J. Math. Tokushima Univ.
Vol. 4. (1970), 33-36

An Expansion Formula for Hypergeometric Functions of Two Variables

By

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(Received September 30, 1970)

1. Introduction: The object of this paper is to derive an expansion formula for a generalised hypergeometric function of two variables in a series of product of generalised hypergeometric functions of two variables and a Meijer's G-function. The result established in this paper is the extension of the results given by Meijer [5, p. 311, Eqn. (237)] and Srivastava [6, p. 246 Eqn. (2.2)].

The following notation due to Chaundy [1] will be used to represent the hypergeometric function of higher order and of two variables

$$F\left[\begin{matrix}(a_p);\ (b_q);\ (c_r);\ x,\ y\\ (d_s);\ (e_h);\ (f_g);\end{matrix}\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n}{[(d_s)]_{m+n}[(e_h)]_m[(f_g)]_n}\frac{x^m}{m!}\frac{y^n}{n!},\qquad (1)$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1, \cdots, a_p$ and $(a_1)_{m+n}, \cdots, (a_p)_{m+n}$ respectively.

2. The following results [1, p. 337] and [2, p. 91] will be required in our investigation.

$$\int_0^\infty x^{\lambda-1}G_{r,s}^{p,q}\left[zx^2\left|\begin{matrix}\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix}\right.\right]dx=\frac{\prod_{j=1}^{p}\Gamma\left(\beta_j+\frac{1}{2}\lambda\right)\prod_{j=1}^{q}\Gamma\left(1-\alpha_j-\frac{1}{2}\lambda\right)}{\prod_{j=p+1}^{s}\Gamma\left(1-\beta_j-\frac{1}{2}\lambda\right)\prod_{j=q+1}^{r}\Gamma\left(\alpha_j+\frac{1}{2}\lambda\right)}\frac{(z)^{-\frac{1}{2}\lambda}}{2},\qquad \cdots\cdots (2)$$

valid for $0\leq p\leq s$, $0\leq q\leq r$, $p+q>\frac{1}{2}(r+s)$, $|\arg z|<\left(p+q-\frac{1}{2}r-\frac{1}{2}s\right)\pi$, $-\min_{1\leq j\leq p}2R(\beta_j)<R(\lambda)<1-\max_{1\leq j\leq q}2R(\alpha_j)$.

$$\int_0^\infty x^{\lambda-1}J_v(ax)G_{r,s}^{p,q}\left[zx^2\left|\begin{matrix}\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{matrix}\right.\right]dx$$

$$=\frac{2^{\lambda-1}}{a^{\lambda}}G_{r+2,s}^{p,q+1}\left[\frac{4z}{a^2}\left|\begin{matrix}1-\frac{1}{2}\nu-\frac{1}{2}\lambda,\alpha_1,\cdots,\alpha_r,1+\frac{1}{2}\nu-\frac{1}{2}\lambda\\ \beta_1,\cdots,\beta_s\end{matrix}\right.\right], \tag{3}$$

valid for $r+s<2(p+q)$, $|\arg z|<\left(p+q-\frac{1}{2}r-\frac{1}{2}s\right)\pi$, $a>0$, $R(\lambda+\nu+2\beta_j)>0$, $j=1, 2, \cdots, p$, $R\left(\lambda+2\alpha_h-\frac{3}{2}\right)>0$, $h=1, 2, \cdots, q$.

3. The first formula to be established here is

$$\sum_{n=0}^{\infty}\frac{1}{n!}G_{s,r+1}^{q+1,p}\left[x\left|\begin{matrix}1-\beta_1,\cdots,1-\beta_s\\ \lambda+n,1-\alpha_1,\cdots,1-\alpha_r\end{matrix}\right.\right]$$

$$=\frac{\prod_{j=1}^{p}\Gamma(\beta_j+\lambda)\prod_{j=1}^{q}\Gamma(1-\alpha_j-\lambda)}{\prod_{j=p+1}^{s}\Gamma(1-\beta_j-\lambda)\prod_{j=q+1}^{r}\Gamma(\alpha_j+\lambda)}x^{\lambda}. \tag{4}$$

PROOF. To prove (4) we start with the result [7, p. 147]

$$z^{\lambda}=\Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{z^n}{n!}J_{\lambda+n}(2z). \tag{5}$$

Multiplying both the sides by

$$z^{-1}G_{r,s}^{p,q}\left[\frac{z^2}{x}\left|\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\right.\right]$$

and integrating with respect to x between the limits 0 and ∞ and evaluating the integrals with the help of the formulae (2) and (3), we get (4).

4. The second formula to be established here is

$$x^{\lambda}F\left[\begin{matrix}(a_m);(b_n);(c_h);xy,xz\\ (d_k);(e_g);(f_w);\end{matrix}\right]=\frac{\prod_{j=p+1}^{s}\Gamma(1-\beta_j-\lambda)\prod_{j=q+1}^{r}\Gamma(\alpha_j+\lambda)}{\prod_{j=1}^{p}\Gamma(\beta_j+\lambda)\prod_{j=1}^{q}\Gamma(1-\alpha_j-\lambda)}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{n!}G_{s,r+1}^{q+1,p}\left[x\left|\begin{matrix}1-\beta_1,\cdots,1-\beta_s\\ \lambda+n,1-\alpha_1,\cdots,1-\alpha_r\end{matrix}\right.\right]$$

$$\times F\left[\begin{matrix}-n,\ (a_m),\ (\alpha_r+\lambda);\ (b_n);\ (c_h);\ (-1)^{p+q-s}y,\ (-1)^{p+q-s}z\\ (d_k),\ (\beta_s+\lambda);\ (e_g);\ (f_w);\end{matrix}\right], \quad (6)$$

provided $m+n=k+g+1$ or $m+n<k+g$, $m+h=k+w+1$ or $m+h<k+w$, $p+q>\frac{1}{2}(r+s-1)$, $|\arg x|<\left(p+q-\frac{1}{2}r-\frac{1}{2}s+\frac{1}{2}\right)\pi$.

PROOF. To prove (6) we substitute the formula (4) in the power series expansions of the hypergeometric series of two variables on the left side, then collect the terms involving the same Meijer's G-functions, we arrive at the result (6).

5. Particular cases: (a) Taking $a=d$ and $m=k$ in (6), we get

$$\frac{\prod_{j=p+1}^{s}\Gamma(1-\beta_j-\lambda)\prod_{j=q+1}^{r}\Gamma(\alpha_j+\lambda)}{\prod_{j=1}^{p}\Gamma(\beta_j+\lambda)\prod_{j=1}^{q}\Gamma(1-\alpha_j-\lambda)}\sum_{n=0}^{\infty}\frac{1}{n!}G_{s,r+1}^{q+1,p}\left[x\left|\begin{matrix}1-\beta_1,\cdots,1-\beta_s\\ \lambda+n,\ 1-\alpha_1,\cdots,1-\alpha_r\end{matrix}\right.\right]$$

$$\times F\left[\begin{matrix}-n,\ (\alpha_r+\lambda);\ (b_n);\ (c_h);\ (-1)^{p+q-s}y,\ (-1)^{p+q-s}z\\ (\beta_s+\lambda);\ (e_g);\ (f_w);\end{matrix}\right]$$

$$=x^{\lambda}{}_nF_g[(b_n);\ (e_g);\ xy]_hF_w[(c_h);\ (f_w);\ xz]. \quad (7)$$

In case $q=0$, $r=1$, $p=s=0$ and using the formula [4, p. 216, eqn. (3)] in (7), it reduces to a result due to Srivastava [6, p. 246, eqn. (2.2)].

(b) Taking $b=e$, $n=g$, $c=f$, $h=w$ in (6), we get after a little simplification.

$$x^{\lambda}{}_mF_k[(a_m);\ (d_k);\ xy]=\frac{\prod_{j=p+1}^{s}\Gamma(1-\beta_j-\lambda)\prod_{j=q+1}^{r}\Gamma(\alpha_j+\lambda)}{\prod_{j=1}^{p}\Gamma(\beta_j+\lambda)\prod_{j=1}^{q}\Gamma(1-\alpha_j-\lambda)}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\times G_{s,r+1}^{q+1,p}\left[x\left|\begin{matrix}1-\beta_1,\cdots,1-\beta_s\\ \lambda+n,\ 1-\alpha_1,\cdots,1-\alpha_r\end{matrix}\right.\right]{}_{m+r+1}F_{k+s}[-n,\ (a_m),\ (\alpha_r+\lambda);$$

$$(d_k),\ (\beta_s+\lambda);\ (-1)^{p+q-s}y]. \quad (8)$$

If we use formula [4, p. 215 eqn. (1)] in (8), it reduces to a result due to Meijer [5, p. 311, eqn. (237)].

(c) Taking $p=0$, $s=0$, $q=1$, $r=1$, $\alpha_1=1$ and using the formula [4, p. 216 eqn (4)]

$$G_{02}^{20}[x\,|\,a, b]=2x^{\frac{1}{2}(a+b)}k_{a-b}(2\sqrt{x}) \tag{9}$$

in (6), we get

$$x^{\lambda}F\begin{bmatrix}(a_m);\,(b_n);\,(c_h);\,x^2y,\,x^2z\\(d_k);\,(e_g);\,(f_w)\end{bmatrix}=\frac{1}{2\Gamma(-\lambda)}\sum_{n=0}^{\infty}\frac{x^n}{n!}K_{\lambda+n}(2x)$$

$$\times F\begin{bmatrix}-n,\,(a_m),\,1+\lambda;\,(b_n);\,(c_h);\,-y,\,-z\\(d_k);\,(e_g);\,(f_w);\end{bmatrix}. \tag{10}$$

If we further specialise the parameters in (10), we obtain

$$x^{\lambda-\alpha-\beta}J_{\alpha}(xy)J_{\beta}(xz)=\frac{y^{\alpha}z^{\beta}}{(2)^{\alpha+\beta+1}\Gamma(-\lambda)\Gamma(1+\alpha)\Gamma(1+\beta)}$$

$$\times\sum_{n=0}^{\infty}\frac{x^n}{n!}K_{\lambda+n}(2x)F_4\left(-n,\,1+\lambda;\,1+\alpha,\,1+\beta;\,\frac{1}{4}y^2,\,\frac{1}{4}z^2\right). \tag{11}$$

In conclusion we mention that many interesting particular cases can be obtained but are not presented for the sake of brevity. We can extend the formula (6) to hypergeometric series of n variables.

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