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The Geometric Method and a Generalized Bairstow Method for Numerical Solution of Polynomial Equation

By

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§1. Introduction

The Bairstow-McAuley type formulas [1, 2] are iterative methods for solving a real polynomial equation by improving approximate quadratic factors of the polynomial on the basis of Newton procedure. Therefore, how to choose the first approximate quadratic factor is in practical very important from the viewpoint of convergence and error analysis.

In the present paper we propose the geometric method which produce systematically the first approximations of all the quadratic factors. If a quadratic factor has been obtained after finite iterations starting from a first approximation, in the Bairstow-McAuley type method, the polynomial is divided by the factor and the same algorithm is then applied to the reduced polynomial. Accordingly the accumulated errors of dividing out the inaccurate factors are propagated to the later factors. To avoid this disadvantage we consider the original polynomial only and do not divide out the obtained approximate quadratic factors. Hence our method serves also for correction of the results obtained by the Bairstow-McAuley type method.

Lastly, we construct a generalized Bairstow method for calculating multiple roots or close roots which produces m-th order factors $x^m+p_1x^{m-1}+p_2x^{m-2}+\cdots+p_{m-1}x+p_m$.

The author expresses his hearty gratitude to Professor Urabe for his kind guidance and constant advice.

§2. Geometric method for two variables

We consider the real polynomial

$$P(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0 \qquad (a_n\neq 0) \tag{2.1}$$

and put

(2.2) $$P(x)=(x^2+px+q)Q(x)+R_1x^{k-1}+R_2x^{k-2} \qquad (2\leqq k\leqq n),$$

where

$$R_1=R_1(p,q),$$

$$R_2=R_2(p,q).$$

Hence our problem is to find p and q so that

(2.3) $$R_1(p,q)=0,$$

(2.4) $$R_2(p,q)=0.$$

Let M be the maximum value of $\sqrt[k]{|a_k|}$ $(k=1, 2, \cdots, n)$, then it is well known that all the zeros of $P(x)$ satisfy the inequality

$$|x|<2M.$$

Since $w=2M>0$, the zeros of the polynomial

$$P(wx)/w^n=x^n+\tilde{a}_1x^{n-1}+\tilde{a}_2x^{n-2}+\cdots+\tilde{a}_{n-1}x+\tilde{a}_n,$$

where

$$\tilde{a}_k=a_k/w^k \qquad (k=1, 2, \cdots, n),$$

lie inside the unit circle $|x|=1$.

Hence for simplicity, we may suppose that all the roots of (2.1) lie inside the unit circle $|x|=1$ and that all the solutions of the simultaneous equations (2.3)$\sim$(2.4) exist in the rectangular region $R=\{(p,q):|p|<2, |q|<1\}$.

In the previous paper [4], the author proposed a new method for numerical solution of a system of nonlinear equations. We shall again apply the method to determine the solutions of the simultaneous equations (2.3)$\sim$(2.4) in the rectangular region R. In geometrically, the method is to calculate all the intersection points of two plane curves (2.3)$\sim$(2.4) in the region R. That is to say, we take a point (p_0, q_0) on the curve

$$C;\ p=p(s),\ q=q(s)$$

which is determined by $R_1(p,q)=0$, and then we compute the curve integrating numerically the initial value problem:

$$(2.5)\qquad \begin{cases} \dfrac{dp}{ds}=\pm\dfrac{\dfrac{\partial R_1}{\partial q}}{\sqrt{\left(\dfrac{\partial R_1}{\partial p}\right)^2+\left(\dfrac{\partial R_1}{\partial q}\right)^2}}, \\ \\ \dfrac{dq}{ds}=\mp\dfrac{\dfrac{\partial R_1}{\partial p}}{\sqrt{\left(\dfrac{\partial R_1}{\partial p}\right)^2+\left(\dfrac{\partial R_1}{\partial q}\right)^2}}, \end{cases}$$

$$p(0)=p_0,\ q(0)=q_0,$$

by a step-by-step method.

Let (p_l, q_l) $(l=1, 2, \dots)$ be an approximate value of $(p(s), q(s))$ obtained at the l-th step by the numerical integration. Then we may have $R_2(p_0, q_0) \cdot R_2(p_1, q_1) \leqq 0$. Otherwise we continue the numerical integration of (2.5) until we have

$$(2.6)\qquad R_2(p_{l-1}, q_{l-1}) \cdot R_2(p_l, q_l) \leqq 0.$$

Once we have had (2.6) for some l, we check if $|R_2(p_{l-1}, q_{l-1})|$ or $|R_2(p_l, q_l)|$ is smaller than a specified positive number ε. If this is not satisfied, we multiply the step-size of the numerical integration by $2^{-p} (p \geqq 1)$ and repeat the numerical integration starting from the point (p_{l-1}, q_{l-1}). If we repeat this process, then after a finite number of repetitions we shall have

$$(2.7)\qquad \begin{cases} R_2(p_{l-1}, q_{l-1}) \cdot R_2(p_l, q_l) \leqq 0, \\ \text{and} \\ |R_2(p_{l-1}, q_{l-1})| \text{ or } |R_2(p_l, q_l)| < \varepsilon, \end{cases}$$

provided on the curve C there is a simple solution of the simultaneous equations (2.3) $\sim$ (2.4), that is, a solution of (2.3) $\sim$ (2.4) for which the Jacobian $\partial(R_1, R_2)/\partial(p, q)$ does not vanish. The values $p=p_{l-1}$, $q=q_{l-1}$ or $p=p_l$, $q=q_l$ satisfying (2.7) give an approximate solution of (2.3) $\sim$ (2.4).

Starting from $p=p_{l-1}$, $q=q_{l-1}$ or $p=p_l$, $q=q_l$, we can compute two roots of the polynomial equation (2.1) by Bairstow method or McAuley method. However, if ε is very small, the quadratic factor $x^2+p_{l-1}x+q_{l-1}$ or $x^2+p_l x+q_l$ will produce sufficiently accurate two roots of the polynomial equation (2.1).

In the rectangular region R the curve $R_1(p, q)=0$ may generally consist of some different disconnected curves. Hence we separate the region R into some subregions D_i:

$$R=D_1+D_2+\cdots+D_{m-1} \text{ (Fig. 1).}$$

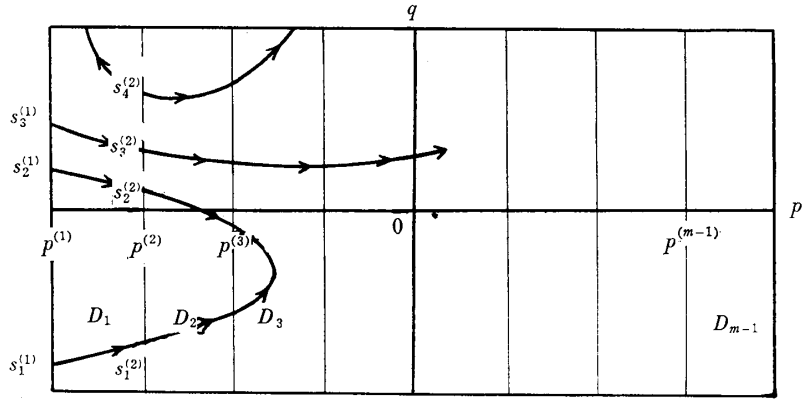

Fig. 1

In order to calculate systematically all the intersection points of the two plane curves (2.3)$\sim$(2.4) in the region R, at first, we calculate all the intersection points $s_1^{(1)}$, $s_2^{(1)}$, $s_3^{(1)}$ of the curve $R_1(p, q)=0$ and the straight line $p=p^{(1)}$. Starting from these intersection points $s_i^{(1)}$ $(i=1, 2, 3)$ we trace the curve $R_1(p, q)=0$ numerically until we reach the boundary of D_1, and store the end points $s_1^{(2)}$, $s_2^{(2)}$, $s_3^{(2)}$ of the curve intersecting with the straight line $p=p^{(2)}$. Next, taking account of the above stored points, we calculate all the intersection points $s_i^{(2)}$ $(i=1, 2, 3, 4)$ of the curve $R_1(p, q)=0$ and the straight line $p=p^{(2)}$ and starting from these points $s_i^{(2)}$ $(i=1, 2, 3, 4)$ we trace the curve $R_1(p, q)=0$ numerically until we reach the boundary of D_2.

Note that starting from the new point $s_4^{(2)}$ we have to trace the curve in two directions (Fig. 1). During this procedure we of course compute each intersection point of the two curves $R_1(p, q)=0$ and $R_2(p, q)=0$ accurately. We repeat the above process.

Thus we obtain all the roots of the real polynomial equation (2.1). When the coefficients a_i $(i=1, 2, \cdots, n)$ of (2.1) are complex numbers, we also have the simultaneous equations with real coefficients:

$$\text{(2.8)} \qquad \begin{cases} f(p, q)=0, \\ g(p, q)=0, \end{cases}$$

where $P(x=p+iq)=f(p, q)+ig(p, q)$.

§3. Convergence of the Bairstow-McAuley type formulas and a generalized Bairstow method

We consider, at first, the convergence of the Bairstow-McAuley type formulas and then study a generalized Bairstow method for computing multiple roots or close roots.

We differentiate the identity (2.2) with respect to p or q. Since $P(x)$ does not depend on p or q, we have

$$\frac{\partial R_1}{\partial p}x^{k-1}+\frac{\partial R_2}{\partial p}x^{k-2}+xG(x)+(x^2+px+q)\frac{\partial G}{\partial p}=0,$$

$$\frac{\partial R_1}{\partial q}x^{k-1}+\frac{\partial R_2}{\partial q}x^{k-2}+G(x)+(x^2+px+q)\frac{\partial G}{\partial q}=0.$$

Let $x^2+\alpha x+\beta$ be the exact quadratic factor of the polynomial $P(x)$ corresponding to the approximate quadratic factor x^2+px+q, and let z_1, z_2 be two roots of $x^2+\alpha x+\beta=0$. Setting $p=\alpha$, $q=\beta$, $x=z_i$, $G_i=G(z_i)$ $(i=1, 2)$, we obtain

(3.1) $$\frac{\partial R_1}{\partial p}z_i^{k-1}+\frac{\partial R_2}{\partial p}z_i^{k-2}=-z_iG_i \qquad (i=1, 2),$$

(3.2) $$\frac{\partial R_1}{\partial q}z_i^{k-1}+\frac{\partial R_2}{\partial q}z_i^{k-2}=-G_i \qquad (i=1, 2).$$

If $z_1 \neq z_2$, we have from (3.1), (3.2),

$$\frac{\partial R_1}{\partial p}=(z_1^{k-3}G_2-z_2^{k-3}G_1)/((z_1z_2)^{k-3}(z_1-z_2)),$$

$$\frac{\partial R_2}{\partial p}=(z_2^{k-2}G_1-z_1^{k-2}G_2)/((z_1z_2)^{k-3}(z_1-z_2)),$$

$$\frac{\partial R_1}{\partial q}=(z_1^{k-2}G_2-z_2^{k-2}G_1)/((z_1z_2)^{k-2}(z_1-z_2)),$$

$$\frac{\partial R_2}{\partial q}=(z_2^{k-1}G_1-z_1^{k-1}G_2)/((z_1z_2)^{k-2}(z_1-z_2)).$$

Hence we obtain

(3.3) $$\frac{\partial(R_1, R_2)}{\partial(p, q)}(\alpha, \beta)=\left(\frac{1}{z_1z_2}\right)^{k-2}G_1\cdot G_2.$$

If $z_1=z_2$, we use the identity

$$P'(x)=(2x+p)G(x)+(x^2+px+q)G'(x)+(k-1)R_1x^{k-2}$$
$$+(k-2)R_2x^{k-3}.$$

Since $P'(x)$ does not depend on p or q, we have

$$(k-1)\,x^{k-2}\frac{\partial R_1}{\partial p}+(k-2)\,x^{k-3}\frac{\partial R_2}{\partial p}+(x^2+px+q)\frac{\partial G'}{\partial p}$$

$$+xG'(x)+(2x+p)\frac{\partial G}{\partial p}+G(x)=0,$$

$$(k-1)\,x^{k-2}\frac{\partial R_1}{\partial q}+(k-2)x^{k-3}\frac{\partial R_2}{\partial q}+(x^2+px+q)\frac{\partial G'}{\partial q}$$

$$+G'(x)+(2x+p)\frac{\partial G}{\partial q}=0.$$

Setting $p=\alpha$, $q=\beta$, $x=z\equiv-\alpha/2$, we have

$$\frac{\partial R_1}{\partial p}z^{k-1}+\frac{\partial R_2}{\partial p}z^{k-2}=-zG(z),$$

$$(k-1)\frac{\partial R_1}{\partial p}z^{k-2}+(k-2)\frac{\partial R_2}{\partial p}z^{k-3}=-G(z)-zG'(z)$$

and

$$\frac{\partial R_1}{\partial q}z^{k-1}+\frac{\partial R_2}{\partial q}z^{k-2}=-G(z),$$

$$(k-1)\frac{\partial R_1}{\partial q}z^{k-2}+(k-2)\frac{\partial R_2}{\partial q}z^{k-3}=-G'(z).$$

Hence we obtain

$$\frac{\partial(R_1, R_2)}{\partial(p, q)}(\alpha, \beta)=\frac{G^2(z)}{z^{2k-4}}\,. \tag{3.4}$$

From (3.3) and (3.4), if $G(z_1)\cdot G(z_2)\neq 0$, a sequence of quadratic polynomial $x^2+p_{(k)}x+q_{(k)}$ $(k=1, 2, \dots)$ converges quadratically to $x^2+\alpha x+\beta$, starting from $p_{(1)}=p_l$, $q_{(1)}=q_l$, which satisfy (2.7). Proof above is a generalization of the result given by Henrici [1].

It will be observed from the above analysis that if z_1 (or z_2) has multiplicity m which is three at least the Bairstow-McAuley type methods will not converge quadratically.

In order to achieve quadratic convergence, we consider the following algorithm:

(3.5) $$P(x)=(x^m+p_1 x^{m-1}+p_2 x^{m-2}+\cdots+p_{m-1} x+p_m)\cdot(x^{n-m}+b_1 x^{n-m-1}+b_{n-m-1}x+b_{n-m})+R_1 x^{m-1}+R_2 x^{m-2}+\cdots+R_{m-1}x+R_m.$$

The quantities $b_1, b_2, \cdots, b_{n-m}, R_1, R_2, \cdots, R_m$ can be found recursively. Conveniently, we introduce the other quantities $b_{n-1}, b_{n-2}, \cdots, b_{n-m+1}$ and define:

$$b_k=a_k-p_1 b_{k-1}-p_2 b_{k-2}-\cdots-p_m b_{k-m} \qquad (k=1, 2, \cdots, n),$$

with

$$b_0=1,\ b_{-1}=b_{-2}=\cdots=b_{-(m-1)}=0.$$

Then we get

(3.6) $$\begin{cases} R_1=b_{n-m+1}, \\ R_2=b_{n-m+2}+p_1 b_{n-m+1}, \\ \vdots \\ R_l=b_{n-m+l}+p_1 b_{n-m+(l-1)}+\cdots+p_j b_{n-m+(l-j)}+\cdots \\ \quad +p_{l-1} b_{n-m+1}, \\ \vdots \\ R_m=b_n+p_1 b_{n-1}+\cdots+p_{m-1} b_{n-m+1}. \end{cases}$$

Differentiating b_k with p_l, we have

$$\frac{\partial b_k}{\partial p_l}=-b_{k-l}-p_1\frac{\partial b_{k-1}}{\partial p_l}-\cdots-p_m\frac{\partial b_{k-m}}{\partial p_l}.$$

Putting $\partial b_k/\partial p_l=-c_{k-l}$, we obtain easily

$$c_{k-l}=b_{k-l}-p_1 c_{k-l-1}-\cdots-p_m c_{k-l-m},$$

with

$$c_0=1,\ c_{-1}=c_{-2}=\cdots=c_{-(m-1)}=0.$$

Hence we have

(3.7) $$\frac{\partial R_l}{\partial p_j}=[b_{n-m+(l-j)}-c_{n-m+(l-j)}]-\sum_{i=1}^{l-1} p_i c_{n-m+l-i-j}.$$

Our problem is then to find $p_1, p_2, \cdots, p_{m-1}$ and p_m, so that

$$
(3.8)\qquad \begin{cases} R_1(p_1, p_2, \cdots, p_m)=0, \\ R_2(p_1, p_2, \cdots, p_m)=0, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ R_l(p_1, p_2, \cdots, p_m)=0, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ R_m(p_1, p_2, \cdots, p_m)=0. \end{cases}
$$

Using (3.6) and (3.7), we apply the Newton method to these equations (3.8). The starting values of p_l $(l=1, 2, \cdots, m)$ may be chosen as follows:

$$
p_l=(-1)^l \cdot {}_mC_l z_1^l \qquad (l=1, 2, \cdots, m).
$$

If we choose m such that the two factors $x^m+p_1 x^{m-1}+\cdots+p_{m-1} x+p_m$, $x^{n-m}+b_1 x^{n-m-1}+\cdots+b_{n-m-1} x+b_{n-m}$ have no common zeros when $R_1=R_2=\cdots=R_m=0$, then the convergence of this method is quadratic [3]. The multiplicity m can be computed easily from the following procedure.

Let L be a simple closed curve in the complex plane and $P(z)\neq 0$ on L. Let the point z march along L in the counterclockwise direction and we put $P(z)=\varphi+i\psi$. Then the number of zeros of $P(z)$ inside L is equal to

$$
m=\frac{1}{2}\sum \chi,
$$

where χ takes the following value whenever $\varphi=0$.

$$
\begin{cases} \chi=-1 & \text{when the sign of } \varphi\cdot\psi \text{ tends from negative to positive,} \\ \chi=+1 & \text{when the sign of } \varphi\cdot\psi \text{ tends from positive to negative,} \\ \chi=0 & \text{when the sign of } \varphi\cdot\psi \text{ does not change before and after of } \varphi=0. \end{cases}
$$

For the proof, see [5], pp. 99–102.

In general, the algorithm (3.5) produces m-th order factors. Specially, when m is equal to three or four, we obtain the exact roots of (2.1) by Cardano's formula or Ferrari's formula.

§4. Numerical examples

1. As a first example, let us consider the polynomial equation

$$
(4.1)\qquad x^5-2x^4+10x^3-9x+3=0.
$$

Setting

$$x = p + iq,$$

we have the following simultaneous equations

$$(4.2) \qquad \begin{cases} f(p, q) = 0, \\ g(p, q) = 0, \end{cases}$$

where

$$f(p, q) = p^5 - 10p^3q^2 + 5pq^4 - 2p^4 + 12p^2q^2 - 2q^4 + 10p^3 - 30pq^2 - 9p + 3,$$

and

$$g(p, q) = q^5 - 10p^2q^3 + 5p^4q - 8p^3q + 8pq^3 + 30p^2q - 10q^3 - 9q.$$

Hence, we apply the geometric method to find all the roots of the equations (4.2).

The graph obtained by plotting of the numerical results is shown in Fig. 2. On the computer TOSBAC 3400, we have five approximations:

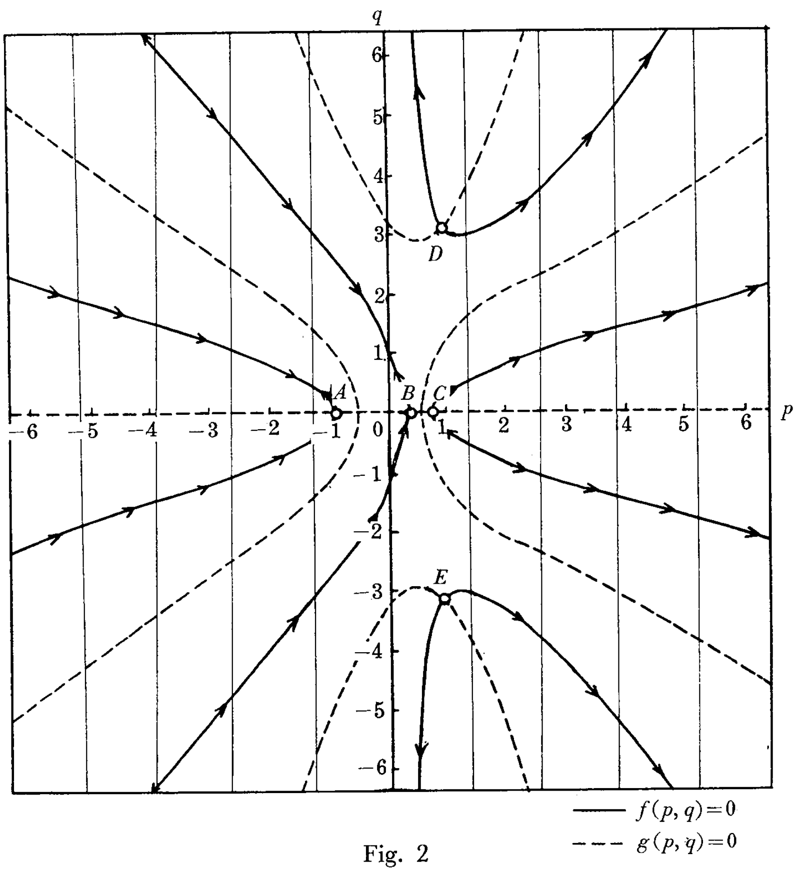

Fig. 2

$A(-0.9691573277, 0)$,

$B(0.3997906784, 0)$,

$C(0.7374430457, 0)$,

$D(0.9159618018, 3.108125866)$,

$E(0.9159618019, -3.108125866)$.

In order to get an error bound for the approximate solution

$$\hat{\boldsymbol{x}} = \{\hat{p}, \hat{q}\} = \{0.3997906784, 0.0000000000\}, \tag{4.3}$$

we apply Urabe's proposition [4, 6] to the equation (4.2) using the Euclidean norms. Put

$$\boldsymbol{x} = \{p, q\}, \ \boldsymbol{F}(\boldsymbol{x}) = \{f(p, q), \ g(p, q)\}, \tag{4.4}$$

then from

$$f(\hat{p}, \hat{q}) = 0.2910383046 \times 10^{-10},$$
$$g(\hat{p}, \hat{q}) = 0.3795507282 \times 10^{-22},$$

readily follows

$$\|\boldsymbol{F}(\hat{\boldsymbol{x}})\| < 0.292 \times \sqrt{2} \times 10^{-10} < r = 0.292 \times 1.415 \times 10^{-10}. \tag{4.5}$$

Put

$$J(\boldsymbol{x}) = \begin{bmatrix} f_p(p, q) & f_q(p, q) \\ g_p(p, q) & g_q(p, q) \end{bmatrix},$$

then for $\boldsymbol{x} = \hat{\boldsymbol{x}}$, we have

$$J(\hat{\boldsymbol{x}}) = \begin{bmatrix} -0.4588486751 \times 10^1 & 0.0000000000 \\ 0.0000000000 & -0.4588486751 \times 10^1 \end{bmatrix}$$

and hence

$$\|J^{-1}(\hat{\boldsymbol{x}})\| < M = 0.309. \tag{4.6}$$

Let Ω be the region such that

$$\Omega=\{\boldsymbol{x}=(p, q): |p-\hat{p}| \leqq H, \ |q-\hat{q}| \leqq H\}$$

where $H=0.0625$. Then, computing the values of $\|J(\boldsymbol{x})-J(\hat{\boldsymbol{x}})\|$ for grid points

$$x=x_{ij}=(p_i, q_j)=\left(\hat{p}+\frac{H}{8}i, \hat{q}+\frac{H}{8}j\right) \qquad (i, j=0, \pm 1, \cdots, \pm 8),$$

we see that

(4.7) $$\|J(\boldsymbol{x})-J(\hat{\boldsymbol{x}})\| \leqq 2.883$$

for any $\boldsymbol{x} \in \Omega$. Put

(4.8) $$\delta=0.0625$$

then evidently

(4.9) $$\Omega \supset \Omega_\delta=\{\boldsymbol{x}: \|\boldsymbol{x}-\hat{\boldsymbol{x}}\| \leqq \delta\},$$

and we have (4.7) for any $\boldsymbol{x} \in \Omega_\delta$. Hence by (4.5), (4.6), (4.7) and (4.8), we see that the conditions (ii) and (iii) in Urabe's proposition [4, 6] are fulfilled if there is a positive number $\kappa<1$ satisfying the following inequalities:

(4.10) $$\begin{cases} 2.883 \leqq \dfrac{\kappa}{0.309}, \\ \dfrac{0.309 \times 0.292 \times 1.415 \times 10^{-10}}{1-\kappa} \leqq 0.0625. \end{cases}$$

These inequalities are equivalent to the inequality

$$2.883 \times 0.309 \leqq \kappa \leqq 1-\frac{0.309 \times 0.292 \times 1.415 \times 10^{-10}}{0.0625},$$

that is,

(4.11) $$0.890847 \leqq \kappa \leqq 0.9999999997\cdots.$$

Hence, indeed, there is a positive number $\kappa<1$ satisfying (4.10). This proves that all the conditions of Urabe's proposition are fulfilled by the approximate solution $\boldsymbol{x}=\hat{\boldsymbol{x}}$. Thus we see that the equation (4.2) posseses one and only one exact solution $\boldsymbol{x}=\bar{\boldsymbol{x}}$ in Ω_δ and that

(4.12) $$\|\hat{\boldsymbol{x}}-\bar{\boldsymbol{x}}\| \leqq \frac{0.309 \times 0.292 \times 1.415 \times 10^{-10}}{1-\kappa},$$

where κ is an arbitrary number satisfying (4.11). From (4.11) and (4.12), we then see that

$$\|\hat{\boldsymbol{x}}-\bar{\boldsymbol{x}}\| \leqq \frac{0.309\times 0.292\times 1.415\times 10^{-10}}{1-0.890847} < 1.17\times 10^{-10} \qquad (4.13)$$

which gives an error bound for the approximate solution $\boldsymbol{x}=\hat{\boldsymbol{x}}=\{\hat{p}, \hat{q}\}$ given by (4.3).

2. As a second example, we choose the polynomial equation

$$x^7-3x^6-x^5+x^4+4x^3+62x^2+96x+40=0,$$

whose exact roots are $3\pm i$, $\pm 2i$ and -1 (triple).

On the computer TOSBAC 3400, our geometric method gave the first approximations as

$$\begin{cases} p_1=-0.6000000000\times 10^1, \\ p_2=0.1000000000\times 10^2, \end{cases}$$

$$\begin{cases} p_1=0.5156141518\times 10^{-11}, \\ p_2=0.4000000000\times 10^1, \end{cases}$$

and

$$\begin{cases} z_1=-0.9985308960, \\ p_1=-3z_1, \\ p_2=3z_1^2, \\ p_3=-z_1^3. \end{cases}$$

Starting from these approximations, the generalized Bairstow method gave the roots as

$$\begin{array}{ll} 0.3000000000\times 10^1 & -0.1000000000\times 10^1\ i, \\ 0.3000000000\times 10^1 & +0.1000000000\times 10^1\ i, \\ -0.2578070759\times 10^{-11} & -0.2000000000\times 10^1\ i, \\ -0.2578070759\times 10^{-11} & +0.2000000000\times 10^1\ i, \\ -0.10000000000\times 10^1 & +0.0000000000\times 10^0\ i, \end{array}$$

$$-0.1000000000 \times 10^{1} \qquad -0.6301164131 \times 10^{-11}\ i,$$

$$-0.1000000000 \times 10^{1} \qquad +0.6301164131 \times 10^{-11}\ i.$$

The McAuley method and the Bairstow method gave very different results according to the different starting values. Starting from a first approximation $p_1=1.0$, $p_2=1.0$, the McAuley method gave the roots as

$$0.3000000000 \times 10^{1} \qquad +0.9999999999 \times 10^{0}\ i,$$

$$0.3000000000 \times 10^{1} \qquad -0.9999999999 \times 10^{0}\ i,$$

$$-0.9997102440 \times 10^{0} \qquad +0.0000000000 \times 10^{0}\ i,$$

$$-0.1000389817 \times 10^{1} \qquad +0.0000000000 \times 10^{0}\ i,$$

$$-0.1474190788 \times 10^{-6} \qquad +0.1999999803 \times 10^{1}\ i,$$

$$-0.1474190788 \times 10^{-6} \qquad -0.1999999803 \times 10^{1}\ i,$$

$$-0.9999002584 \times 10^{0} \qquad +0.0000000000 \times 10^{0}\ i,$$

and the Bairstow method gave the roots as

$$-0.1000107275 \times 10^{1} \qquad +0.1858270563 \times 10^{-3}\ i,$$

$$-0.1000107275 \times 10^{1} \qquad -0.1858270563 \times 10^{-3}\ i,$$

$$0.3713741546 \times 10^{1} \qquad +0.0000000000 \times 10^{0}\ i,$$

$$-0.9997854486 \times 10^{0} \qquad +0.0000000000 \times 10^{0}\ i,$$

$$0.7321240536 \times 10^{0} \qquad +0.1941595805 \times 10^{1}\ i,$$

$$0.7321240536 \times 10^{0} \qquad -0.1941595805 \times 10^{1}\ i,$$

$$0.8220103452 \times 10^{0} \qquad +0.0000000000 \times 10^{0}\ i.$$

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