsymbol{u}_p - \boldsymbol{u}_{p-1})]\} \, (\boldsymbol{u}_p - \boldsymbol{u}_{p-1}) d\theta.$$

Then, by (7.1) and (7.11), we have

$$\|\boldsymbol{u}_{p+1} - \boldsymbol{u}_p\|_n \leq \mu \frac{\kappa}{\mu} \|\boldsymbol{u}_p - \boldsymbol{u}_{p-1}\|_n = \kappa \|\boldsymbol{u}_p - \boldsymbol{u}_{p-1}\|_n. \tag{7.19}$$

Hence, by our assumption of induction, we have

$$\|\boldsymbol{u}_{p+1} - \boldsymbol{u}_p\|_n \leq \kappa \cdot \kappa^{p-1} \|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n = \kappa^p \|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n,$$

which proves (7.15) for p.

Since

$$\|\boldsymbol{u}_{p+1} - \boldsymbol{u}_0\|_n \leq \|\boldsymbol{u}_{p+1} - \boldsymbol{u}_p\|_n + \|\boldsymbol{u}_p - \boldsymbol{u}_{p-1}\|_n + \cdots + \|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n,$$

by (7.15) we have

$$\|\boldsymbol{u}_{p+1} - \boldsymbol{u}_0\|_n \leq (\kappa^p + \kappa^{p-1} + \cdots + \kappa + 1) \|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n \leq \frac{1}{1-\kappa} \|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_n.$$

Hence we have by (7.17) that

$$\|\boldsymbol{u}_{p+1} - \boldsymbol{u}_0\|_n \leq \delta. \tag{7.20}$$

This proves (7.16) for p.

Thus we see that the iterative process (7.14) really yields an infinite sequence $\{\boldsymbol{u}_p\}$ and for this sequence we have (7.15) and (7.16) for all p.

From (7.15) and (7.16), we readily see that the sequence $\{\boldsymbol{u}_p\}$ is a fundamental sequence in $D_\delta \subset C[I] \times R^1$ with respect to the norm $\|\cdots\|_n$.

By the completeness of the space $C[I] \times R^1$, there exists a vector function $\hat{\boldsymbol{u}}(t) \in C[I] \times R^1$ such that

$$\|\boldsymbol{u}_p - \hat{\boldsymbol{u}}\|_n \longrightarrow 0 \qquad \text{as} \quad p \longrightarrow \infty.$$

Since $\boldsymbol{u}_p \in D_\delta$, it is evident that

$$\|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_n \leq \delta$$

and

$$\hat{\boldsymbol{u}} \in D'_\delta \subset S.$$

Hence we have

$$\boldsymbol{V}(\hat{\boldsymbol{u}}) - A(t)\hat{\boldsymbol{u}} \in C[I]$$

and

$$L\hat{\boldsymbol{u}} - \boldsymbol{f}(\hat{\boldsymbol{u}}) \in R^{d+1}.$$

However, by (2.5), (7.8) and (7.14) we have

$$\begin{aligned} T\boldsymbol{u}_{p+1} &= T\boldsymbol{u}_p - \boldsymbol{F}(\boldsymbol{u}_p) \\ &= [\boldsymbol{V}(\boldsymbol{u}_p) - A(t)\boldsymbol{u}_p,\ L\boldsymbol{u}_p - \boldsymbol{f}(\boldsymbol{u}_p)] \end{aligned}$$

and hence we have

$$\boldsymbol{u}_{p+1} = T^{-1}[\boldsymbol{V}(\boldsymbol{u}_p) - A(t)\boldsymbol{u}_p,\ L\boldsymbol{u}_p - \boldsymbol{f}(\boldsymbol{u}_p)] \tag{7.21}$$

for $p=0, 1, 2, \ldots$. Letting $p \to \infty$ in (7.21), we have

$$\hat{\boldsymbol{u}} = T^{-1}[\boldsymbol{V}(\hat{\boldsymbol{u}}) - A(t)\hat{\boldsymbol{u}},\ L\hat{\boldsymbol{u}} - \boldsymbol{f}(\hat{\boldsymbol{u}})]. \tag{7.22}$$

Since T^{-1} is a linear operator mapping W into $C^1[I] \times R^1$, the relation (7.22) shows that $\hat{\boldsymbol{u}} \in C^1[I] \times R^1$. Hence, by Proposition 2, we see that $\boldsymbol{u} = \hat{\boldsymbol{u}}$ is a unique solution of (7.9) in D_δ and we have

$$\|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_n \leq \frac{\mu r}{1-\kappa}. \tag{7.23}$$

Q. E. D.

The error estimation (7.7) has been used in the papers [4], [5]. But, it will be natural to give the error estimation which consists of an error bound of periodic solution and the one of period, separately. For this purpose we shall introduce the product space $C[I] \times R^1$ a new norm defined by

$$\|\boldsymbol{u}(t)\|_\infty = \|\boldsymbol{x}(t)\|_n + |\omega|,$$

where $\boldsymbol{u}(t)=(\boldsymbol{x}(t), \omega)$. Put $\boldsymbol{y}=(\boldsymbol{h}_1(t), h_{d+1})$ and $A(t)=X_u(\boldsymbol{u}(t))$ in (2.5), then the linear operator T can be written as follows:

$$Ty=\left[\frac{d\boldsymbol{h}_1(t)}{dt}-A(t)\begin{pmatrix}\boldsymbol{h}_1(t)\\ h_{d+1}\end{pmatrix}, Ly\right] \tag{7.24}$$

for any $y(t)\in C^1[I]\times R^1$. Then the equality (2.6) defines two projective operators, such as P^1 with domain W and range $C^1[I]$ and P^2 with domain W and range R^1, by

$$P^1\boldsymbol{w}=\boldsymbol{h}_1(t),\quad P^2\boldsymbol{w}=h_{d+1},$$

where $\boldsymbol{w}=(\boldsymbol{g}, \boldsymbol{v})\in W$. Thus we have

$$T^{-1}\boldsymbol{w}=(P^1\boldsymbol{w}, P^2\boldsymbol{w}). \tag{7.25}$$

As can be seen, P^1 and P^2 are linear operators. We shall put

$$H^1\boldsymbol{g}=\begin{pmatrix}H_{11}\boldsymbol{g}\\ H_{12}\boldsymbol{g}\end{pmatrix},\quad H^2\boldsymbol{v}=\begin{pmatrix}H_{21}\boldsymbol{v}\\ H_{22}\boldsymbol{v}\end{pmatrix},$$

where $H_{11}\boldsymbol{g}$ and $H_{21}\boldsymbol{v}$ are d-dimensional vectors, $H_{12}\boldsymbol{g}$ and $H_{22}\boldsymbol{v}$ are scalars. Since

$${}^t\{T^{-1}(\boldsymbol{g}, \boldsymbol{v})\}=H^1\boldsymbol{g}+H^2\boldsymbol{v}=\begin{pmatrix}\boldsymbol{h}_1(t)\\ h_{d+1}\end{pmatrix}=\begin{pmatrix}P^1(\boldsymbol{g}, \boldsymbol{v})\\ P^2(\boldsymbol{g}, \boldsymbol{v})\end{pmatrix},$$

we have

$$\begin{aligned}\|P^1(\boldsymbol{g}, \boldsymbol{v})\|_n&=\|\boldsymbol{h}_1(t)\|_n=\|H_{11}\boldsymbol{g}+H_{21}\boldsymbol{v}\|_n\\ &\le\|H_{11}\|_n\cdot\|\boldsymbol{g}\|_n+\|H_{21}\|_n\cdot\|\boldsymbol{v}\|\\ &\le\max(\|H_{11}\|_n, \|H_{21}\|_n)(\|\boldsymbol{g}\|_n+\|\boldsymbol{v}\|).\end{aligned}$$

This inequality implies

$$\|P^1\|_n\le\max(\|H_{11}\|_n, \|H_{21}\|_n). \tag{7.26}$$

On the other hand, we have

$$\begin{aligned}|P^2(\boldsymbol{g}, \boldsymbol{v})|=|h_{d+1}|&=|H_{12}\boldsymbol{g}+H_{22}\boldsymbol{v}|\le|H_{12}|\cdot\|\boldsymbol{g}\|_n+|H_{22}|\cdot\|\boldsymbol{v}\|\\ &\le\max(|H_{12}|, |H_{22}|)(\|\boldsymbol{g}\|_n+\|\boldsymbol{v}\|).\end{aligned}$$

This inequality implies

$$|P^2|\le\max(|H_{12}|, |H_{22}|). \tag{7.27}$$

From (7.26) and (7.27), we have

$$\|T^{-1}(g, v)\|_\infty \leq (\|P^1\|_n + |P^2|)(\|g\|_n + \|v\|)$$

and

$$\|T^{-1}\|_\infty \leq \|P^1\|_n + |P^2|, \tag{7.28}$$

where $\|\cdots\|_\infty$ denotes the induced operator norm.

Now the weak Fréchet differential of $F(u)$ at $u=\bar{u}(t)=(\bar{x}(t), \bar{\omega})$ can be written from (1.18) as follows:

$$F'(\bar{u})y = \left[\frac{dh_1}{dt} - X_u(\bar{u})y,\ f'(\bar{u})y\right], \tag{7.29}$$

where $X_u(\bar{u}) = \left(\frac{\bar{\omega}}{2\pi} X_x(\bar{x})\ \frac{1}{2\pi} X(\bar{x})\right)$ is a $d\times(d+1)$ matrix, $X_x(\bar{x})$ denotes the Jacobian matrix at $x=\bar{x}$ and $f'(\bar{u})y=\mathrm{col}\,(h_1(0)-h_1(2\pi),\ l(y))$.

In (7.23) we take $A(t)$ and Ly such that $A(t)=X_u(\bar{x}(t))$ and $Ly=f'(\bar{u})\,y$, then we have $T=F'(\bar{u})$ and the following theorem.

Theorem 3. *Assume that the equation* (1.18) *possesses an approximate solution* $u=\bar{u}(t)$ *in* S^1 *such that* $\det G=\det f'(\bar{u})[\Phi(t)]\neq 0$, *where* $\Phi(t)$ *is the fundamental matrix of the linear homogenous differential system*

$$\frac{dy}{dt} = \Xi[\bar{u}(t)]\,y$$

satisfying the initial condition $\Phi(0)=E$ *(unit matrix). Let* μ_1, μ_2, *and* r *be the positive numbers such that*

$$\mu_1 = \max(\|H_{11}\|_n,\ \|H_{21}\|_n),\quad \mu_2=\max(|H_{12}|,\ |H_{22}|) \tag{7.30}$$

and

$$r \geq \|\|F(\bar{u})\|\| = \|\frac{d\bar{x}}{dt} - \frac{\bar{\omega}}{2\pi} X(\bar{x})\|_n + \|f(\bar{u})\|. \tag{7.31}$$

If there exist the positive numbers δ_1, δ_2, *and the non-negative number* $\kappa<1$ *such that*

$$D'_\delta = \{u(t) \mid \|x(t)-\bar{x}(t)\|_n \leq \delta_1,\ |\omega-\bar{\omega}|\leq\delta_2,\ u(t)\in C[I]\times R^1\}\subset S, \tag{7.32}$$

$$\|X_u(u)-X_u(\bar{u})\|_n + \|f'(u)-f'(\bar{u})\| \leq \frac{\kappa}{\mu_1+\mu_2} \quad \text{on } D'_\delta, \tag{7.33}$$

$$\frac{\mu_1 r}{1-\kappa} \leq \delta_1,\quad \frac{\mu_2 r}{1-\kappa}\leq\delta_2, \tag{7.34}$$

then the equation (1.18) *has one and only one* 2π*-periodic solution* $u=\hat{u}(t)$ *in the region*

$$D_\delta=\{\boldsymbol{u}(t)\mid \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_n\le\delta_1,\ |\omega-\bar{\omega}|\le\delta_2,\ \boldsymbol{u}(t)\in C^1[I]\times R^1\}$$

and for this solution $\hat{\boldsymbol{u}}(t)$ *we have an error estimation*

$$\|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(t)\|_n\le\frac{\mu_1 r}{1-\kappa},\quad |\hat{\omega}-\bar{\omega}|\le\frac{\mu_2 r}{1-\kappa}. \tag{7.35}$$

For the proof, see [12].

§8. Isolatedness of solution

In section 2 we called the solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in S^1 of (1.9) *isolated* if

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}})\,[\Phi(t)]\ne 0, \tag{8.1}$$

where $\boldsymbol{f}'(\hat{\boldsymbol{u}})$ denotes the weak Fréchet derivative of $\boldsymbol{f}(\boldsymbol{u})$. This definition comes from the following proposition.

Proposition 3. *If the condition* (8.1) *is satisfied with the solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$, *then there is no other solution of* (1.9) *in a sufficiently small neighborhood of* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$.

PROOF. By Proposition 1 the operator $T(\hat{\boldsymbol{u}})=\boldsymbol{F}'(\hat{\boldsymbol{u}})$ has a linear inverse operator T^{-1}. Let ε be an arbitrary positive number such that

$$\varepsilon<\frac{1}{\|T^{-1}\|_\infty}. \tag{8.2}$$

For this ε, by the openness of S^1 and the continuity of $T(\boldsymbol{u})$, there exist the positive numbers δ_1 and δ_2 such that

$$\hat{D}_\delta=\{\boldsymbol{u}(t)\mid \|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)\|_n<\delta_1,\ |\omega-\hat{\omega}|<\delta_2,\ \boldsymbol{u}(t)\in C^1[I]\times R^1\}\subset S^1 \tag{8.3}$$

and

$$|||T(\boldsymbol{u})-T(\hat{\boldsymbol{u}})|||<\varepsilon \qquad on \quad \hat{D}_\delta. \tag{8.4}$$

Now suppose (1.9) has another solution $\boldsymbol{u}=\hat{\boldsymbol{u}}'(t)$ in $\hat{D}_\delta$. Then

$$\boldsymbol{F}(\hat{\boldsymbol{u}}')=\boldsymbol{F}(\hat{\boldsymbol{u}})=\boldsymbol{0}.$$

Hence, by mean value theorem, we have

$$\int_0^1 T[\hat{\boldsymbol{u}}+\theta(\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}})]\,(\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}})d\theta=0.$$

Hence we have

$$\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}=T^{-1}(\hat{\boldsymbol{u}})\int_0^1\{T(\hat{\boldsymbol{u}})-T[\hat{\boldsymbol{u}}+\theta(\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}})]\}\,(\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}})d\theta.$$

By (8.4) we then have

$$\|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}\|_\infty \leq |||T^{-1}(\hat{\boldsymbol{u}})|||\cdot\varepsilon\cdot\|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}\|_\infty.$$

Since $\varepsilon\cdot|||T^{-1}(\hat{\boldsymbol{u}})|||<1$ by (8.2), we get $\|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}\|_\infty=0$, that is, $\hat{\boldsymbol{u}}'(t)\equiv\hat{\boldsymbol{u}}(t)$. Q. E. D.

Theorem 4. *The solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *obtained in* Theorem 3 *is an isolated solution.*

PROOF. If $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ is not an isolated solution, then

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\Phi(t)]=0. \tag{8.5}$$

By (8.5) we then have a non-zero constant vector $\hat{\boldsymbol{c}}\in R^{d+1}$ such that

$$\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Phi(t)]\hat{\boldsymbol{c}}=\boldsymbol{0}.$$

Put $\hat{\boldsymbol{y}}=\Phi(t)\hat{\boldsymbol{c}}$, then evidently we have

$$\frac{d\hat{\boldsymbol{y}}}{dt}-X_u(\hat{\boldsymbol{u}})\hat{\boldsymbol{y}}=\left[\frac{d\Phi(t)}{dt}-X_u(\hat{\boldsymbol{u}})\Phi(t)\right]\hat{\boldsymbol{c}}=\boldsymbol{0}$$

and

$$\boldsymbol{f}'(\hat{\boldsymbol{u}})\hat{\boldsymbol{y}}=\boldsymbol{0}.$$

These facts imply from (7.10) that

$$\boldsymbol{F}'(\hat{\boldsymbol{u}})\hat{\boldsymbol{y}}=\boldsymbol{0}.$$

Then we have

$$\hat{\boldsymbol{y}}=T^{-1}(T-\boldsymbol{F}'(\hat{\boldsymbol{u}}))\hat{\boldsymbol{y}}.$$

On the other hand, we have from the condition (7.33) that

$$|||T-\boldsymbol{F}'(\hat{\boldsymbol{u}})|||\leq\frac{\kappa}{\mu_1+\mu_2} \qquad on \quad D'_\delta.$$

Hence we have

$$\|\hat{\boldsymbol{y}}\|_\infty\leq(\mu_1+\mu_2)\frac{\kappa}{\mu_1+\mu_2}\|\hat{\boldsymbol{y}}\|_\infty=\kappa\|\hat{\boldsymbol{y}}\|_\infty,$$

that is,

$$(1-\kappa)\|\hat{\boldsymbol{y}}\|_\infty\leq 0.$$

Since $\kappa<1$, we have

$$\|\hat{\boldsymbol{y}}\|_\infty=0,$$

that is, $\hat{\boldsymbol{y}}(t)\equiv\boldsymbol{0}$, which implies $\hat{\boldsymbol{c}}=\boldsymbol{0}$. This is contradiction. Q. E. D.

For autonomous systems, the isolatedness of solution means that characteristic multipliers of the first variation equation are all different from one except one characteristic multiplier. The reason why the terminology "*isolated*" is employed is that there is no other periodic solution near the periodic solution in question, if the above condition is fulfilled.

On the other hand, as can be seen from Theorem 4, the isolatedness of the solution $\hat{\boldsymbol{u}}(t)$ of (1.9) means that the condition (8.1) is valid.

The additional linear functional $l(\boldsymbol{u})$ given in (1.6) is related to the isolatedness of solution. The fact can be seen from the following theorem given by M. Urabe [10] without proof.

Theorem 5. *The isolatedness of a periodic solution* $\hat{\boldsymbol{x}}(t)$ *of* (1.3) *is equivalent to the one of a corresponding solution* $\hat{\boldsymbol{u}}(t)$ *of* (1.9) *if and only if*

$$l[\boldsymbol{X}(\hat{\boldsymbol{x}})]\neq 0, \tag{8.6}$$

where $\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t),\ \hat{\omega}(t))$.

PROOF. Put

$$\Theta(t)=\begin{pmatrix} \Phi(t) & \boldsymbol{p}(t) \\ \boldsymbol{q}^*(t) & s(t) \end{pmatrix}, \tag{8.7}$$

where $\Phi(t)$ is a $d\times d$ matrix, $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ are d-dimensional column vectors and $s(t)$ is a scalar function. The first variation equation of (1.3) clearly reads as follows:

$$\frac{d\boldsymbol{y}}{dt}=\begin{pmatrix} \frac{\omega}{2\pi}X_x(\hat{\boldsymbol{x}}) & \frac{1}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}) \\ 0 & 0 \end{pmatrix}\boldsymbol{y}.$$

Replacing $\boldsymbol{y}$ by $\Theta(t)$, we have the differential equations in Φ, $\boldsymbol{p}$, $\boldsymbol{q}$, and s. Making use of the initial conditions

$$\Phi(0)=E \text{ (unit matrix)},\ \boldsymbol{p}(0)=\boldsymbol{q}(0)=\boldsymbol{0},\ s(0)=1, \tag{8.8}$$

we see that

(i) $\Phi(t)$ is the fundamental matrix of the equation

$$\frac{d\boldsymbol{z}}{dt}=\frac{\omega}{2\pi}X_x(\hat{\boldsymbol{x}})\boldsymbol{z} \tag{8.9}$$

satisfying the initial condition $\Phi(0)=E$ and then we have

$$\Phi(t)=\Phi_0\left(\frac{\omega}{2\pi}t\right),$$

where $\Phi_0(t)$ is a fundamental matrix of the equation

$$\frac{d\boldsymbol{z}}{dt}=X_x(\hat{\boldsymbol{x}})\boldsymbol{z}$$

satisfying $\Phi_0(0)=E$.

$\boldsymbol{p}(t)$, $\boldsymbol{q}(t)$, and $s(t)$ are obtained as follows:

(ii) $$\boldsymbol{p}(t)=\frac{1}{2\pi}\Phi(t)\int_0^t \Phi^{-1}(\xi)\boldsymbol{X}[\hat{\boldsymbol{x}}(\xi)]d\xi,$$

(iii) $\boldsymbol{q}(t)\equiv\boldsymbol{0}$, and $s(t)\equiv 1$.

Now let us note that

$$\frac{d}{dt}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=X_x[\hat{\boldsymbol{x}}(t)]\cdot\frac{\omega}{2\pi}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)],$$

that is, $\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]$ is a 2π-periodic solution to (8.9). Hence we have

(8.10) $$\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=\Phi(t)\boldsymbol{c}$$

for some constant vector $\boldsymbol{c}\neq\boldsymbol{0}$. Equation (8.10) implies

$$\Phi^{-1}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=\boldsymbol{c}.$$

Hence, from (8.10) we have

$$\boldsymbol{p}(t)=\frac{t}{2\pi}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)].$$

Hence we have

$$\Theta(t)=\begin{pmatrix}\Phi(t) & \frac{t}{2\pi}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]\\ \boldsymbol{0} & 1\end{pmatrix}.$$

From (8.10) it is evident that

$$\Phi(2\pi)\boldsymbol{c}=\boldsymbol{c}.$$

By (1.8) we then have

(8.11) $$\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta(t)]=\begin{pmatrix}E-\Phi(2\pi) & -\boldsymbol{c}\\ l[\Phi] & \frac{1}{2\pi}l[t\boldsymbol{X}(\hat{\boldsymbol{x}})]\end{pmatrix}.$$

Let us set

$$\boldsymbol{c}_1=\frac{\boldsymbol{c}}{\|\boldsymbol{c}\|}$$

and Q be an orthogonal matrix whose first column vector is $\boldsymbol{c}_1$. Consider

$$K=\begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},$$

then K is also an orthogonal matrix. Write Q as $Q=[\boldsymbol{c}_1, Q_1]$, where Q_1 is a $d\times(d-1)$ matrix whose column vector are unit vectors and moreover they are orthogonal each other. By (8.11) we then have

$$K^*\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta(t)]K=\begin{pmatrix} Q^*[E-\Phi(2\pi)]Q & -Q^*\boldsymbol{c} \\ l[\Phi]Q & \frac{1}{2\pi}l[t\boldsymbol{X}(\hat{\boldsymbol{x}})] \end{pmatrix}.$$

However

$$(8.12)\qquad Q^*\Phi(2\pi)Q=\begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}\Phi(2\pi)(\boldsymbol{c}_1, Q_1)=\begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}[\Phi(2\pi)\boldsymbol{c}_1, \Phi(2\pi)Q_1]$$

$$=\begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}[\boldsymbol{c}_1, \Phi(2\pi)Q_1]$$

$$=\begin{pmatrix} 1 & \boldsymbol{c}_1^*\Phi(2\pi)Q_1 \\ \mathbf{0} & Q_1^*\Phi(2\pi)Q_1 \end{pmatrix}.$$

Hence the eigenvalues of $\Phi(2\pi)$ are 1 and those of $Q_1^*\Phi(2\pi)Q_1$.

$$Q^*[E-\Phi(2\pi)]Q=\begin{pmatrix} 0 & -\boldsymbol{c}_1^*\Phi(2\pi)Q_1 \\ \mathbf{0} & E-Q_1^*\Phi(2\pi)Q_1 \end{pmatrix}$$

and

$$Q^*\boldsymbol{c}=\begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}\boldsymbol{c}_1\|\boldsymbol{c}\|=\begin{pmatrix} \|\boldsymbol{c}\| \\ \mathbf{0} \end{pmatrix}.$$

By the linearity of $l(\boldsymbol{x})$ we have

$$\begin{aligned} l[\Phi]Q &= l[\Phi](\boldsymbol{c}_1, Q_1) \\ &= (l[\Phi\boldsymbol{c}_1], l[\Phi\boldsymbol{Q}_1]) \\ &= \left(\frac{1}{\|\boldsymbol{c}\|}l[\boldsymbol{X}(\hat{\boldsymbol{x}})], l[\Phi Q_1]\right). \end{aligned}$$

Hence we have

$$K^*\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta(t)]K=\begin{pmatrix} 0 & -\boldsymbol{c}_1^*\Phi(2\pi)Q_1 & -\|\boldsymbol{c}\| \\ \boldsymbol{0} & E-Q_1^*\Phi(2\pi)Q_1 & \boldsymbol{0} \\ \frac{1}{\|\boldsymbol{c}\|}l[\boldsymbol{X}(\hat{\boldsymbol{x}})] & l[\Phi Q_1] & \frac{1}{2\pi}l[t\boldsymbol{X}(\hat{\boldsymbol{x}})] \end{pmatrix}.$$

This tells us that

$$\begin{aligned}\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta] &= \det[K^*\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta]K] \\ &= l[\boldsymbol{X}(\hat{\boldsymbol{x}})]\cdot\det[E-Q_1^*\Phi(2\pi)Q_1].\end{aligned}$$

As can be seen from (8.12), the isolatedness of a periodic solution $\hat{\boldsymbol{x}}(t)$ to (1.3) is equivalent to $\det[E-Q_1^*\Phi(2\pi)Q_1]\neq 0$.

On the other hand, the isolatedness of solution to the boundary value problem (1.9) is $\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta]\neq 0$. Hence the both of conditions are equivalent if and only if $l[\boldsymbol{X}(\hat{\boldsymbol{x}})]\neq 0$. This completes the proof. Q. E. D.

In his paper [6], A. Stokes has established an existence and error estimation theorem for non-critical approximate solution to the equation

$$\omega\frac{d\boldsymbol{x}}{dt}=\boldsymbol{X}(\boldsymbol{x})$$

which is a consequence of (1.1). His theorem is also very important. But he failed to obtain the Galerkin approximation to van der Pol equation with damping coefficient $\varepsilon=1.0$. Moreover, in the same example, he gave an approximate period $T=2\pi/\bar{\omega}=2\pi/0.9620=6.66368152\cdots$ with error bound 5×10^{-13}. But this error bound is not good but its exact one may be 4×10^{-4}.

As for the numerical results of van der Pol equation, see also the papers [2], [4], [5], [7], and [12].

Department of Mathematics
Faculty of Engineering
Tokushima University

References

[1] Cesari, L., Functional anlaysis and periodic solutions of nonlinear differential equations, "*Contribution to Differential Equations*", **1** (1963), 149–187.

[2] Krogdahl, W. S., Numerical solutions of van der Pol equations, "*Z. Angew. Math. Phys.*", vol. **11** (1960), 59–63.

[3] Shinohara, Y., A geometric method of numerical solution of nonlinear equations and its application to nonlinear oscillations, "*Publ. RIMS Kyoto Univ.*", vol. **8** (1972), 13–42.

[4] Shinohara, Y., Numerical anaysis of periodic solutions and their periods to autonomous differential systems, "*J. Math. Tokushima Univ.*", **11** (1977), 11–32.

[5] Shinohara, Y. and Yamamoto, N., Galerkin approximation of periodic solution and its period to van der Pol equation, "*J. Math. Tokushima Univ.*", **12** (1978), 19–42.

[6] Stokes, A., On the approximation of nonlinear oscillations, "*J. Differential Equations*", **12** (1972), 535–558.

[7] Strasberg, M., Recherche de solutions périodique d'équations différentielles non linéaires par des méthodes de discratisation, "*Équations différentielles et fonctionelles non linéaires*", ed. Janssens, P., Mawhin, J. et Rouche N., Hermann, 1973.

[8] Urabe, M., Galerkin's procedure for nonlinear periodic systems, "*Arch. Rational Mech. Anal.*", **20** (1965), 120–152.

[9] Urabe, M., The Newton method and its application to boundary value problems with nonlinear boundary conditions, "*Pro. U. S.–Japan Seminar on Differential and Functional equations*", Benjamin, New York, (1967), 383–410.

[10] Urabe, M., A remark to the computation of periodic solution to autonomous differential systems, "*Qualitative study of nonlinear oscillations*", (1974), 27–29.

[11] Urabe, M., Yanagiwara, H. and Shinohara, Y., Periodic solutions to van der Pol's equation with damping coefficient $\lambda=2-10$, "*J. Sci. Hiroshima Univ.*", Ser. A, **23** (1960), 325–366.

[12] Yamamoto, N., An error analysis of Galerkin approximations of periodic solution and its period to autonomous differential system, "*J. Math. Tokushima Univ.*", **13** (1979), 53–77.