J. Math. Tokushima Univ.
Vol. 11 (1977), 11–32

Numerical Analysis of Periodic Solutions and their Periods to Autonomous Differential Systems

By

Yoshitane SHINOHARA

(Received May 15, 1977)

§1. Introduction

Consider a n-dimensional autonomous differential system

$$\frac{d\boldsymbol{x}}{d\tau}=\boldsymbol{X}(\boldsymbol{x}), \tag{1.1}$$

where $\boldsymbol{X}(\boldsymbol{x})\in C^1_{\boldsymbol{x}}[D]$, D being a domain in the $\boldsymbol{x}$-space. Let $\boldsymbol{x}(\tau)$ be a desired ω-periodic solution of the autonomous system (1.1).

For the numerical computation of the periodic solution $\boldsymbol{x}(\tau)$, we transform τ to t by

$$\tau=\frac{\omega}{2}t, \tag{1.2}$$

then equation (1.1) is rewritten in the following form:

$$\frac{d\boldsymbol{x}}{dt}=\frac{\omega}{2}\boldsymbol{X}(\boldsymbol{x}). \tag{1.3}$$

The problem then is reduced to the one of finding a 2-periodic solution of (1.3), but in our case ω is also unknown. Hence, we consider the following differential system:

$$\begin{cases}\dfrac{d\boldsymbol{x}}{dt}=\dfrac{\omega}{2}\boldsymbol{X}(\boldsymbol{x}),\\[2mm] \dfrac{d\omega}{dt}=0,\end{cases} \tag{1.4}$$

where $\boldsymbol{x}$ and ω are unknown functions. The periodic boundary condition for (1.4) is then as follows:

$$\boldsymbol{x}(-1)=\boldsymbol{x}(1). \tag{1.5}$$

As is well known, when $\boldsymbol{x}(t)$ is a solution of the autonomous system (1.3), $\boldsymbol{x}(t+\alpha)$

is also a solution for an arbitrary constant α. The fact tells us that no 2-periodic solution of (1.3) is uniquely determined by the boundary condition (1.5). Hence, we add one more condition, say,

$$l(\boldsymbol{x})=\beta, \tag{1.6}$$

where $l(\boldsymbol{x})$ is a linear functional and β is a constant number.

We shall write the set of boundary conditions (1.5) and (1.6) in the following form:

$$\boldsymbol{f}(\boldsymbol{u})=\boldsymbol{0}, \tag{1.7}$$

where

$$\boldsymbol{u}=\begin{pmatrix}\boldsymbol{x}(t)\\ \omega(t)\end{pmatrix}, \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix}\boldsymbol{x}(-1)-\boldsymbol{x}(1)\\ l(\boldsymbol{x})-\beta\end{pmatrix}. \tag{1.8}$$

Then the boundary value problem (1.4)–(1.6) can be rewritten as follows:

$$\begin{cases}\dfrac{d\boldsymbol{u}}{dt}=\boldsymbol{V}(\boldsymbol{u}), \\ \boldsymbol{f}(\boldsymbol{u})=\boldsymbol{0}, \end{cases} \tag{1.9}$$

where $\boldsymbol{V}(\boldsymbol{u})=\mathrm{col}\left[\frac{\omega}{2}\boldsymbol{X}(\boldsymbol{x}), 0\right]$.

In the present paper, we firstly consider a more general boundary value problem

$$\begin{cases}\dfrac{d\boldsymbol{u}}{dt}=\boldsymbol{V}(\boldsymbol{u}), \\ \boldsymbol{f}(\boldsymbol{u})=\boldsymbol{0}, \end{cases} \tag{1.10}$$

where $\boldsymbol{u}$ and $\boldsymbol{V}(\boldsymbol{u})$ are real $(n+1)$-dimensional vectors and $\boldsymbol{f}(\boldsymbol{u})$ is a operator mapping some set of $C[I]$ into R^{n+1}. Here $C[I]$ is the space consisting of $(n+1)$-dimensional vectors whose components are continuous functions defined on the interval $I=[-1, 1]$.

In the next section, we shall establish an existence theorem of the boundary value problem (1.10) and give a method of calculating an error bound on the approximate solution obtained.

A criterion how to choose the additional linear functional $l(\boldsymbol{x})$ will also be given.

Finally, in the last section, we shall apply our results to compute the periodic solution of van der Pol equation in Chebyshev-series.

§2. Basic Theorems

Let D be a domain in the $\boldsymbol{u}$-space. Consider a product space $\Omega = I \times D$ and put

$$S = \{\boldsymbol{u}(t) | (t, \boldsymbol{u}(t)) \in \Omega \quad \text{for all } t \in I,\ \boldsymbol{u}(t) \in M \equiv C^1[I]\},$$

$$S' = \{\boldsymbol{u}(t) | (t, \boldsymbol{u}(t)) \in \Omega \quad \text{for all } t \in I,\ \boldsymbol{u}(t) \in C[I]\}.$$

In (1.10), we assume that $\boldsymbol{V}(\boldsymbol{u})$ is defined and continuously Fréchet differentiable on S'. By $\boldsymbol{V}_u(\hat{\boldsymbol{u}})$ and $\boldsymbol{f}'(\hat{\boldsymbol{u}})$ we denote the Jacobian matrix of $\boldsymbol{V}(\boldsymbol{u})$ and the Fréchet derivative of $\boldsymbol{f}(\boldsymbol{u})$ at $\hat{\boldsymbol{u}}$ respectively.

We shall denote the Euclidean norm by $\|\cdots\|$, and for any $\boldsymbol{u}(t) \in C[I]$ we define its norm $\|\boldsymbol{u}\|_c$ by

$$\|\boldsymbol{u}\|_c = \sup_{t \in I} \|\boldsymbol{u}(t)\|.$$

Consider also a product space $N \equiv C[I] \times R^{n+1}$, and for any $\boldsymbol{n} = [\boldsymbol{u}(t), \boldsymbol{v}] \in N$ we define its norm $|||\boldsymbol{n}|||$ by

$$|||\boldsymbol{n}||| = \|\boldsymbol{u}\|_c + \|\boldsymbol{v}\|. \tag{2.1}$$

Then the product space N is evidently a Banach space with respect to the norm $|||\cdots|||$.

Now we consider an additive operator T mapping M into N of the following form:

$$T\boldsymbol{h} = \left[\frac{d\boldsymbol{h}}{dt} - A(t)\boldsymbol{h},\ L\boldsymbol{h}\right], \tag{2.2}$$

where $A(t)$ is an $(n+1) \times (n+1)$ matrix continuous on I and L is a linear operator mapping $C[I]$ into R^{n+1}. By $\Psi(t)$, let us denote an arbitrary fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt} = A(t)\boldsymbol{z},$$

and by $L[\Psi(t)]$ we denote the matrix whose column vectors are $L[\boldsymbol{\psi}_i(t)]$ $(i = 1, 2, \ldots, n+1)$, where $\boldsymbol{\psi}_i(t)$ $(i = 1, 2, \ldots, n+1)$ are column vectors of the matrix $\Psi(t)$.

Let $\boldsymbol{F}(\boldsymbol{u})$ be a continuously Fréchet differentiable operator mapping an open set D of a linear normed space M into a Banach space N. Then applying the Newton method to the equation

$$\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{0}, \tag{2.3}$$

we get the following theorem.

Theorem 1 (Urabe [5]).

Suppose that equation (2.3) *has an approximate solution* $\boldsymbol{u}=\bar{\boldsymbol{u}}\in D$ *for which there are an additive operator* T *mapping* M *into* N, *a positive number* δ *and a non-negative number* $\kappa<1$ *such that*

(i) T *has a linear inverse operator* T^{-1},

(ii) $D_{\delta}=\{\boldsymbol{u}|\ \|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_{c}\leqq\delta,\ \boldsymbol{u}\in M\}\subset D$,

(iii) $|||\boldsymbol{F}'(\boldsymbol{u})-T|||\leqq\dfrac{\kappa}{\mu}$ on D_{δ},

(iv) $\mu r/(1-\kappa)\leqq\delta$,

where $r(\geqq 0)$ *and* $\mu(>0)$ *are the numbers such that*

$$|||\boldsymbol{F}(\bar{\boldsymbol{u}})|||\leqq r,$$

$$\|T^{-1}\|_{c}\leqq\mu.$$

Then the Newton iterative process

$$\boldsymbol{u}_{p+1}=\boldsymbol{u}_{p}-T^{-1}\boldsymbol{F}(\boldsymbol{u}_{p})\quad(p=0,\ 1,\ 2,\ldots),\qquad \boldsymbol{u}_{0}=\bar{\boldsymbol{u}}$$

yields a fundamental sequence $\{\boldsymbol{u}_{p}\}$ $(p=0,\ 1,\ 2,\ldots)$ *in* D_{δ} *and we have*

$$\|\boldsymbol{u}_{p}-\bar{\boldsymbol{u}}\|_{c}\leqq\frac{\mu r}{1-\kappa}\qquad(p=0,1,\ 2,\ldots).$$

If the above fundamental sequence $\{\boldsymbol{u}_{p}\}$ $(p=0,\ 1,\ 2,\ldots)$ *converges in* M, *namely, there is an* $\hat{\boldsymbol{u}}\in M$ *such that*

$$\|\boldsymbol{u}_{p}-\hat{\boldsymbol{u}}\|_{c}\longrightarrow 0\quad as\quad p\longrightarrow\infty,$$

then $\hat{\boldsymbol{u}}$ *is an unique solution of* (2.3) *in* D_{δ} *and we have*

$$\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_{c}\leqq\frac{\mu r}{1-\kappa}.$$

We have also the following theorem.

Theorem 2 (Urabe [5]).

If the matrix $G\equiv L[\Psi(t)]$ *is non-singular, namely,*

$$\det G=\det L[\Psi(t)]\neq 0, \tag{2.4}$$

then the operator T *defined by* (2.2) *has a linear inverse operator* T^{-1}, *and for* $\|T^{-1}\|_{c}$ *we have*

$$\|T^{-1}\|_{c}\leqq\max(\|H_{1}\|_{c},\ \|H_{2}\|_{c}). \tag{2.5}$$

Here H_1 is the linear operator mapping $C[I]$ into $M=C^1[I]\subset C[I]$ such that

$$H_1\boldsymbol{\phi}=\Psi(t)\int_{-1}^{t}\Psi^{-1}(s)\boldsymbol{\phi}(s)ds-\Psi(t)G^{-1}L[\Psi(t)\int_{-1}^{t}\Psi^{-1}(s)\boldsymbol{\phi}(s)ds] \tag{2.6}$$

and H_2 is the linear operator mapping R^{n+1} into M such that

$$H_2\boldsymbol{v}=\Psi(t)G^{-1}\boldsymbol{v}\,. \tag{2.7}$$

When an approximate solution $\bar{\boldsymbol{u}}(t)$ of the boundary value problem (1.10) has been obtained, it is necessary to find an error bound on $\bar{\boldsymbol{u}}(t)$. For this purpose, we take $A(t)$ and L respectively such that

$$A(t)=\boldsymbol{V}_u(\bar{\boldsymbol{u}}(t))\,, \tag{2.8}$$

$$L=\boldsymbol{f}'(\bar{\boldsymbol{u}}(t))\,. \tag{2.9}$$

Then we have the following theorem.

Theorem 3.

Assume that the boundary value problem (1.10) *possesses an approximate solution $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ in S such that the matrix*

$$G=\boldsymbol{f}'(\bar{\boldsymbol{u}})\,[\Psi(t)] \tag{2.10}$$

is non-singular, where $\Psi(t)$ is the fundamental matrix of the following linear system satisfying the initial condition $\Psi(-1)=E$ (E is the unit matrix):

$$\frac{d\boldsymbol{z}}{dt}=\boldsymbol{V}_u(\bar{\boldsymbol{u}}(t))\boldsymbol{z}\,. \tag{2.11}$$

Let μ and r be the positive numbers such that

$$\mu=\max(\|H_1\|_c,\ \|H_2\|_c)\geqq\|T^{-1}\|_c\,, \tag{2.12}$$

$$r\geqq\left\|\frac{d\bar{\boldsymbol{u}}}{dt}-\boldsymbol{V}(\bar{\boldsymbol{u}})\right\|_c+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|\,. \tag{2.13}$$

If there exist a positive number δ and a non-negative number $\kappa<1$ such that

(2.14) (i) $D'_\delta=\{\boldsymbol{u}\,|\,\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_c\leqq\delta,\ \boldsymbol{u}\in C[I]\}\subset S'$,

(2.15) (ii) $\|\boldsymbol{V}_u(\boldsymbol{u})-\boldsymbol{V}_u(\bar{\boldsymbol{u}})\|_c+\|\boldsymbol{f}'(\boldsymbol{u})-\boldsymbol{f}'(\bar{\boldsymbol{u}})\|\leqq\frac{\kappa}{\mu}$ on D'_δ,

(2.16) (iii) $\frac{\mu r}{1-\kappa}\leqq\delta$,

then the boundary value problem (1.10) *has one and only one solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in*

$$D_\delta=\{\boldsymbol{u}|\ \|\boldsymbol{u}-\bar{\boldsymbol{u}}\|_c\leqq\delta,\ \boldsymbol{u}\in M\}\,, \tag{2.17}$$

and for this exact solution $\hat{\boldsymbol{u}}(t)$ *we have*

$$\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_c\leqq\frac{\mu r}{1-\kappa}\,. \tag{2.18}$$

PROOF. Let us put

$$\boldsymbol{F}(\boldsymbol{u})=\left[\frac{d\boldsymbol{u}}{dt}-\boldsymbol{V}(\boldsymbol{u}),\ \boldsymbol{f}(\boldsymbol{u})\right], \tag{2.19}$$

then $\boldsymbol{F}(\boldsymbol{u})$ maps $S\subset M$ into N and the boundary value problem (1.10) is rewritten to the equation

$$\boldsymbol{F}(\boldsymbol{u})=\boldsymbol{0}\,. \tag{2.20}$$

The Fréchet derivative $\boldsymbol{F}'(\boldsymbol{u})$ of $\boldsymbol{F}(\boldsymbol{u})$ at $\boldsymbol{u}$ can be written as follows:

$$\boldsymbol{F}'(\boldsymbol{u})\boldsymbol{h}=\left[\frac{d\boldsymbol{h}}{dt}-\boldsymbol{V}_u(\boldsymbol{u})\boldsymbol{h},\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{h}\right], \tag{2.21}$$

where $\boldsymbol{h}$ is an arbitrary element belonging to M. Then, by (2.2), (2.8), (2.9), (2.15) and (2.21) we have

$$|||\boldsymbol{F}'(\boldsymbol{u})-T|||\leqq\frac{\kappa}{\mu}\qquad\text{on } D_\delta\subset D'_\delta\,. \tag{2.22}$$

For the approximate solution $\bar{\boldsymbol{u}}(t)\in S$ we have from (2.13) that

$$|||\boldsymbol{F}(\bar{\boldsymbol{u}})|||=\left\|\frac{d\bar{\boldsymbol{u}}}{dt}-\boldsymbol{V}(\bar{\boldsymbol{u}})\right\|_c+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|\leqq r\,. \tag{2.23}$$

From (2.14) and (2.17), we have

$$D_\delta\subset D'_\delta$$

and

$$D'_\delta\subset S'\,.$$

Hence we have

$$D_\delta\subset S'\cap M=S\,. \tag{2.24}$$

By Theorem 2, the operator T has a linear inverse operator T^{-1} satisfying (2.12). The facts tell us that the Newton iterative process

$$\boldsymbol{u}_{p+1}=\boldsymbol{u}_p-T^{-1}\boldsymbol{F}(\boldsymbol{u}_p),\quad \boldsymbol{u}_0=\bar{\boldsymbol{u}}\qquad(p=0,\ 1,\ 2,\dots) \tag{2.25}$$

is well-defined in D_δ and we have a fundamental sequence $\{\boldsymbol{u}_p\}$ in $D_\delta\subset M\subset C[I]$ with respect to the norm $\|\cdots\|_c$.

By the completeness of the space $C[I]$, there exists a vector-function $\hat{\boldsymbol{u}}(t)\in C[I]$ such that

$$\|\boldsymbol{u}_p-\hat{\boldsymbol{u}}\|_c\longrightarrow 0 \quad \text{as} \quad p\longrightarrow\infty.$$

Since $\boldsymbol{u}_p\in D_\delta$, it is evident that

$$\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_c\leqq\delta$$

and $\hat{\boldsymbol{u}}\in D'_\delta\subset S'$.
Hence we have

$$\boldsymbol{V}(\hat{\boldsymbol{u}})-A(t)\hat{\boldsymbol{u}}\in C[I]$$

and $L\hat{\boldsymbol{u}}-\boldsymbol{f}(\hat{\boldsymbol{u}})\in R^{n+1}$. However, by (2.2), (2.19) and (2.25) we have

$$\begin{aligned}T\boldsymbol{u}_{p+1}&=T\boldsymbol{u}_p-\boldsymbol{F}(\boldsymbol{u}_p)\\&=[\boldsymbol{V}(\boldsymbol{u}_p)-A(t)\boldsymbol{u}_p,\ L\boldsymbol{u}_p-\boldsymbol{f}(\boldsymbol{u}_p)]\end{aligned}$$

and hence

(2.26) $$\boldsymbol{u}_{p+1}=T^{-1}[\boldsymbol{V}(\boldsymbol{u}_p)-A(t)\boldsymbol{u}_p,\ L\boldsymbol{u}_p-\boldsymbol{f}(\boldsymbol{u}_p)] \qquad (p=0,\ 1,\ 2,\ldots).$$

Letting $p\to\infty$ in (2.26), we have

(2.27) $$\hat{\boldsymbol{u}}=T^{-1}[\boldsymbol{V}(\hat{\boldsymbol{u}})-A(t)\hat{\boldsymbol{u}},\ L\hat{\boldsymbol{u}}-\boldsymbol{f}(\hat{\boldsymbol{u}})].$$

Since T^{-1} is a linear operator mapping $N=C[I]\times R^{n+1}$ into $M=C^1[I]$, the relation (2.27) shows that $\hat{\boldsymbol{u}}\in M$.
Hence, by Theorem 1 we see that $\boldsymbol{u}=\hat{\boldsymbol{u}}$ is a unique solution of (2.20) in D_δ and we have

$$\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_c\leqq\frac{\mu r}{1-\kappa}.$$

This completes the proof.

In the numerical computation, a desired periodic solution of (1.1) is usually assumed to be "*isolated*". For autonomous systems, the isolatedness means that characteristic multipliers of the first variation equation are all different from one except one characteristic multiplier. The reason why the "*isolated*" is employed is that there is no other periodic solution near the periodic solution in question, if the above condition is fulfilled.

On the other hand, the isolatedness of a solution $\hat{\boldsymbol{u}}(t)$ of the boundary value problem (1.10) means that

(2.28) $$\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta(t)]\neq 0,$$

where $\Theta(t)$ is an arbitrary fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt}=\boldsymbol{V}_u(\hat{\boldsymbol{u}})\boldsymbol{z}. \tag{2.29}$$

(Urabe [5]).

The additional linear functional $l(\boldsymbol{x})$ given in (1.6) is closely related to the isolatedness of solution.

Theorem 4.

The isolatedness of a periodic solution $\hat{\boldsymbol{x}}(t)$ *of* (1.3) *is equivalent to the one of a corresponding solution* $\hat{\boldsymbol{u}}(t)$ *of the boundary value problem* (1.9) *if and only if*

$$l[\boldsymbol{X}(\hat{\boldsymbol{x}})]\neq 0, \tag{2.30}$$

where $\hat{\boldsymbol{u}}(t)=\mathrm{col}\,[\hat{\boldsymbol{x}}(t),\ \hat{\omega}(t)]$.

PROOF. Let

$$\Theta(t)=\begin{pmatrix}\Psi(t) & \boldsymbol{p}(t)\\ \boldsymbol{q}^*(t) & s(t)\end{pmatrix}, \tag{2.31}$$

where $\Psi(t)$ is an $n\times n$ matrix, $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ are n-dimensional column vectors, $s(t)$ is a scalar function and $\boldsymbol{q}^*(t)$ is the transpose of $\boldsymbol{q}(t)$.

The first variation equation of (1.4) clearly reads as follows:

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix}\frac{\omega}{2}\boldsymbol{X}_x(\hat{\boldsymbol{x}}) & \frac{1}{2}\boldsymbol{X}(\hat{\boldsymbol{x}})\\ 0 & 0\end{pmatrix}\boldsymbol{z}.$$

Replacing $\boldsymbol{z}$ by $\Theta(t)$, we have the differential equations in Ψ, $\boldsymbol{p}$, $\boldsymbol{q}$ and s. Making use of the initial conditions

$$\Psi(-1)=E,\quad \boldsymbol{p}(-1)=\boldsymbol{q}(-1)=\boldsymbol{0},\quad s(-1)=1, \tag{2.32}$$

we see that

(i) $\Psi(t)$ is the fundamental matrix of the equation

$$\frac{d\boldsymbol{y}}{dt}=\frac{\omega}{2}\boldsymbol{X}_x(\hat{\boldsymbol{x}})\boldsymbol{y} \tag{2.33}$$

satisfying the initial condition $\Psi(-1)=E$ and then we obtain

$$\Psi(t)=\Psi_0\left(\frac{\omega}{2}t\right), \tag{2.34}$$

where $\Psi_0(\tau)$ is the fundamental matrix of the equation

(2.35) $$\frac{dy}{d\tau}=\boldsymbol{X}_x(\hat{\boldsymbol{x}})\boldsymbol{y}$$

satisfying the initial condition $\Psi_0\left(-\frac{\omega}{2}\right)=E$,

(ii) $\boldsymbol{p}(t)$, $\boldsymbol{q}(t)$ and $s(t)$ are obtained as follows:

(2.36) $$\boldsymbol{p}(t)=\frac{1}{2}\Psi(t)\int_{-1}^{t}\Psi^{-1}(\xi)\boldsymbol{X}[\hat{\boldsymbol{x}}(\xi)]\,d\xi\,,$$

(iii) $\boldsymbol{q}(t)\equiv\boldsymbol{0}$ and $s(t)\equiv 1$.

Now let us note that

$$\frac{d}{dt}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=\boldsymbol{X}_x[\hat{\boldsymbol{x}}(t)]\frac{\omega}{2}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]\,,$$

that is, $\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]$ is a 2-periodic solution of (2.33). Hence we have

(2.37) $$\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=\Psi(t)\boldsymbol{c}$$

for some constant vector $\boldsymbol{c}\neq\boldsymbol{0}$. Equation (2.37) implies

$$\Psi^{-1}(t)\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]=\boldsymbol{c}\,.$$

Hence, from (2.37) we have

(2.38) $$\boldsymbol{p}(t)=\frac{1}{2}(t+1)\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]\,.$$

Hence we have

(2.39) $$\Theta(t)=\begin{pmatrix}\Psi(t) & \frac{(t+1)}{2}\boldsymbol{X}[\hat{\boldsymbol{x}}(t)]\\ \boldsymbol{0} & 1\end{pmatrix}.$$

From (2.37), it is clear that

(2.40) $$\Psi(1)\boldsymbol{c}=\boldsymbol{c}\,.$$

By (1.8), we then have

(2.41) $$\boldsymbol{f}'(\hat{\boldsymbol{u}})\ [\Theta(t)]=\begin{pmatrix}E-\Psi(1) & -\boldsymbol{c}\\ l[\Psi] & \frac{1}{2}l[(t+1)\boldsymbol{X}(\hat{\boldsymbol{x}})]\end{pmatrix}.$$

Let us set $\boldsymbol{c}_1=\boldsymbol{c}/\|\boldsymbol{c}\|$ and Q be an orthogonal matrix whose the first column vector is $\boldsymbol{c}_1$.
Consider the matrix

$$K=\begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},$$

then K is also an orthogonal matrix. Write Q as

$$Q=[\boldsymbol{c}_1, Q_1],$$

where Q_1 is an $n\times(n-1)$ matrix whose column vectors are unit vectors and moreover they are mutually orthogonal. By (2.41), we then have

$$K^*\boldsymbol{f}'(\hat{\boldsymbol{u}})[\Theta(t)]K=\begin{pmatrix} Q^*[E-\Psi(1)]Q & -Q^*\boldsymbol{c} \\ l[\Psi]Q & \frac{1}{2}l[(t+1)\boldsymbol{X}(\hat{\boldsymbol{x}})] \end{pmatrix}. \tag{2.42}$$

However

$$\begin{aligned} Q^*\Psi(1)Q &= \begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}\Psi(1)[\boldsymbol{c}_1, Q_1] \\ &= \begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}[\Psi(1)\boldsymbol{c}_1, \Psi(1)Q_1] \\ &= \begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}[\boldsymbol{c}_1, \Psi(1)Q_1] \\ &= \begin{pmatrix} 1 & \boldsymbol{c}_1^*\Psi(1)Q_1 \\ \mathbf{0} & Q_1^*\Psi(1)Q_1 \end{pmatrix}. \end{aligned} \tag{2.43}$$

Hence the eigenvalues of $\Psi(1)$ are 1 and those of $Q_1^*\Psi(1)Q_1$.

Now

$$Q^*[E-\Psi(1)]Q=\begin{pmatrix} 0 & -\boldsymbol{c}_1^*\Psi(1)Q_1 \\ \mathbf{0} & E-Q_1^*\Psi(1)Q_1 \end{pmatrix}$$

and

$$Q^*\boldsymbol{c}=\begin{pmatrix} \boldsymbol{c}_1^* \\ Q_1^* \end{pmatrix}\boldsymbol{c}_1\cdot\|\boldsymbol{c}\|=\begin{pmatrix} \|\boldsymbol{c}\| \\ \mathbf{0} \end{pmatrix}.$$

By the linearity of $l(\boldsymbol{x})$, we have

$$\begin{aligned} l[\Psi]Q &= l[\Psi][\boldsymbol{c}_1, Q_1] \\ &= [l[\Psi\boldsymbol{c}_1], l[\Psi Q_1]] \\ &= \left[\frac{1}{\|\boldsymbol{c}\|}l[\boldsymbol{X}(\hat{\boldsymbol{x}})], l[\Psi Q_1]\right]. \end{aligned}$$

Hence we have

$$K^*f'(\hat{u})\,[\Theta(t)]K=\begin{pmatrix} 0 & -c_1^*\Psi(1)Q_1 & -\|c\| \\ \mathbf{0} & E-Q_1^*\Psi(1)Q_1 & \mathbf{0} \\ \frac{1}{\|c\|}l[X(\hat{x})] & l[\Psi Q_1] & \frac{1}{2}l[(t+1)X(\hat{x})] \end{pmatrix}.$$

This tells us that

$$\begin{aligned}\det f'(\hat{u})\,[\Theta]&=\det[K^*f'(\hat{u})\,[\Theta]K]\\ &=l[X(\hat{x})]\cdot\det[E-Q_1^*\Psi(1)Q_1].\end{aligned}$$

As is seen from (2.43), the isolatedness of a periodic solution $\hat{x}(t)$ of (1.3) is equivalent to $\det[E-Q_1^*\Psi(1)Q_1]\neq 0$.

On the other hand, by (2.28) the isolatedness of a solution of the boundary value problem (1.9) is $\det f'(\hat{u})[\Theta]\neq 0$. Hence, the both of conditions are equivalent mutually if and only if $l[X(\hat{x})]\neq 0$. This completes the proof.

REMARK. Theorem 4 is given by M. Urabe [9] without proof.

§3. Application to van der Pol equation

In the previous paper [3], M. Urabe, H. Yanagiwara and the author have computed the $\omega(\lambda)$-periodic solutions of van der Pol equation

$$\frac{d^2x}{d\tau^2}-\lambda(1-x^2)\frac{dx}{d\tau}+x=0, \tag{3.1}$$

where $\omega(\lambda)$ (>0) is the period of the desired periodic solution of (3.1). In the paper [3], we rewrote equation (3.1) in the following first order system

$$\begin{cases}\dfrac{dx}{d\tau}=y, \\[2mm] \dfrac{dy}{d\tau}=-x+\lambda(1-x^2)y\end{cases} \tag{3.2}$$

and used the predictor-corrector method for computing the periodic solution of (3.2). But, we gave no error bound for the numerical results obtained. Hence, we could guarantee no significant figures of the numerical results.

The present paper will present a practical method for computing a Chebyshev-series-approximation to the periodic solution of van der Pol equation (3.1) and give an error bound on the Chebyshev-series-approximation obtained.

Now, by the transformation $\tau=\omega t/2$ equation (3.1) is rewritten in the following form

$$\frac{d^2x}{dt^2}-\lambda(1-x^2)\frac{\omega}{2}\frac{dx}{dt}+\left(\frac{\omega}{2}\right)^2x=0 \tag{3.3}$$

and the problem is reduced to the one of finding a 2-periodic solution of the boundary value problem:

$$\begin{cases} \dfrac{dx}{dt}=y\,, \\ \dfrac{dy}{dt}=-\left(\dfrac{\omega}{2}\right)^2x+\dfrac{\omega}{2}\lambda(1-x^2)y\,, \\ \dfrac{d\omega}{dt}=0\,, \end{cases} \tag{3.4}$$

$$\begin{cases} x(-1)-x(1)=0\,, \\ y(-1)-y(1)=0\,. \end{cases} \tag{3.5}$$

As is shown in the first section, the boundary value problem (3.4)–(3.5) is clearly incomplete. Hence, we consider an additional condition of the form

$$l(\boldsymbol{u})\equiv\frac{2}{\pi}\int_{-1}^{1}\frac{x(t)}{\sqrt{1-t^2}}T_{\hat{n}}(t)dt=\beta\,, \tag{3.6}$$

where $\boldsymbol{u}=\mathrm{col}\,[x(t),\ y(t),\ \omega(t)]$ and $T_{\hat{n}}(t)$ is a Chesbyshev polynomial of degree $\hat{n}$ such that $T_{\hat{n}}(\cos\theta)=\cos\hat{n}\theta$.

In order to get a Chebychev-series-approximation to the boundary value problem (3.4)–(3.6), let us consider a finite Chebyshev-series

$$\boldsymbol{u}_m(t)=\sum_{p=0}^{m}\varepsilon_p\cdot\boldsymbol{a}_p\cdot T_p(t) \tag{3.7}$$

with undetermined coefficients $\boldsymbol{a}_0$, $\boldsymbol{a}_1,\dots,\ \boldsymbol{a}_m$, where

$$\varepsilon_p=\begin{cases} 0 & \text{for}\quad p<0\,, \\ \dfrac{1}{2} & \text{for}\quad p=0\,, \\ 1 & \text{for}\quad p>0\,. \end{cases} \tag{3.8}$$

By the formula in Fourier analysis, we see that coefficients $\boldsymbol{a}_p$ $(p=0,\ 1,\ 2,\dots)$ can be evaluated by the formula

$$\boldsymbol{a}_p\doteqdot\frac{2}{N}\sum_{i=1}^{N}\boldsymbol{u}_m(\cos\theta_i)\cos p\theta_i \qquad (p=0,\ 1,\ 2,\dots)\,, \tag{3.9}$$

where N is a non-small positive integer greater than p and

$$\theta_i = \frac{2i-1}{2N}\pi \qquad (i=1, 2, \dots, N).$$

In our cases we have chosen N always so that $N=64$.

The boundary value problem (3.4)–(3.6) can be written briefly as

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{X}(\boldsymbol{u}), \tag{3.10}$$

$$\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}. \tag{3.11}$$

Here,

$$\boldsymbol{X}(\boldsymbol{u}) = \begin{pmatrix} y \\ -\left(\frac{\omega}{2}\right)^2 x + \frac{\omega}{2}\lambda(1-x^2)y \\ 0 \end{pmatrix}, \tag{3.12}$$

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} x(-1)-x(1) \\ y(-1)-y(1) \\ \frac{2}{\pi}\int_{-1}^{1} \frac{x(t)}{\sqrt{1-t^2}} T_{\hat{n}}(t)\,dt \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \tag{3.13}$$

$$= L_1\boldsymbol{u}(-1) - L_1\boldsymbol{u}(1) + L_2\int_{-1}^{1} \frac{2}{\pi}\frac{\boldsymbol{u}(t)}{\sqrt{1-t^2}} T_{\hat{n}}(t)\,dt - \boldsymbol{\beta},$$

where L_1 and L_2 are matrices such that

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\boldsymbol{\beta} = \mathrm{col}\,[0, 0, \beta]$.

The Fréchet derivative $\boldsymbol{f}'(\boldsymbol{u})$ of $\boldsymbol{f}(\boldsymbol{u})$ at $\boldsymbol{u}$ reads

$$\boldsymbol{f}'(\boldsymbol{u})[\boldsymbol{h}] = L_2\int_{-1}^{1} \frac{2}{\pi}\frac{\boldsymbol{h}(t)}{\sqrt{1-t^2}} T_{\hat{n}}(t)\,dt + L_1[\boldsymbol{h}(-1) - \boldsymbol{h}(1)]. \tag{3.14}$$

Let

$$\frac{d\boldsymbol{u}_m(t)}{dt} = \sum_{p=0}^{m-1} \varepsilon_p \cdot \boldsymbol{a}'_p \cdot T_p(t),$$

then we have

(3.15) $$\boldsymbol{a}'_p = \boldsymbol{a}'_p(\boldsymbol{\alpha}) = \sum_{s=0}^{m} \varepsilon_{s-p} \cdot v_{s-p} \cdot s \cdot \boldsymbol{a}_p \qquad (p=0, 1, 2, \dots, m-1),$$

where

$$\boldsymbol{\alpha} = \mathrm{col}\,[\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_m]$$

and

(3.16) $$v_i = 1 - (-1)^i .$$

For the equations (3.10), (3.11) and the finite Chebyshev-series (3.7), we consider the following determining equations

(3.17) $$\frac{d\boldsymbol{u}_m(t)}{dt} = P_{m-1}\boldsymbol{X}(\boldsymbol{u}_m(t)) ,$$

(3.18) $$\boldsymbol{f}(\boldsymbol{u}_m(t)) = \boldsymbol{0} ,$$

where P_{m-1} is an operator which expresses the truncation of the Chebyshev series of the operand discarding the terms of the order higher than $m-1$. Then the equations (3.17) and (3.18) are equivalent to the equations

(3.19) $$\begin{cases} \boldsymbol{F}_p(\boldsymbol{\alpha}) \equiv \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \boldsymbol{X}[\boldsymbol{u}_m(\cos\theta)] \cos p\theta d\theta - \boldsymbol{a}'_p(\boldsymbol{\alpha}) = \boldsymbol{0} \ (p=0, 1, \dots\ m-1) , \\ \boldsymbol{F}_m(\boldsymbol{\alpha}) \equiv \boldsymbol{f}(\boldsymbol{u}_m(t)) = \boldsymbol{0} . \end{cases}$$

Put

(3.20) $$\boldsymbol{F}(\boldsymbol{\alpha}) = \mathrm{col}\,[\boldsymbol{F}_0(\boldsymbol{\alpha}), \boldsymbol{F}_1(\boldsymbol{\alpha}), \dots, \boldsymbol{F}_m(\boldsymbol{\alpha})] ,$$

then the determining equation (3.19) can be written briefly as

(3.21) $$\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0} .$$

Since the function $\boldsymbol{X}(\boldsymbol{u})$ is nonlinear in $\boldsymbol{u}$, $\boldsymbol{F}(\boldsymbol{\alpha})$ is also a nonlinear equation in $\boldsymbol{\alpha}$. Hence, for numerical solution of the nonlinear equation (3.21) the Newton method will be used.

Starting from a certain approximate solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$, we compute the sequence $\{\boldsymbol{\alpha}_p\}$ successively by the iterative process

(3.22) $$\begin{cases} J(\boldsymbol{\alpha}_p)\boldsymbol{h}_p + \boldsymbol{F}(\boldsymbol{\alpha}_p) = \boldsymbol{0} , \\ \boldsymbol{\alpha}_{p+1} = \boldsymbol{\alpha}_p + \boldsymbol{h}_p \qquad (p=0, 1, 2, \dots), \end{cases}$$

where $J(\boldsymbol{\alpha})$ is the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$. In order to practice the iterative process (3.22) on a computer, it suffices to evaluate $\boldsymbol{F}(\boldsymbol{\alpha})$ and $J(\boldsymbol{\alpha})$ for known $\boldsymbol{\alpha}$.

Given a suitable small positive number Δ and a positive integer ι (maximum number of iterations), if the convergence criterion

$$\|\boldsymbol{\alpha}_{p+1}^{(k)}-\boldsymbol{\alpha}_p^{(k)}\|\leqq\Delta$$

holds for some $p\leqq\iota$ and for all k $(0\leqq k\leqq m)$, then we regard that the sequence $\{\boldsymbol{\alpha}_p\}$ has converged and stop the iterative computation (3.22), where $\boldsymbol{\alpha}_p=\mathrm{col}\,[\boldsymbol{\alpha}_p^{(0)}, \boldsymbol{\alpha}_p^{(1)},\dots, \boldsymbol{\alpha}_p^{(m)}]$. In our example, we shall set $\Delta=10^{-10}$ and $\iota=20$, respectively.

Let $J_{ij}(\boldsymbol{\alpha})$ $(0\leqq i, j\leqq m)$ be the following 3×3 matrices:

$$J_{ij}(\boldsymbol{\alpha})=\frac{2}{\pi}\int_0^\pi \boldsymbol{X}_u[\boldsymbol{u}_m(\cos\theta)]\cos i\,\theta\cos j\,\theta\,d\theta-\varepsilon_{j-i}\cdot v_{j-i}\cdot j\cdot E_3 \tag{3.23}$$

$$(0\leqq i\leqq m-1,\ 0\leqq j\leqq m)$$

and

$$J_{mj}(\boldsymbol{\alpha})=\boldsymbol{f}'(\boldsymbol{u}_m)\,[\varepsilon_j T_j(t)E_3] \qquad (0\leqq j\leqq m), \tag{3.24}$$

then we have

$$J(\boldsymbol{\alpha})=(J_{ij}(\boldsymbol{\alpha})), \tag{3.25}$$

where E_3 is the unit matrix such that

$$E_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

The starting value $\boldsymbol{\alpha}_0=\boldsymbol{\alpha}_0(\bar{\bar{\lambda}})$ necessary for the Newton method for small $\lambda=\bar{\bar{\lambda}}$ can be obtained by the Chebyshev expansion of

$$\begin{cases} x(t)=2\cos\dfrac{\omega}{2}t, \\ y(t)=-\omega\sin\dfrac{\omega}{2}t, \\ \omega=2\pi, \end{cases} \tag{3.26}$$

which is a periodic solution of the equations (3.4) and (3.5) with sufficiently small $\bar{\bar{\lambda}}$. (See [6]).

For not small λ, tracing the curve

$$\boldsymbol{F}_p(\boldsymbol{\alpha}, \lambda)=0 \qquad (p=0, 1, 2,\dots, m)$$

through the point $\boldsymbol{\alpha}_0(\bar{\bar{\lambda}})$, we can obtain the starting value $\boldsymbol{\alpha}_0$ for not small λ. (See [8]).

Table

Chebyshev-series-approximation

$$x=\sum_{n=0}^{40}\varepsilon_n a_n T_n(t),\ \dot{x}=\sum_{n=0}^{40}\varepsilon_n b_n T_n(t),\ \omega=\sum_{n=0}^{40}\varepsilon_n c_n T_n(t)$$

$$\text{to}\ \frac{d^2x}{dt^2}-0.01\,(1-x^2)\frac{\omega}{2}\frac{dx}{dt}+\left(\frac{\omega}{2}\right)^2x=0\,.$$

n	a_n	b_n	c_n
0	−0.31321 50376 12953	−3.69992 70699 17898	12.56644 91539 70990
1	1.10057 49015 49166	−0.91649 59756 09040	0.0
2	−0.50000 00000 00000	−5.90107 68730 16231	0.0
3	−1.28841 62680 65687	1.08350 40243 90959	0.0
4	0.15717 29832 02010	1.82942 07353 77892	0.0
5	0.20098 89889 80537	−0.17387 98412 25113	0.0
6	−0.01459 36964 26223	−0.18046 91544 27488	0.0
7	−0.01426 14793 94856	0.00124 45158 89560	0.0
8	−0.00041 84671 20066	0.01919 15571 00496	0.0
9	0.00137 42894 17900	0.00793 99898 10619	0.0
10	0.00053 81299 15730	−0.00554 56524 21709	0.0
11	−0.00031 27228 72987	−0.00282 26085 03982	0.0
12	−0.00014 26724 37307	0.00133 42507 83998	0.0
13	0.00005 91335 36510	0.00060 15299 91390	0.0
14	0.00002 54248 93182	−0.00020 32211 65269	0.0
15	−0.00000 73609 70422	−0.00011 03670 17692	0.0
16	−0.00000 41672 02188	0.00001 76079 47381	0.0
17	0.00000 05067 60232	0.00002 29834 52312	0.0
18	0.00000 07806 53341	0.00000 03780 99495	0.0
19	0.00000 00247 73548	−0.00000 51200 67952	0.0
20	−0.00000 01534 60277	−0.00000 05632 95308	0.0
21	−0.00000 00177 02283	0.00000 10183 43109	0.0
22	0.00000 00270 48869	0.00000 01802 00561	0.0
23	0.00000 00050 01463	−0.00000 01718 07146	0.0
24	−0.00000 00041 08327	−0.00000 00498 66715	0.0
25	−0.00000 00012 53744	0.00000 00253 92565	0.0
26	0.00000 00005 53622	0.00000 00128 20474	0.0
27	0.00000 00002 92379	−0.00000 00033 95801	0.0
28	−0.00000 00000 67373	−0.00000 00029 67998	0.0
29	−0.00000 00000 61786	0.00000 00003 77066	0.0
30	0.00000 00000 06525	0.00000 00006 15568	0.0
31	0.00000 00000 11818	−0.00000 00000 14459	0.0
32	−0.00000 00000 00075	−0.00000 00001 17136	0.0
33	−0.00000 00000 02093	−0.00000 00000 09665	0.0
34	−0.00000 00000 00205	0.00000 00000 21024	0.0
35	0.00000 00000 00351	0.00000 00000 04245	0.0
36	0.00000 00000 00076	−0.00000 00000 03584	0.0
37	−0.00000 00000 00056	−0.00000 00000 01211	0.0
38	−0.00000 00000 00019	0.00000 00000 00566	0.0
39	0.00000 00000 00007	0.00000 00000 00235	0.0
40	0.00000 00000 00003	−0.00000 00000 00070	0.0

Taking account of the Chebyshev expansion of (3.26) and (2.30), we shall set $\hat{n}=2$ and $\beta=-0.5$ in (3.13). Numerical result is shown in Table. The computations in the present paper have been carried out by the use of *FACOM* 230 at Tokushima University.

After having found an approximate solution $\boldsymbol{u}_m(t)$, it is necessary to verify the existence of the exact periodic solution $\hat{\boldsymbol{u}}(t)$ and to give a posteriori error estimation for $\boldsymbol{u}_m(t)$. For this purpose we begin with checking the conditions in Theorem 3.

In the present case, from (3.14) and (3.12) we have

$$\|\boldsymbol{f}'(\boldsymbol{u})-\boldsymbol{f}'(\bar{\boldsymbol{u}})\|=0 \tag{3.27}$$

and

$$\boldsymbol{X}_u(x, y, \omega)=\begin{pmatrix} 0 & 1 & 0 \\ -\lambda xy\omega-\frac{\omega^2}{4} & \frac{\omega}{2}\lambda(1-x^2) & \frac{1}{2}\lambda(1-x^2)y-\frac{\omega}{2}x \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Therefore, for a Chebyshev-series-approximation $\bar{\boldsymbol{u}}=\boldsymbol{u}_m(t)$ we have

$$\begin{aligned} \|\boldsymbol{X}_u(x, y, \omega)-\boldsymbol{X}_u(\bar{x}, \bar{y}, \bar{\omega})\| = \Big\{&\Big[\lambda(\bar{x}\bar{y}\bar{\omega}-xy\omega)+\frac{1}{4}(\bar{\omega}+\omega)(\bar{\omega}-\omega)\Big]^2 \\ &+\Big(\frac{\lambda}{2}\Big)^2[(1-x^2)\omega-(1-\bar{x}^2)\bar{\omega}]^2+\frac{1}{4}\Big\{\lambda[(1-x^2)y-(1-\bar{x}^2)\bar{y}] \\ &+(\bar{x}\bar{\omega}-x\omega)\Big\}^2\Big\}^{\frac{1}{2}}. \end{aligned} \tag{3.28}$$

Then, if we assume that

$$[(x-\bar{x})^2+(y-\bar{y})^2+(\omega-\bar{\omega})^2]^{\frac{1}{2}}\leqq\delta, \tag{3.29}$$

then using

$$x=(x-\bar{x})+\bar{x}, \quad y=(y-\bar{y})+\bar{y}, \quad \omega=(\omega-\bar{\omega})+\bar{\omega},$$

we have

$$\begin{aligned} &\Big[\lambda(\bar{x}\bar{y}\bar{\omega}-xy\omega)+\frac{1}{4}(\bar{\omega}+\omega)(\bar{\omega}-\omega)\Big]^2 \\ &\leqq\Big\{\lambda^2\{(|\bar{y}||\bar{\omega}|)^2+|\bar{\omega}|^2(\delta^2+2|\bar{x}|\delta+|\bar{x}|^2)+[\delta^2+\delta(|\bar{x}|+|\bar{y}|) \\ &\qquad\qquad +|\bar{x}||\bar{y}|]^2\} \\ &\quad+\frac{\lambda}{2}\{|\bar{\omega}||\bar{y}|+|\bar{\omega}|(\delta+|\bar{x}|)+(\delta+|\bar{x}|)(\delta+|\bar{y}|)\}(\delta+2|\bar{\omega}|) \\ &\quad+\frac{1}{16}(\delta^2+4|\bar{\omega}|\delta+4|\bar{\omega}|^2)\Big\}\delta^2, \end{aligned} \tag{3.30}$$

$$(3.31)\qquad \left(\frac{\lambda}{2}\right)^2[(1-x^2)\omega-(1-\bar{x}^2)\bar{\omega}]^2$$
$$\leqq \frac{\lambda^2}{4}[(1+\delta^2+2|\bar{x}|\delta+|\bar{x}|^2)^2+|\bar{\omega}|^2(\delta+2|\bar{x}|)^2]\delta^2$$

and

$$(3.32)\qquad \frac{1}{4}\{\lambda[(1-x^2)y-(1-\bar{x}^2)\bar{y}]+(\bar{x}\bar{\omega}-x\omega)\}^2$$
$$\leqq \frac{1}{4}\{\lambda^2[(\delta+2|\bar{x}|)^2(\delta+|\bar{y}|)^2+(1+|\bar{x}|^2)^2]$$
$$+2\lambda[(\delta+2|\bar{x}|)(\delta+|\bar{y}|)+(1+|\bar{x}|^2)](|\bar{x}|+\delta+|\bar{\omega}|)$$
$$+[|\bar{x}|^2+(\delta+|\bar{\omega}|)^2]\}\delta^2 .$$

However, for $\lambda=0.01$, $m=40$ we have from Table that

$$(3.33)\qquad \begin{cases} |\bar{x}(t)|\leqq \sum_{p=0}^{m}\varepsilon_p|a_p|<3.43550 , \\ |\bar{y}(t)|\leqq \sum_{p=0}^{m}\varepsilon_p|b_p|<11.97386 , \\ |\bar{\omega}(t)|<6.28323 . \end{cases}$$

Thus from (3.28), (3.30), (3.31), (3.32) and (3.33), we have

$$(3.34)\qquad \|\boldsymbol{X}_u(x, y, \omega)-\boldsymbol{X}_u(\bar{x}, \bar{y}, \bar{\omega})\|\leqq\delta[0.000015\delta^4+0.0143677\delta^3+0.67833248\delta^2$$
$$+6.74771126\delta+36.9841858]^{\frac{1}{2}}.$$

On the other hand, from (2.4), (2.9) and (3.14), we have

$$(3.35)\qquad G=\boldsymbol{f}'(\bar{\boldsymbol{u}})\,[\Psi(t)]=L_1\,[E-\Psi(1)]+L_2\int_{-1}^{1}\frac{2}{\pi}\frac{\Psi(t)}{\sqrt{1-t^2}}\,T_{\hat{n}}(t)\,dt$$

and from (2.6), if $\det G\neq 0$, we have

$$H_1\boldsymbol{\phi}=\int_{-1}^{1}H_1(t, s)\boldsymbol{\phi}(s)ds ,$$

where

$$H_1(t, s)=\begin{cases} \Psi(t)\Big\{E-G^{-1}\Big[-L_1\Psi(1)+L_2\int_s^1\frac{2}{\pi}\frac{\Psi(\xi)}{\sqrt{1-\xi^2}}\,T_{\hat{n}}(\xi)\,d\xi\Big]\Big\}\Psi^{-1}(s) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } s<t), \\ -\Psi(t)G^{-1}\Big[-L_1\Psi\;(1)+L_2\int_s^1\frac{2}{\pi}\frac{\Psi(\xi)}{\sqrt{1-\xi^2}}\,T_{\hat{n}}(\xi)\,d\xi\Big]\Psi^{-1}(s) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } s\geqq t) \end{cases}$$

or

$$(3.36)\qquad H_1(t,s)=\begin{cases}\Psi(t)G^{-1}\Big[L_1+L_2\displaystyle\int_{-1}^{s}\frac{2}{\pi}\frac{\Psi(\xi)}{\sqrt{1-\xi^2}}T_{\hat{n}}(\xi)d\xi\Big]\Psi^{-1}(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{if } -1\leqq s<t\leqq 1),\\ \Psi(t)\Big\{G^{-1}\Big[L_1+L_2\displaystyle\int_{-1}^{s}\frac{2}{\pi}\frac{\Psi(\xi)}{\sqrt{1-\xi^2}}T_{\hat{n}}(\xi)d\xi\Big]-E\Big\}\Psi^{-1}(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{if } -1\leqq t\leqq s\leqq 1).\end{cases}$$

Hence, we may set

$$(3.37)\qquad \mu=\max(\|H_1\|_c,\ \sup_{t\in I}\|\Psi(t)G^{-1}\|).$$

Let

$$\frac{d\bar{\boldsymbol{u}}(t)}{dt}-\boldsymbol{X}(\bar{\boldsymbol{u}}(t))=\sum_{p=0}^{\infty}\varepsilon_p\boldsymbol{c}_pT_p(t),$$

then inequality (2.13) is valid if

$$(3.38)\qquad [\sum_k(\sum_{p=0}^{m'}\varepsilon_p|c_p^k|)^2]^{\frac{1}{2}}+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|<r$$

with $m'=m+10$, where c_p^k are components of vectors $\boldsymbol{c}_p$. For computation of $\boldsymbol{c}_0$, $\boldsymbol{c}_1,\dots,\boldsymbol{c}_m$, we have used the formulas (3.9) and (3.15).

Now, we readily see that the conditions (2.14), (2.15) and (2.16) are fulfilled if

$$(3.39)\qquad \begin{cases}\delta[0.00015\delta^4+0.0143677\delta^3+0.6783324\delta^2+6.7477126\delta+36.9841858]^{\frac{1}{2}}\leqq\dfrac{\kappa}{\mu},\\ \dfrac{\mu r}{1-\kappa}\leqq\delta.\end{cases}$$

In (3.35) and (3.36), $\Psi(t)$ is given in Chebyshev series by solving the linear system

$$\frac{d\boldsymbol{z}}{dt}=\boldsymbol{X}_u(\bar{\boldsymbol{u}}(t))\boldsymbol{z}$$

satisfying the initial condition $\Psi(-1)=E$.

Let $H_{ij}(t, s)$ and $M_{ij}(t)$ denote the elements of the matrices $H(t, s)$ and $\Psi(t)G^{-1}$, respectively.

Then we have

$$\|H_1\|_c\leqq[2\cdot\max_p\int_{-1}^{1}\sum_{i,j}H_{ij}^2(t_p, s)ds]^{\frac{1}{2}}$$

and

$$\|\Psi(t)G^{-1}\| \leqq [\max_p \sum_{i,j} M_{ij}^2(t_p)]^{\frac{1}{2}} \qquad (p=0, 2, 4, \ldots, 128).$$

By (3.37), a number slightly greater than the quantity

$$\max\left([2\cdot\max_p \int_{-1}^{1} \sum_{i,j} H_{ij}^2(t_p, s)ds]^{\frac{1}{2}},\quad [\max_p \sum_{i,j} M_{ij}^2(t_p)]^{\frac{1}{2}}\right)$$

may be taken for the number μ, where the above integral may be evaluated by Simpson's rule with mesh-size 1/64 and $t_p = -1 + p/64$.

By the above way, we obtain

$$\det G = 0.189\times 10^{-2}, \quad r = 0.638\times 10^{-11} \quad \text{and} \quad \mu = 0.101\times 10^4.$$

Therefore, (3.39) can be written as

(3.40) $$\delta[0.00015\delta^4 + 0.0143677\delta^3 + 0.6783324\delta^2 + 6.7477126\delta + 36.9841858]^{\frac{1}{2}} \leqq \frac{\kappa}{0.101\times 10^4},$$

(3.41) $$\frac{6.45\times 10^{-9}}{1-\kappa} \leqq \delta.$$

Since we expect $\kappa \ll 1$, from (3.41) we suppose

(3.42) $$\delta \leqq 10^{-8}.$$

Then (3.40) is valid if

$$\delta[36.9841858 + 6.7477126\times 10^{-8} + \cdots]^{\frac{1}{2}} \leqq \frac{\kappa}{0.101\times 10^4}.$$

This inequality is valid if

$$\delta[36.9841859]^{\frac{1}{2}} \leqq \frac{\kappa}{0.101\times 10^4},$$

that is,

$$6.08146\cdots\times\delta \leqq \frac{\kappa}{0.101\times 10^4}.$$

This inequality is also valid if

$$6.082\times\delta \leqq \frac{\kappa}{0.101\times 10^4},$$

that is,

(3.43) $$\delta \leqq \frac{\kappa}{6.082\times 0.101\times 10^4} \leqq \frac{\kappa}{6.142\times 10^3}.$$

Then from (3.41) and (3.43) we have

(3.44) $$\frac{6.45\times 10^{-9}}{1-\kappa}\leqq\delta\leqq\frac{\kappa}{6.142\times 10^{3}},$$

which implies

$$6.45\times 6.142\times 10^{-6}\leqq\kappa(1-\kappa),$$

that is,

$$3.96159\times 10^{-5}\leqq\kappa(1-\kappa)<\kappa.$$

Hence we suppose

(3.45) $$\kappa=4\times 10^{-5}.$$

Then for this value of κ, we have

(3.46) $$\begin{cases}\dfrac{6.45\times 10^{-9}}{1-\kappa}=6.450258\cdots\times 10^{-9}, \\ \dfrac{\kappa}{6.142\times 10^{3}}=6.51253\cdots\times 10^{-9}.\end{cases}$$

Thus taking into account (3.42), by (3.44), (3.45) and (3.46), we see that (3.39) is valid for κ and δ such that

(3.47) $$\kappa=4\times 10^{-5},\quad 6.46\times 10^{-9}\leqq\delta\leqq 6.51\times 10^{-9},$$

in other words, the conditions of Theorem 3 are fulfilled by δ and κ specified in (3.47).

In conclusion, we thus see that the boundary value problem (3.10)–(3.11) possesses a unique exact solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in the region

$$\{[x-\bar{x}(t)]^2+[y-\bar{y}(t)]^2+[\omega-\bar{\omega}(t)]^2\}^{\frac{1}{2}}\leqq 6.51\times 10^{-9}$$

and moreover

$$\{[\hat{x}(t)-\bar{x}(t)]^2+[\hat{y}(t)-\bar{y}(t)]^2+[\hat{\omega}-\bar{\omega}]^2\}^{\frac{1}{2}}\leqq\eta=6.46\times 10^{-9}.$$

The quantity $\eta=6.46\times 10^{-9}$ gives an error bound to the Chebyshev-series-approximation $\boldsymbol{u}=\bar{}(t)$ given in Table. Hence, for $\lambda=0.01$ we obtain $\bar{\omega}=6.2832246$ which approximates the exact period $\hat{\omega}$ to eight significant figures.

Acknowledgement

The author would like to dedicate the present paper to the late Professor Minoru Urabe.

Department of Applied Mathematics
Faculty of Engineering
Tokushima University

References

[1] Drodnitsyn, A., Asymptotic solution of the van der Pol equation, *Prikl. Mat. Mekh., Inst. Mech. Acad. Sci. USSR.* **11** (1947), 313.

[2] Urabe, M., Periodic solution of van der Pol's equation with damping coefficient $\lambda=0(0.2)1.0$, *J. Sci. Hiroshima Univ., Ser. A*, **21** (1958), 193–207.

[3] Urabe, M., Yanagiwara, H. and Shinohara Y., Periodic solutions of van der Pol's equation with damping coefficient $\lambda=2$–10, *J. Sci. Hiroshima Univ., Ser. A,* **23** (1960), 325–366.

[4] Krogdahl, W. S., Numerical solutions of van der Pol equation, *Z. Angew. Math. Phys.*, Vol. **11** (1960), 59–63.

[5] Urabe, M., The Newton method and its application to boundary value problems with nonlinear boundary conditions, *Pro. US-Japan Seminar on Differential and Functional equations,* Benjamin, New York, (1967), 383–410.

[6] Urabe, M., Nonlinear autonomous oscillations, Analytical Theory, Academic Press, 1967.

[7] Urabe, M., Numerical solution of boundary value problems in Chebyshev series — A method of computation and error estimation, Conference on the numerical solution on differential equations, *Lecture Notes in Mathematics* **109** (1969), 40–86.

[8] Shinohara, Y., A geometric method of numerical solution of nonlinear equations and its application to nonlinear oscillations, *Publ. RIMS. Kyoto Univ.*, Vol. **8** (1972), 13–42.

[9] Urabe, M., A remark to the computation of periodic solution to autonomous differential systems, *Qualitative study of nonlinear oscillations,* (1974), 27–29.

[10] Fujii, M., Numerical solution of boundary value problems with nonlinear boundary conditions in Chebyshev series, *Bulletin of Fukuoka Univ. of Education,* Vol. **25** (1975), 27–45.