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Harmonic Sections of Tangent Bundles

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Let M be an m dimensional smooth Riemannian manifold with metric g. The tangent bundle $T(M)$ over M is endowed with the Riemannian metric g^D, the diagonal lift of g [3], [5]. Let X be a vector field on M. Then it is regarded as a mapping ϕ_X of M to $T(M)$. The purpose of this paper is to study under what conditions the mapping ϕ_X of Riemannian manifolds is harmonic.

§1 is devoted to describe some basic facts on geometry of tangent bundles. We will see in §2 that the natural projection, π: $T(M)\to M$ is a totally geodesic submersion. In the last section, it is proved that when M is compact and orientable, ϕ_X: $M\to T(M)$ is harmonic iff the first covariant derivative of X vanishes.

§1. Diagonal lifts of Riemannian metrics to tangent bundles

We will review differential geometry of tangent bundles briefly. For details, compare [5].

Let $\{U, x^i\}$ be a coordinate neighborhood, where (x^i) is a system of local coordinate defined in the open set U. Then we can introduce a system of local coordinates (x^i, y^j) in the open set $\pi^{-1}(U)$ of $T(M)$ in such a way that for each $p\in U$, $(x^i(p), y^j)\mapsto \sum_{j=1}^{m} y^j\left(\frac{\partial}{\partial x^j}\right)_p \in T(M)$, where π: $T(M)\to M$ is the natural projection. (x^i, y^j) are called the induced coordinates in $\pi^{-1}(U)$.

The Riemannian metric of M is given locally by

$$ds_M^2=\sum_{i=1}^{m}(\theta^i)^2,$$

where θ^i are local 1-forms such that

$$\theta^i=\sum_{j=1}^{m}\xi_j^i dx^j.$$

(In the paper, the indices $i, j, k,\dots$ run over the range $\{1,\dots, m\}$ and the indices A, B, C,... the range $\{1,\dots, m,\dots, 2m\}$. We also use the notation $i^*=m+i$.) Let ω^i, ω^{i^*} be vertical lifts and horizontal lifts of the local 1-forms θ^i, i.e.,

$$(1)\qquad \begin{cases} \omega^i = (\theta^i)^V = \pi^*\theta^i = \sum_{j=1}^m \xi^i_j \cdot \pi dx^j, \\ \omega^{i*} = (\theta^i)^H = \sum_{j=1}^m \xi^i_j \cdot \pi(dy^j + \sum_{k,l=1}^m \Gamma^j_{kl} y^k dx^l), \end{cases}$$

where Γ^j_{kl} are local components of the Riemannian connection in M. Then the diagonal lift g^D of g is written locally as

$$(2)\qquad ds^2_{T(M)} = \sum_{A=1}^{2m} (\omega^A)^2 = \sum_{i=1}^m (\omega^i)^2 + \sum_{i=1}^m (\omega^{i*})^2.$$

Let $X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}$ be a vector field on M. The vertical lift X^V and the horizontal lift X^H of X are written locally as

$$X^V = \sum_{i=1}^m X^i \frac{\partial}{\partial y^i},$$

$$X^H = \sum_{j=1}^m X^j \left(\frac{\partial}{\partial x^k} - \sum_{k,l=1}^m \Gamma^k_{jl} y^l \frac{\partial}{\partial y^k} \right).$$

The structure equations in M are

$$(3)\qquad \begin{cases} d\theta^i = \sum_{j=1}^m \theta^j \wedge \theta^i_j, \\ d\theta^i_j = \sum_{k=1}^m \theta^k_j \wedge \theta^i_k - \frac{1}{2} \sum_{k,l=1}^m R^i{}_{jkl} \theta^k \wedge \theta^l, \end{cases}$$

where θ^i_j are the Riemannian connection forms and R^i_{jkl} are the coefficients of the Riemannian curvature tensor. Let ω^A_B be the Riemannian connection forms in $T(M)$. Then,

$$(4)\qquad d\omega^A = \sum_{B=1}^{2m} \omega^B \wedge \omega^A_B.$$

From the basic properties of vertical lifts [5], it follows

$$d\omega^i = d(\theta^i)^V = (d\theta^i)^V = \sum_{j=1}^m (\theta^j)^V \wedge (\theta^i_j)^V = \sum_{j=1}^m \omega^j \wedge \pi^*\theta^i_j.$$

On the other hand, a direct calculation shows

$$d\omega^{i*} = \sum_{j=1}^m \omega^{j*} \wedge \pi^*\theta^i_j + \frac{1}{2} \sum_{j,k=1}^m R^i_{ljk} \xi_h y^h \omega^{j*} \wedge \omega^{k*}.$$

Comparing with (4), we get

Proposition 1. *Let* $Y^i = \sum_{j=1}^m \xi^i_j y^j$.

$$
(5)\qquad
\begin{cases}
\omega^i_j = \pi^*\theta^i_j - \dfrac{1}{2}\displaystyle\sum_{l,k=1}^{m} R^i_{jkl} Y^l \omega^{k^*}, \\[2ex]
\omega^i_{j^*} = -\,\omega^{j^*}_i = -\dfrac{1}{2}\displaystyle\sum_{l,k=1}^{m} R^i_{kjl} Y^l \omega^k, \\[2ex]
\omega^{i^*}_{j^*} = \pi^*\theta^i_j.
\end{cases}
$$

§ 2. Riemannianian submersion

Let N be an n-dimensional Riemannian manifold with metric ds^2_N. We assume $n > m$. Let $f: N \to M$ be a smooth mapping. If for every point p of N, we can choose local 1-forms $\omega^1, \dots, \omega^n$ in a neighborhood of p in N and $\theta^1, \dots, \theta^m$ in a neighborhood of $f(p)$ in M such that $ds^2_N = \sum_{a=1}^{n} (\omega^a)^2$, $ds^2_M = \sum_{i=1}^{m} (\theta^i)^2$ and

$$
(6)\qquad f^*\theta^i = \omega^i, \quad i = 1, \dots, m,
$$

$f: N \to M$ is called a Riemannian submersion. (In this section, the indices a, b, c run from 1 to n and α, β from $m+1$ to n.) Let ω^a_b be the connection forms in N, i.e.,

$$
d\omega^a = \sum_{b=1}^{n} \omega^b \wedge \omega^a_b.
$$

Then we can put

$$
(7)\qquad
\begin{aligned}
f^*\theta^i_j - \omega^i_j &= \sum_{\alpha=m+1}^{n} L^i_{j\alpha} \omega^\alpha, \\
\omega^i_\alpha &= \sum_{\beta=m+1}^{n} L^i_{\alpha\beta} \omega^\beta.
\end{aligned}
$$

$L^i_{j\alpha}$, $L^i_{\alpha\beta}$ are called the structure tensors of the Riemannian submersion f. If $\sum_{\alpha=m+1}^{n} L^i_{\alpha\alpha} = 0$ (resp. $L^i_{\alpha\beta} = 0$), f is said to be *minimal* (resp. *totally geodesic*) [2].

Now we will return to the natural projection $\pi: T(M) \to M$. Since we have $\pi^*\theta^i = \omega^i$, it is a Riemannian submersion. Morevoer, Proposition 1 implies

Proposition 2. *The natural projection $\pi: T(M) \to M$ is a totally geodesic Riemannian submersion with structure tensors*

$$
L^i_{jk^*} = \frac{1}{2}\sum_{l=1}^{m} R^i_{jkl} Y^l, \quad L^i_{j^*k^*} = 0.
$$

§ 3. Sections of tangent bundles

Let $\phi_X: M \to T(M)$ be a section of the tangent bundle. We can put locally

$X=\sum_{i=1}^{n} X^i e_i$ with respect to the dual base $\{e_i\}$ of $\{\theta^i\}$. Define F_i^A by

$$\phi_X^*(\omega^A)=\sum_{i=1}^{n} F_i^A\theta^i. \tag{8}$$

Then it holds

$$\phi_X^*(\omega^i)=\phi_X^*\pi^*(\theta^i)=\theta^i.$$

By a calculation, we get

$$\phi_X^*(\omega^{i^*})=\sum_{k=1}^{n} X_k^i\theta^k,$$

where X_k^i are components of the first covariant differential of X given by

$$\sum_{k=1}^{n} X_k^i\theta^k=dX^i+\sum_{j=1}^{n} X^j\theta_j^i.$$

Thus it is evident

$$F_j^i=\delta_j^i,\ F_j^{i^*}=X_j^i. \tag{9}$$

The fundamental tensor F_{ij}^A of the mapping ϕ_X is defined to be

$$\sum_{j=1}^{m} F_{ij}^A\theta^j=dF_i^A+\sum_{B=1}^{2m} F_i^B\omega_B^A-\sum_{j=1}^{m} F_j^A\theta_i^j.$$

If $\sum_{i=1}^{m} F_{ii}^A=0$, ϕ_X is called a harmonic mapping [1]. Using (5) and (9) we obtain

Proposition 3. *The components F_{ij}^A of the fundamental tensor of the mapping ϕ_X: $M\to T(M)$ are given by*

$$\begin{cases} F_{ij}^k=\dfrac{1}{2}\sum_{l,h}(R_{ilh}^k X_j^h+R_{jlh}^k X_i^h)X^l, \\ F_{ij}^{k^*}=X_{ij}^k+\dfrac{1}{2}\sum_{l=1}^{m} R_{lij}^k X^l, \end{cases} \tag{10}$$

where X_{ij}^k are the components of the second covariant differential of the vector field X.

Proposition 4. *ϕ_X: $M\to T(M)$ is a harmonic mapping iff*

$$\sum_{i=1}^{m} X_{ii}^k=0,\qquad \sum_{j,i=1}^{m} R_{ilj}^k X_i^j=0.$$

If M is compact and orientable, we have the following integral formula [4, p. 39]

$$\int_M\{\sum_{i,k=1}^{m} X_{ii}^k X^k+\sum_{i,j=1}^{m}(X_j^i)^2\}dV=0,$$

where dV is the Riemannian volume element. Hence, $\sum_{i=1}^{m} X^k_{ii}=0$ $(k=1,\dots, m)$ imply $X^i_j=0 (i, j=1,\dots, m)$. Thus we get

Proposition 5. *Assume that M is compact and orientable. $\phi_X: M \to T(M)$ is harmonic iff X has the vanishing covariant derivative.*

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References

[1] S. I. Goldberg and T. Ishihara, Harmonic quasiconformal mappings of Riemannian manifolds, Amer. J. Math., **98** (1976), 225–240.
[2] S. I. Goldberg and T. Ishihara, Riemannian submersion commuting with the Laplacian, J. Differential Geometry, **13** (1978), 149–154.
[3] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. Jour., **10** (1958), 238–254.
[4] K. Yano, Integral formulas in Riemannian Geometry, Marcel Dekker, Inc. New York, 1970.
[5] K. Yano and S. Ishihara, Tangent and cotangent bundles, Marcel Dekker, Inc. New York, 1973.