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A Note on General Projective Spaces of Paths and Tangent Bundles I

By

Yoshihiro ICHIJYÔ

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§1. φ-spaces and φ-paths

First of all, we consider a φ-space $(M, T(M), \tau, \varphi, \tilde{\mathfrak{X}}(M))$, which is defined in the preceding paper[1]. Of course M is an n-dimensional differentiable C^∞ manifold, $T(M)$ is its tangent bundle, τ is the canonical projection $T(M)\rightarrow M$, φ is a homogeneous φ-connection (i.e., non-linear connection) and $\tilde{\mathfrak{X}}(M)=d\tau\cdot\mathfrak{X}(T(M))$[2].

Take a point $z=(x, y)$ on $T(M)$ where x is a point of M and y is a tangent vector of M at the point x, then we can consider a vertical lift of y to the point $z=(z, y)$ and denote it $(Y)_z$, i.e.,

$$(Y)_z=(y)^v_{z=(x,y)}. \tag{1.1}$$

Then Y_z form a vector field Y on $T(M)$. Similarly, we can take a vector field X_φ defined by a horizontal lift of y to (x, y), i.e.,

$$(X_\varphi)_z=(y)^h_{z=(x,y)} \tag{1.2}$$

In a local canonical coordinate system, X_φ and Y can be represented by

$$(X_\varphi)_z=y^i\Big(\frac{\partial}{\partial x^i}\Big)_{(z=x,y)}-\varphi^i_m(x, y)y^m\Big(\frac{\partial}{\partial y^i}\Big)_{z=(x,y)},\ (Y)_z=y^i\Big(\frac{\partial}{\partial y^i}\Big)_{z=(x,y)}. \tag{1.3}$$

Now we define over $T(M)$ a 2-dimensional distribution D^2_φ which is spanned by X_φ and Y. Direct calculation leads us to $[X_\varphi, Y]=-X_\varphi$. Hence this distribution D^2_φ is integrable. Thus we denote by S^2_φ its integral submanifold.

Next, we shall define *a path* (or *φ-path*) in M. A curve C in M is called a path with respect to φ (or a φ-path) if the curve C satisfies

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1) Y. ICHIJYÔ: Almost complex structures of tangent bundles and Finsler metrics, J. Math. Kyoto Univ. 6 (1967) 419-452.

2) The terminology and signes of the preceding paper will be used in this paper without too much comment.

(1.4) $$\nabla^{\#}_{\dot{x}}\dot{x} = p\dot{x}$$

where $\dot{x}$ is a tangent vector along C, p is a scalar and $\nabla^{\#}$ is a covariant differential with respect to the φ-connection along the curve C.

Proposition 1. *A curve C in M is a path if and only if the natural lift $\tilde{C}$ of C is included in the submanifold S^2_φ in $T(M)$.*

Proof. Take a tangent vector U_z of $\tilde{C}$ at any point $z=(x, \dot{x})$ of $\tilde{C}$ in $T(M)$, then U_z is written in a canonical coordinate system as $U_z=\left(\dot{x}^i\frac{\partial}{\partial x^i}+\ddot{x}^i\frac{\partial}{\partial y^i}\right)_{z=(x,\dot{x})}$. If C is a path, then U_z is rewritten as

$$\begin{aligned}U_z&=\left(\dot{x}^i\frac{\partial}{\partial x^i}+\{p\dot{x}^i-\varphi^i_m(x, \dot{x})\dot{x}^m\}\frac{\partial}{\partial y^i}\right)_{z=(x,\dot{x})}\\&=(X_\varphi+pY)_z.\end{aligned}$$

Thus we have $U_z\in D^2_\varphi$.

Conversely, if $U_z\in D^2_\varphi$, then it follows that

$$U_z=(\alpha X_\varphi+\beta Y)_{z=(x,\dot{x})},$$

which implies $\alpha=1$ and $\nabla^{\#}_{\dot{x}}\dot{x}=\beta\dot{x}$.

§2. Affine vector field

The horizontal vector field X_φ defined by (1.3) has a form $(X_\varphi)_z=\left(y^i\frac{\partial}{\partial x^i}-\gamma^i\frac{\partial}{\partial y^i}\right)_{z=(x,y)}$, where $\gamma^i=\varphi^i_m(x, y)y^m$. The quantities γ^i have a law of transformation

(2.1) $$\bar{\gamma}^{i'}=\frac{\partial\bar{x}^{i'}}{\partial x^l}\gamma^l-\frac{\partial^2\bar{x}^{i'}}{\partial x^l\partial x^m}y^l y^m.$$

Hence it follows directly that the quantities

$$\gamma^i_k=\frac{1}{2}\dot{\partial}_k\gamma^i=\frac{1}{2}(\varphi^i_k(x, y)+\dot{\partial}_k\varphi^i_m(x, y)y^m)$$

define a new non-linear connection $\gamma(\varphi)$ satisfying

(2.2) $$T_\gamma=0, \quad X_\gamma=X_\varphi,$$

where T_γ is a torsion with respect to $\gamma(\varphi)$.

Conversely, as for a vector field X_φ, if any non-linear connection φ' satisfies $T_{\varphi'}=0$ and $X_{\varphi'}=X_\varphi$, then $\varphi'=\gamma(\varphi)$. Because the relation $X_{\varphi'}=X_\varphi$ gives $\gamma^i_k = \frac{1}{2}(\partial_k\varphi'^i_m y^m+\varphi'^i_k)$. And $T_{\varphi'}=0$ leads us to $\gamma^i_k=\frac{1}{2}(\partial_m\varphi'^i_k y^m+\varphi'^i_k)=\varphi'^i_k$.

Thus we call the $\gamma(\varphi)$ *a symmetric non-linear connection derived from* φ. Of course, the relation $D^2_\varphi=D^2_\gamma$ holds good.

A vector field A on $T(M)$ is called *an affine vector field* on $T(M)$ if A satisfies the following conditions;

$$\begin{cases} d\tau\cdot A_{(x,y)}=y_x, \\ d\lambda^*\cdot A_{(x,y)}=A_{(x,\lambda y)} \end{cases} \tag{2.3}$$

where λ^* is a mapping $T(M)\to T(M)((x, y)\to(x, \lambda y), \lambda>0)$.

An affine vector field A has, therefore, components of $(y^i, -\gamma^i)$ in a canonical coordinate system. Since A is a vector field on $T(M)$, γ^i satisfy the law of transformation (2.1). Thus $\gamma^i_k=\frac{1}{2}\partial_k\gamma^i$ give a symmetric non-linear connection γ. Of course $(2.3)_2$ gives that $X_\gamma=A$.

Conversely, the above mentioned result shows us that γ is a uniquely given non-linear connection which preserves A horizontal and is symmetric. Hence the non-linear connection γ thus defined is called, hereafter, *a symmetric non-linear connection derived from A*. On the other hand if an affine vector field A is given, M becomes a general affine space of path with respect to γ^i, $\gamma^i_k=\frac{1}{2}\partial_k\gamma^i$ and $\gamma^i_{kj}=\partial_j\gamma^i_k$. Hence we obtain directly the

Proposition 2. *In order that a manifold M is a general affine space of path, it is necessary and sufficient that the $T(M)$ admits an affine vector field satisfying* (2.3).

§3. General projective space of path

Let us now assume that two non-linear connections φ and $\bar\varphi$ are given. If any path with respect to φ is, at the same time, a path with respect to $\bar\varphi$ and vice versa, then φ and $\bar\varphi$ are called *projective* and denoted by $\varphi\wedge\!\!\!\!\nwarrow\bar\varphi$.

Proposition 3. *In order that non-linear connections φ and $\bar\varphi$ are mutually projective, it is necessary and sufficient that the relation $D^2_\varphi=D^2_{\bar\varphi}$ holds good.*

Proof. If $\varphi\wedge\!\!\!\!\nwarrow\bar\varphi$, the relation $\bar\varphi^i_m y^m-\varphi^i_m y^m=py^i$ is true. Hence we have $X_\varphi=X_{\bar\varphi}+pY$. This leads us to $D^2_\varphi=D^2_{\bar\varphi}$. The converse is evident.

A symmetric non-linear connection $\gamma(\varphi)$ derived from a non-linear connection φ is, of course, projective to φ.

A 2-dimensional distribution D^2 in $T(M)$ is called *projective distribution* if it satisfies

$$(3.1)\quad \begin{cases} (1) & Y\in D^2, \\ (2) & D^2 \text{ admits, at least, a vector field } X \text{ satisfying } d\tau\cdot X_{(x,y)}=y_x. \end{cases}$$

Then D^2_φ is evidently an example of a projective distribution.

Let us now assume that a tangent bundle $T(M)$ admits a projective distribution D^2. Then the basic vector field X in D^2 is a kind of affine vector field. In a canonical coordinate system, we represent X as $(y^i, -A^i)$. Then the quantities $G^j = A^j - (\partial_l A^l)y^j/_{n+1}$ become invariant with respect to the choice of the basic vector field X. But the law of transformation of the G^j is given by

$$(3.3)\qquad \bar{G}^{j'} = \frac{\partial \bar{x}^{j'}}{\partial x^l} G^l - \frac{\partial^2 \bar{x}^{j'}}{\partial x^p \partial x^q} y^p y^q + \frac{2y^l \partial_l \log \Delta}{n+1} \bar{y}^{j'},$$

where we put $\Delta = \left| \dfrac{\partial \bar{x}^{i'}}{\partial x^j} \right|$

Now take a canonical parameter ρ^* in $T(M)$ which satisfies

$$(3.4)\quad \begin{cases} (1) & \rho^* \text{ is positively homogeneous of degree 1 with respect to } y, \\ (2) & \rho^* \text{ is independent to the choice of } X, \\ (3) & \text{the law of transformation of } \rho^* \text{ is given by} \\ & \qquad \bar{\rho}^* = \rho^* + 2y^l \partial_l \log \Delta. \end{cases}$$

If we put

$$(3.5)\qquad \Gamma^i = G^i - \frac{\rho^*}{n+1} y^i,$$

then $\left(y^i \dfrac{\partial}{\partial x^i} - \Gamma^i \dfrac{\partial}{\partial y^i} \right)$ form an affine vector field over $T(M)$, which we call *an affine vector field with respect to* D^2 *and* ρ^*, and denote it by Γ. The vector field Γ is, of course, independent to the choice of X.

Proposition 4. *Let a tangent bundle* $T(M)$ *admit a projective distribution* D^2 *and a canonical parameter* ρ^*. *A 2-dimensional distribution* D^2_Γ, *which is spanned by the vector field* Y *and the affine vector field* Γ *with respect to the* D^2 *and* ρ^*, *coincides with the given projective distribution* D^2, i.e., *the relation*

$D^2_\Gamma = D^2$ *holds good.*

PROOF. In order to prove the Proposition, it is sufficient to show that $\Gamma \in D^2$. This follows at once from the relation

$$\Gamma = X + \frac{\dot{\partial}_l A^l + \rho^*}{n+1} Y.$$

The above arguments show that a manifold whose tangent bundle $T(M)$ admits a projective distribution D^2 is a so-called general projective space of path.

§4. Natural almost complex structures in a general projective space of path

If a tangent bundle $T(M)$ admits a non-linear connection φ, the $T(M)$ also admits a family of almost complex structures $J_\varphi(\rho, \alpha)$ which is defined by

$$\text{(4.1)} \qquad \begin{cases} J_\varphi(\rho, \alpha)\cdot u^h = \alpha u^h - \dfrac{1+\alpha^2}{\rho} u^v, \\ J_\varphi(\rho, \alpha)\cdot u^v = \rho u^h - \alpha u^v, \end{cases}$$

where ρ and α are any scalar fields on the $T(M)$.

Especially the family of almost complex structures $J_\varphi(-1, \alpha)$ is called a family of natural almost complex structures and is denoted by $J(\varphi, \alpha)$. The components of $J(\varphi, \alpha)$ in a canonical coordinate system are given by

$$\text{(4.2)} \qquad (J^A_B(\varphi, \alpha)) = \begin{pmatrix} \alpha E_n - \varphi, & -E_n \\ \varphi^2 - 2\alpha\varphi + (1+\alpha^2) E_n, & \varphi - \alpha E_n \end{pmatrix}.$$

If another non-linear connection φ' is given and the relation $J(\varphi', \alpha') = J(\varphi, \alpha)$ holds good, then the straightforward calculation gives us that $\varphi' - \alpha' E_n = \varphi - \alpha E_n$. And the converse is also true.

Proposition 5. *As for two non-linear connections φ and φ' the projective distribution D^2_φ in a tangent bundle $T(M)$ is preserved invariant by the family of natural almost complex structures $J(\varphi', \alpha)$ if and only if the relation $\varphi \overline{\wedge} \varphi'$ holds good.*

PROOF. If the relation $\varphi \overline{\wedge} \varphi'$ holds good then $D^2_\varphi = D^2_{\varphi'}$ is also true. Hence D^2_φ is spanned by the vector fields Y and $X_{\varphi'}$. Now we have

$$\begin{cases} J(\varphi', \alpha)\cdot X_{\varphi'} = \alpha X_{\varphi'} + (1+\alpha^2)Y \in D_\varphi^2, \\ J(\varphi', \alpha)\cdot Y = -X_{\varphi'} - \alpha Y \in D_\varphi^2. \end{cases}$$

Hence $J(\varphi', \alpha)$ preserves D_φ^2 invariant.

Conversely, if $J(\varphi', \alpha)$ preserves D_φ^2 invariant, the relation $J(\varphi', \alpha)\cdot Y \in D_\varphi^2$ holds good. On the other hand the relation $J(\varphi', \alpha) = -X_{\varphi'} - \alpha Y$ also holds good. Thus we obtain $X_{\varphi'} \in D_\varphi^2$, i.e., $\varphi \overline{\wedge} \varphi'$.

College of General Education
University of Tokushima