博士論文　　(要約)

論文題目

Numerical and mathematical analysis for blow-up phenomena to nonlinear wave equations

(非線形波動方程式の爆発現象に関する数値・数学解析)

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Preface

Blow-up phenomena are one of important problems in the theory of nonlinear partial differential equations (PDEs). Since the behavior of solutions of PDEs near the blow-up time is a meaningful study, the numerical study of them is also crucial from the standpoint of mathematical study. In this paper, we study numerical analysis of blow-up phenomena for nonlinear wave equations focusing on the blow-up time.

In practical applications, it is desirable to use numerical methods which are mathematically guaranteed their validity. This is because it is hard to distinguish the numerical results which exactly simulate blow-up phenomena of PDEs from failure of computations.

Moreover, convergence analysis of numerical method used for the simulations is important for the numerical analysis of blow-up phenomena. In this paper, we consider a splitting method which is a time-discretization numerical method. It is often used for Schrödinger equations.

On the other hand, we analytically show continuous differentiability of the blow-up curve of a wave equation with a nonlinear term involving the derivative of unknown functions by applying the idea of numerical analysis in Chapter 1. We also simulate these results. Moreover, we present numerical results that showed the blow-up curves have singular points.

In Chapter 1, we consider the following wave equation.

$$\begin{cases} u_{tt} - u_{xx} = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{0.1}$$

Here, $S_L = \mathbb{R}/L\mathbb{Z}$ and $p > 1$ is a constant such that the function s^p $(s \geq 0)$ is of class C^4. The solution of (0.1) blows up in finite time if the initial values are large enough. The aims of this Chapter are to construct the numerical method of the blow-up time and to give the error estimates of them. In this paper, we call the approximation of the blow-up time numerical blow-up time. We divide the proof of convergence of the numerical blow-up time into 2 steps.

(Step 1.) Proof of convergence of numerical method for wave equations.
(Step 2.) Proof of convergence of numerical blow-up time.

There are almost no studies on numerical blow-up time for wave equations, while there are lots of such studies for heat equations. In resent years, construction of numerical blow-up time and convergence analysis of it for wave equations were done by Cho [10]. However, the proof of (Step 1.) is still open at present. He proved (Step 2.) holds under the assumption that (Step 1.) holds.

We need to take sufficiently small time increments near the blow-up time in order to compute the blow-up phenomena. That is, we use the variable time increments. There are many results of convergence analysis of numerical methods using variable time increments for heat equations. However, there is no such study for wave equations. The reason is that wave equations have the second derivative by time. Thus, we construct the numerical methods and corresponding numerical blow-up time for (0.1) and prove both (Step 1.) and (Step 2.).

We rewrite (0.1) as the following first order system.

$$\begin{cases} u_t + u_x = \phi, & t > 0,\ x \in S_L, \\ \phi_t - \phi_x = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad \phi(0,x) = u_1(x) + u_0'(x), & x \in S_L. \end{cases} \tag{0.2}$$

We present numerical method using variable time increments for (0.2). We show our numerical methods satisfy (Step 1.) by using the idea of [32]. We also prove our numerical blow-up time satisfies (Step 2.). Moreover, we present numerical results of blow-up time of (0.2).

In Chapter 2, we consider error analysis of semilinear evolution equations. As mentioned above, such study is important from the viewpoint of numerical analysis of blow-up phenomena. Let X be a Hilbert space and let A be an m-dissipative operator in X. For $u_0 \in D(A)$, we consider the following Cauchy problem for semilinear evolution equation:

$$\begin{cases} u_t = Au + F(u), \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{0.3}$$

The splitting method is one of time-discretization methods. Let $S(t)$ be the solution operator of (0.3). The idea behind splitting methods is to approximate the solution $u(t) = S(t)u_0$ of (0.3) by $\Phi_A(t)$ and $\Phi_F(t)$, which are solution operators of $\partial_t v = Av$ and $\partial_t w = F(w)$, respectively. The splitting method is useful when $\Phi_A(t)$ and $\Phi_F(t)$ are easy to compute, while $S(t)u_0$ is difficult to compute. In particular, the approximation $\Psi(t) = \Phi_A(t/2)\Phi_F(t)\Phi_A(t/2)$ is called the Strang-type splitting method. The Strang-type spitting method is numerically known as a second order convergent scheme. In addition, splitting method retains the dissipation or conservation properties of (0.3). Hence their ease of calculation and the dissipation or conservation properties, the splitting method is in common used as a numerical method for solving various differential equations. However, there are many open problems on error analysis of (0.3). In particular, for (0.3), whether the Strang-type splitting method is second order convergent or not was an open question in a rigorous manner.

The splitting method which is split into 2 parts is used on many occasions. On the other hand, sometimes there are cases that we should use the splitting method which is split into 3 parts. Therefore, we demonstrate that the convergence of our Strang-type splitting method which is split into 3 parts is a second order rate.

In Chapter 3, we consider a blow-up curve for the following nonlinear wave equation.

$$\begin{cases} u_{tt} - u_{xx} = F(u), & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{0.4}$$

where $F(u) = |u_t|^p$. Here, $p > 1$ is a constant such that the function s^p $(s \geq 0)$ is of class C^4. It is well known that the solution of (0.4) blows up in finite time if the initial values are large enough. Let R^* and T^* be positive constants. We set $B_{R^*} = \{x \mid |x| < R^*\}$. We consider

$$T(x) = \sup\{t \in (0, T^*) \mid \ |u_t(t,x)| < \infty\} \qquad (x \in B_{R^*}).$$

We call $\Gamma = \{(T(x), x) \mid x \in B_{R^*}\}$ blow-up curve. Below, we will identify Γ with T itself. We have 2 purposes of this Chapter. First, we analytically show that $T \in C^1(B_{R^*})$. Second, we present numerical examples of blow-up curve. We numerically show that the blow-up curve is smooth if the initial values of (0.4) are large and smooth enough. Moreover, we show that the case where the blow-up curve has singular points even the initial values are smooth. In previous study, the cases of $F(u) = |u|^p,\ e^u$ and the following blow-up curve are considered (for example, [6], [7], [18]).

$$\tilde{T}(x) = \sup\{t \in (0, T^*) \mid \ |u(t,x)| < \infty\} \qquad (x \in B_{R^*}).$$

It was shown that $\tilde{T} \in C^1(B_{R^*})$ under suitable initial values. The method introduced by Caffarelli-Friedman [7] are used in the proof of regularity of the blow-up curve. However, we cannot directly apply their method to (0.4) in the case of $F(u) = |u_t|^p$. For these reasons, the mathematical analysis of blow-up curve for the wave equation with a nonlinear term involving the derivative of unknown functions is not well understood.

On the other hand, Ohta-Takamura [30] studied the blow-up curve in the case of $F(u) = (u_t)^2 - (u_x)^2$. The key point of their proof is the transformation $v = e^{-u}$. We see that v satisfies $v_{tt} - v_{xx} = 0$. Thanks to the linearization, we can study the blow-up curve in the case of $F(u) = (u_t)^2 - (u_x)^2$. However, we cannot use this transformation in the case of $F(u) = |u_t|^p$.

Thus, we rewrite (0.4) as the following first order system by using the idea of Chapter 1.

$$\begin{cases} D_-\phi = 2^{-p}|\phi + \psi|^p, & t > 0,\ x \in \mathbb{R}, \\ D_+\psi = 2^{-p}|\phi + \psi|^p, & t > 0,\ x \in \mathbb{R}, \\ \phi(x,0) = f(x), \quad \psi(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $D_-v = v_t - v_x$, $D_+v = v_t + v_x$ and $f = u_1 + \partial_x u_0$, $g = u_1 - \partial_x u_0$. Such rewriting makes it easier to analyze the blow-up curve, not to mention ease of analysis of numerical methods. We also offer an alternative proof of [7] for showing that the blow-up curve of the blow-up limits is an affine function. Our proof is more elementary and easy to read. Moreover, we show some numerical examples of the blow-up curve of (0.4) in the case of $F(u) = |u_t|^p$. From the numerical results, the blow-up curve sometimes has singular points even the initial values are smooth if the initial values are not large. The analytical proof is still open in the case of $F(u) = |u_t|^p$.

In order that we want to readers to avoid to confuse the formulations, we explicitly write the definitions in each chapter. Although multiple same definitions may appear through the thesis, the arguments in each chapter become self contained. This helps readers understand the detailed content of each chapter separately.

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Contents

1 Blow-up of finite-difference solutions to nonlinear wave equations

Finite-difference schemes for computing blow-up solutions of one dimensional nonlinear wave equations are presented. By applying time increments control technique, we can introduce a numerical blow-up time which is an approximation of the exact blow-up time of the nonlinear wave equation. After having verified the convergence of our proposed schemes, we prove that solutions of those finite-difference schemes actually blow up in the corresponding numerical blow-up times. Then, we prove that the numerical blow-up time converges to the exact blow-up time as the discretization parameters tend to zero. Several numerical examples that confirm the validity of our theoretical results are also offered.

1.1 Introduction

The purpose of this chapter is to establish numerical methods for computing blow-up solutions of one space dimensional nonlinear wave equations with power nonlinearlities. In order to avoid unessential difficulties about boundary conditions, we concentrate our attention to L-periodic functions of x with $L > 0$. That is, setting $S_L = \mathbb{R}/L\mathbb{Z}$, we consider the following initial value problem for the function $u = u(t, x)$ $(t \geq 0,\ x \in S_L)$,

$$\begin{cases} u_{tt} - u_{xx} = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{1.1}$$

Before stating assumptions on nonlinearlity and initial values, we recall a general result for nonlinear wave equations. Set $Q_{T,L} = [0, T] \times S_L$ for $T > 0$.

Proposition 1.1.1. *Let $u_0, u_1 \in C^3(S_L)$ and $f \in C^4(\mathbb{R})$ be given. Then, there exists $T > 0$ and a unique classical solution $u \in C^3(Q_{T,L})$ of*

$$\begin{cases} u_{tt} - u_{xx} = f(u), & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{1.2}$$

Moreover, there exists a positive and continuous function $C_{ml}(\eta)$ of $\eta > 0$ satisfying

$$\left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \leq C_{ml} \left(\|u\|_{L^\infty(Q_{T,L})} \right)$$

for non-negative integers m, l such that $m+l \leq 3$. Furthermore, if $f(s) \geq 0$ for $s \geq 0$ and $u_0(x) \geq 0$, $u_1(x) \geq 0$ for $x \in S_L$, then we have $u(t,x) \geq 0$ for $(t,x) \in Q_{T,L}$.

This proposition is proved by the standard argument based on the contraction mapping principle (cf. [15, §12.3]) with the aid of the explicit solution formula given as

$$u(t,x) = \frac{1}{2}[u_0(x-t) + u_0(x-t)] \\ + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi + \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(u(s,y))\, dyds.$$

Throughout this paper, we make the following assumptions:

$$f(u) = |u|^p \text{ with } p > 1 \text{ is of class } C^4; \tag{1.3}$$

$$u_0,\ u_1 \in C^3(S_L); \tag{1.4}$$

$$u_0(x) \geq 0, \quad u_1(x) \geq 0, \quad x \in S_L. \tag{1.5}$$

Thanks to Proposition 1.1.1, the problem (1.1) admits a unique non-negative solution $u \in C^3(Q_{T,L})$, which we will call simply a *solution* hereinafter. We note that the condition (1.3) is equivalently written as

$$p = 2 \text{ or } p \text{ is a real number} \geq 4. \tag{1.6}$$

See also Remark 1.2.10.

The supremum of T in Proposition 1.1.1 is called the lifespan of a solution and is denoted by T_∞. If $T_\infty = \infty$, then we say that the solution u of (1.1) exists globally-in-time. On the other hand, if $T_\infty < \infty$, we say that u blows up in finite time and call T_∞ the blow-up time of a solution.

As a readily obtainable consequence of Proposition 1.1.1, we deduce the following proposition.

Proposition 1.1.2. *Let u be the solution of* (1.1). *Then, the following (i) and (ii) are equivalent.*

(i) *u blows up in finite time $T_\infty < \infty$.*

(ii) $\lim_{t \uparrow T_\infty} \|u(t)\|_{L^\infty(S_L)} = \infty$.

Any solution u of (1.1) actually blows up. To verify this fact, the functional

$$K(v) = \frac{1}{L}\int_0^L v(x)\, dx \qquad (v \in C(S_L))$$

plays an important role. Obviously, we have

$$K(v) \leq \|v\|_{L^\infty(S_L)} \quad (0 \leq v \in C(S_L)). \tag{1.7}$$

Proposition 1.1.3. *Assume that*

$$\alpha = K(u_0) \geq 0, \quad \beta = K(u_1) > 0. \tag{1.8}$$

Then, there exists $T_\infty \in (0, \infty)$ *such that the solution* u *of* (1.1) *blows up in finite time* T_∞*.*

As a matter of fact, the key point of the proof is that the solution u of (1.1) satisfies, whenever it exists,

$$\frac{d}{dt}K(u(t)) \geq \beta + \int_0^t K(u(s))^p \, ds > 0, \tag{1.9}$$

$$\left[\frac{d}{dt}K(u(t))\right]^2 \geq \frac{2}{p+1}K(u(t))^{p+1} + M_1 \geq 0, \tag{1.10}$$

where $M_1 = \beta^2 - \frac{2}{p+1}\alpha^{p+1}$ and $K(u(t)) = K(u(t, \cdot))$.

These inequalities, together with the following elementary proposition, implies that $K(u(t))$ cannot exist beyond T_K, which is defined below. Thus, $u(t, x)$ blows up in finite time $T_\infty \in (0, T_K]$, which completes the proof of Proposition 1.1.3.

Proposition 1.1.4. *Let a* C^1 *function* $w = w(t)$ *satisfy a differential inequality*

$$\frac{d}{dt}w(t) \geq \sqrt{\frac{2}{p+1}w(t)^{p+1} + M_1} \qquad (t > 0) \tag{1.11}$$

with $w(0) = \alpha \geq 0$*. Then,* $w(t)$ *blows up in finite time* $T_K \in (0, T_1)$*, where*

$$T_1 = \int_\alpha^\infty \left[\beta^2 + \frac{2}{p+1}(s^{p+1} - \alpha^{p+1})\right]^{-\frac{1}{2}} ds < \infty.$$

Inequalities (1.9) and (1.10) are derived in the following manner. First, we derive by using Jensen's inequality

$$\frac{d^2}{dt^2}K(u(t)) \geq K(u(t))^p, \tag{1.12}$$

which gives (1.9). Multiplying the both-sides of (1.12) by $(d/dt)K(u(t))$, we have

$$\frac{d}{dt}K(u(t))\frac{d^2}{dt^2}K(u(t)) \geq \frac{d}{dt}K(u(t))K(u(t))^p.$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d}{dt}K(u(t))\right)^2 - \int_\alpha^{K(u(t))} \xi^p \, d\xi\right] \geq 0.$$

Therefore, we get

$$\left[\frac{d}{dt}K(u(t))\right]^2 \geq \beta^2 + \frac{2}{p+1}\left[K(u(t))^{p+1} - \alpha^{p+1}\right],$$

which implies (1.10).

There are a large number of works devoted to blow-up of positive solutions for nonlinear wave equations. To our best knowledge, the first result was obtained by Kawarada [24]. He studied a nonlinear wave equation

$$u_{tt} - \Delta u = f(u) \quad (x \in \Omega,\ t > 0) \tag{1.13}$$

in a smooth bounded domain Ω in $\mathbb{R}^d$ and proved a positive solution actually blows up in finite time if the initial values are sufficiently large. (He did not consider a positive solution explicitly, but as a readily obtainable corollary of his theorem we could obtain the blow-up of a positive solution.) Those results are referred as "large data blow-up" results. After Kawarada's work, a lost of results have been reported. For example, Glassey's papers [16], [17] are well-known. On the other hand, "small data blow-up" results were presented, for example, F. John ([22]) and T. Kato ([23]). See an excellent survey by S. Alinhac ([2]) for more details on blow-up results for nonlinear hyperbolic equations. In contrast to parabolic equations, it seems that there is a little work devoted to asymptotic profiles and blow-up rates of blow-up solutions for hyperbolic equations. Therefore, numerical methods would be important tools to study blow-up phenomena in hyperbolic equations.

However, the computation of blow-up solutions is a difficult task. We do not state here the detail of those issues; see, for example, [13] and [10]. In order to surmount those obstacles, various techniques for computing blow-up solutions of various nonlinear partial differential equations are developed so far. Among them, Δt_n is of use. The pioneering work is done by Nakagawa [28] in 1976. He considered the explicit Euler/finite difference scheme to a semilinear heat equation $u_t - u_{xx} = u^2$ ($t > 0, 0 < x < 1$) with $u(t, 0) = u(t, 1) = 0$. The crucial point of his strategy is that the time increment and the discrete time are given, respectively, as

$$\Delta t_n = \tau \min\left\{1, \frac{1}{\|u_h(t_n)\|_{L^2}}\right\}, \quad t_{n+1} = t_n + \Delta t_n = \sum_{k=0}^{n} \Delta t_k$$

with some $\tau > 0$, where $u_h(t_n)$, h being the size of space grids, denotes the piecewise constant interpolation function of the finite-difference solution at $t = t_n$ and $\|u_h(t_n)\|_{L^2}$ its $L^2(0, 1)$ norm. Then, he succeeded in proving that, for a sufficiently large initial value, the finite-difference solution $u_h(t_n)$ actually blows up in finite time

$$T(\tau, h) = \sum_{n=1}^{\infty} \Delta t_n < \infty$$

and

$$\lim_{\tau,h\to 0} T(\tau, h) = T_\infty, \tag{1.14}$$

where τ denotes the size of a time discretization and T_∞ the blow-up time of the equation under consideration. $T(\tau, h)$ is called the *numerical blow-up time*. Later, Nakagawa's result has been extend to several directions; see, for example, Chen [9], Abia et al. [1], Nakagawa and Ushijima [29] and Cho et al. [13]. However, those papers are concerned only with parabolic equations. On the other hand, it seems that little is known for hyperbolic equations and C. H. Cho's work ([10]) is the first result on the subject. He studied the initial-boundary value problem for a nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = u^2 \qquad (t > 0,\ x \in (0,1)), \\ u = 0 \quad (t \geq 0,\ x = 0,1), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases}$$

and the explicit Euler/finite-difference scheme

$$\begin{cases} \dfrac{1}{\tau_n}\left(\dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} - \dfrac{u_j^n - u_j^{n-1}}{\Delta t_{n-1}}\right) = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + (u_j^n)^2, \\ u_0^n = u_N^n = 0, \quad u_j^0 = u_0(x_j), \quad u_j^1(x_j) = u_0(x_j) + \Delta t_0 u_1(x_j), \end{cases} \tag{1.15}$$

where the time and space variable are discretized as $t_n = \Delta t_0 + \Delta t_1 + \cdots + \Delta t_{n-1}$, $x_j = j/N$ and $N \in \mathbb{N}$, and u_j^n denotes the approximation of $u(t_n, x_j)$. He proposed the following time-increments control strategy

$$\Delta t_n = \tau \min\left\{1,\ \frac{1}{\|u_h(t_n)\|_{L^2}^{1/2}}\right\}, \qquad \tau_n = \frac{\Delta t_n + \Delta t_{n-1}}{2}. \tag{1.16}$$

Then, he succeeded in proving that (1.15) actually holds true under some assumptions. One of the crucial assumptions in his theorem is convergence of the finite-difference solutions, that is,

$$\lim_{h\to 0} \max_{0 \leq t_n \leq T} |u_j^n - u(t_n, x_j)| = 0 \tag{1.17}$$

for any $T \in (0, T_\infty)$. The proof of this convergence result is still open at present. As a matter of fact, we need some a priori estimates or stability in a certain norm in order to prove (1.17). However, as Cho mentioned in [10, page 487], it is quite difficult to prove a stability that remains true even when $\Delta t_n \to 0$.

Recently, K. Matsuya reported some interesting results on global existence and blow-up of solutions of a discrete nonlinear wave equation in [26]. However, it seems that his results are not directly related with approximation of partial differential equations.

This paper is motivated by the paper [10] and devoted to a study of the finite-difference method applied to (1.1). Thus, we propose finite-difference schemes and prove convergence results (cf. Theorems 1.2.4 and 1.2.5) for those schemes even when time-increments approaches to zero. To accomplish this purpose, we rewrite the equation as

$$u_t + u_x = \phi, \quad \phi_t - \phi_x = |u|^p,$$

which is based on the formal factorization $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u = |u|^p$, and then follow the method of convergence analysis proposed by [32] that is originally developed to study time-discretizations for a system of nonlinear Schrödinger equations. Actually, it suffices to prove local stability results in a certain sense (cf. Theorems 1.2.2 and 1.2.3) in order to obtain convergence results. Moreover, we show that discrete analogues of (1.9) and (1.10) holds true, and therefore, we can deduce approximation of blow-up time (1.14) (cf. Theorem 1.2.8).

Notation

For $\boldsymbol{v} = (v_1, \ldots, v_J)^{\mathrm{T}} \in \mathbb{R}^J$, we set $\|\boldsymbol{v}\| = \max_{1\le j\le J} |v_j|$, where $\cdot^{\mathrm{T}}$ indicates the transpose of a matrix. We write $\boldsymbol{v} \ge \boldsymbol{0}$ if and only if $v_i \ge 0$ $(1 \le i \le J)$. We use the matrix ∞ norm

$$\|E\| = \max_{\boldsymbol{v}\in\mathbb{R}^J} \frac{\|E\boldsymbol{v}\|}{\|\boldsymbol{v}\|} = \max_{1\le i\le J} \sum_{j=1}^{J} |E_{ij}|$$

for a matrix $E = (E_{ij}) \in \mathbb{R}^{J\times J}$. Moreover, we write $E \ge O$ if and only if $E_{i,j} \ge 0$ $(1 \le i, j \le J)$. The set of all positive integers is denoted by $\mathbb{N}$.

1.2 Schemes and main results

Introducing a new variable $\phi = u_t + u_x$, we first convert (1.1) into the first order system as follows:

$$\begin{cases} u_t + u_x = \phi & (t,x) \in Q_{T,L}, \\ \phi_t - \phi_x = |u|^p & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad \phi(0,x) = u_1(x) + u_0'(x), & x \in S_L. \end{cases} \tag{1.18}$$

Take a positive integer J and set $x_j = jh$ with $h = L/J$. As a discretization of the time variable, we take positive constants $\Delta t_0, \Delta t_1, \ldots$ and set

$$t_0 = 0, \quad t_n = \sum_{k=0}^{n-1} \Delta t_k = t_{n-1} + \Delta t_{n-1} \quad (n \ge 1).$$

Then, our explicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^n \approx \phi(t_n, x_j) \qquad (1 \le j \le J,\ t \ge 0)$$

reads as

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{u_j^n - u_{j-1}^n}{h} = \phi_j^n \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t_n} - \dfrac{\phi_{j+1}^n - \phi_j^n}{h} = |u_j^{n+1}|^p \end{cases} \qquad (1 \le j \le J,\ n \ge 0) \tag{1.19}$$

where u_0^n and ϕ_{J+1}^n are set as $u_0^n = u_J^n$ and $\phi_{J+1}^n = \phi_1^n$.

We also consider an implicit scheme for the purpose of comparison. However, we do not prefer fully implicit schemes since we need iterative computations for solving resulting nonlinear system. Instead, we consider a linearly-implicit scheme by introducing dual time grids

$$t_{n+\frac{1}{2}} = \frac{\Delta t_0}{2} + t_n \quad (n \ge 0). \tag{1.20}$$

Then, our implicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^{n+\frac{1}{2}} \approx \phi(t_{n+\frac{1}{2}}, x_j) \qquad (1 \le j \le J,\ n \ge 0)$$

reads as

$$
\begin{cases}
\dfrac{u_j^{n+1}-u_j^n}{\Delta t_n}+\dfrac{1}{2}\left(\dfrac{u_j^{n+1}-u_{j-1}^{n+1}}{h}+\dfrac{u_j^n-u_{j-1}^n}{h}\right)=\phi_j^{n+\frac{1}{2}},\\
\dfrac{\phi_j^{n+\frac{3}{2}}-\phi_j^{n+\frac{1}{2}}}{\Delta t_n}-\dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}}-\phi_j^{n+\frac{3}{2}}}{h}+\dfrac{\phi_{j+1}^{n+\frac{1}{2}}-\phi_j^{n+\frac{1}{2}}}{h}\right)=|u_j^{n+1}|^p,
\end{cases}
\qquad (1\le j\le J,\ n\ge 0), \tag{1.21}
$$

where u_0^n and $\phi_{J+1}^{n+\frac{1}{2}}$ are set as $u_0^n=u_J^n$ and $\phi_{J+1}^{n+\frac{1}{2}}=\phi_1^{n+\frac{1}{2}}$.

Remark 1.2.1. It is possible to take

$$
t_{\frac{1}{2}}=\frac{\Delta t_0}{2},\quad t_{n+\frac{1}{2}}=\frac{\Delta t_0}{2}+\sum_{k=1}^{n}\tau_k\quad (n\ge 1)
$$

as dual time grids instead of (1.20), where $\tau_k=(\Delta t_{k-1}+\Delta t_k)/2$. With this choice, the implicit scheme is modified as

$$
\begin{cases}
\dfrac{u_j^{n+1}-u_j^n}{\Delta t_n}+\dfrac{1}{2}\left(\dfrac{u_j^{n+1}-u_{j-1}^{n+1}}{h}+\dfrac{u_j^n-u_{j-1}^n}{h}\right)=\phi_j^{n+\frac{1}{2}},\\
\dfrac{\phi_j^{n+\frac{3}{2}}-\phi_j^{n+\frac{1}{2}}}{\tau_n}-\dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}}-\phi_j^{n+\frac{3}{2}}}{h}+\dfrac{\phi_{j+1}^{n+\frac{1}{2}}-\phi_j^{n+\frac{1}{2}}}{h}\right)=|u_j^{n+1}|^p,
\end{cases}
\qquad (1\le j\le J,\ n\ge 0). \tag{1.22}
$$

Then, we can deduce all the results presented below with obvious modifications.

For $n\ge 0$, we set

$$
\boldsymbol{u}^n=(u_1^n,\dots,u_J^n)^{\mathrm{T}}\in\mathbb{R}^J,
$$
$$
\boldsymbol{\phi}^n=(\phi_1^n,\dots,\phi_J^n)^{\mathrm{T}}\in\mathbb{R}^J,\qquad \boldsymbol{\phi}^{n+\frac{1}{2}}=(\phi_1^{n+\frac{1}{2}},\dots,\phi_J^{n+\frac{1}{2}})^{\mathrm{T}}\in\mathbb{R}^J.
$$

Theorem 1.2.2 (Local stability of the explicit scheme)**.** *Let $\tau=\gamma h$ with some $\gamma\in(0,1)$ and assume that $\Delta t_n\le\tau$ for $n\ge 0$. Let $\boldsymbol{a}\ge\mathbf{0},\boldsymbol{b}\ge\mathbf{0}\in\mathbb{R}^J$. Then, the solution $(\boldsymbol{u}^n,\boldsymbol{\phi}^n)$ of the explicit scheme* (1.19) *with $\boldsymbol{u}^0=\boldsymbol{a}$ and $\boldsymbol{\phi}^0=\boldsymbol{b}$ satisfies $\boldsymbol{u}^n\ge\mathbf{0}$ and $\boldsymbol{\phi}^n\ge\mathbf{0}$ for $n\ge 1$. Furthermore, for any $N\in\mathbb{N}$, there exists a constants $h_{R,N}>0$ depending only on N and $R=\|\boldsymbol{a}\|+\|\boldsymbol{b}\|$ such that, if $h\in(0,h_{R,N}]$, we have*

$$
\sup_{1\le n\le N}(\|\boldsymbol{u}^n\|+\|\boldsymbol{\phi}^n\|)\le 2R. \tag{1.23}
$$

Theorem 1.2.3 (Well-posedness and local stability of the implicit scheme)**.** *Let $\tau=2\gamma h$ with some $\gamma\in(0,1)$ and assume that $\Delta t_n\le\tau$ for $n\ge 0$. Let $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^J$. Then, the implicit scheme* (1.21) *admits a unique solution $(\boldsymbol{u}^n,\boldsymbol{\phi}^{n+\frac{1}{2}})$ for any $n\ge 1$, where $\boldsymbol{u}^0=\boldsymbol{a}$ and $\boldsymbol{\phi}^{\frac{1}{2}}=\boldsymbol{b}$. Moreover, if $\boldsymbol{a}\ge\mathbf{0}$ and $\boldsymbol{b}\ge\mathbf{0}$, then we have $\boldsymbol{u}^n\ge\mathbf{0}$ and $\boldsymbol{\phi}^{n+\frac{1}{2}}\ge\mathbf{0}$ for $n\ge 1$. Furthermore, for any $N\in\mathbb{N}$, there exists a constants*

$h_{R,N} > 0$ *depending only on* N *and* $R = \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ *such that, if* $h \in (0, h_{R,N}]$, *we have*

$$\sup_{1\le n\le N} \left(\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\|\right) \le 2R. \tag{1.24}$$

In order to state convergence results, we introduce $\boldsymbol{e}^n = (e_j^n)$, $\boldsymbol{\varepsilon}^n = (\varepsilon_j^n)$ and $\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = (\varepsilon_j^{n+\frac{1}{2}})$ which are given as

$$e_j^n = u(t_n, x_j) - u_j^n, \quad \varepsilon_j^n = \phi(t_n, x_j) - \phi_j^n, \quad \varepsilon_j^{n+\frac{1}{2}} = \phi(t_{n+\frac{1}{2}}, x_j) - \phi_j^{n+\frac{1}{2}}.$$

Recall that T_∞ denotes the blow-up time of the solution $u(t,x)$ of (1.1).

Theorem 1.2.4 (Convergence of the explicit scheme)**.** *Let* $\tau = \gamma h$ *with some* $\gamma \in (0,1)$ *and assume that* $\Delta t_n \le \tau$ *for* $n \ge 0$. *Suppose that* $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ *is the solution of the explicit scheme* (1.19) *for* $n \ge 1$, *where* $(\boldsymbol{u}^0, \boldsymbol{\phi}^0)$ *is defined as*

$$u_j^0 = u_0(x_j), \quad \phi_j^0 = u_1(x_j) + u_0'(x_j) \quad (1 \le j \le J). \tag{1.25}$$

Let $T \in (0, T_\infty)$ *be arbitrarily. Then, there exists positive constants* h_0 *and* M_0 *which depend only on*

$$p, \quad T, \quad \gamma, \quad M = \max_{0\le m+l\le 3} \left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \tag{1.26}$$

such that we have

$$\max_{0\le t_n\le T} (\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^n\|) \le M_0(\tau + h)$$

for any $h \in (0, h_0]$.

Theorem 1.2.5 (Convergence of the implicit scheme)**.** *Let* $\tau = 2\gamma h$ *with some* $\gamma \in (0,1)$ *and assume that* $\Delta t_n \le \tau$ *for* $n \ge 0$. *Suppose that* $(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})$ *is the solution of the implicit scheme* (1.21) *for* $n \ge 1$, *where* $(\boldsymbol{u}^0, \boldsymbol{\phi}^{\frac{1}{2}})$ *is defined as*

$$u_j^0 = u_0(x_j), \quad \phi_j^{\frac{1}{2}} = u_1(x_j) + u_0'(x_j) \quad (1 \le j \le J). \tag{1.27}$$

Let $T \in (0, T_\infty)$ *be arbitrarily. Then, there exists positive constants* h_0 *and* M_0, *which depend only on* (1.26), *such that we have*

$$\max_{0\le t_{n+1}\le T} \left(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|\right) \le M_0(\tau + h) \tag{1.28}$$

for any $h \in (0, h_0]$.

Remark 1.2.6. If taking constant time-increments $\Delta t_n = \tau$ and suitable initial value $\boldsymbol{\phi}^{\frac{1}{2}}$, we can prove

$$\max_{0\le t_{n+1}\le T} \left(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|\right) \le M_0(\tau^2 + h)$$

instead of (1.29).

By using the solutions of the explicit scheme (1.19) and the implicit scheme (1.21), we can calculate the blow-up time T_∞ of the solution of (1.1). To this purpose, we fix

$$1 \le q < \infty, \qquad 0 < \gamma < 1 \tag{1.29}$$

and choose the time increments $\Delta t_0, \Delta t_1, \ldots$ as

$$\Delta t_n = \tau \cdot \min \left\{ 1, \ \frac{1}{\|\boldsymbol{u}^n\|^q} \right\} \qquad (n \ge 0), \tag{1.30}$$

where τ is taken as

$$\tau = \begin{cases} \gamma h & \text{for the explicit scheme (1.19)} \\ 2\gamma h & \text{for the implicit scheme (1.21).} \end{cases} \tag{1.31}$$

Definition 1. Let $\boldsymbol{u}^n$ be the solution of the explicit scheme (1.19) or the implicit scheme (1.21) with the time increment control (1.30) and (1.31). Then, we set

$$T(h) = \sum_{n=0}^{\infty} \Delta t_n.$$

If $T(h) < \infty$, we say that $\boldsymbol{u}^n$ blows up in finite time $T(h)$.

Remark 1.2.7. The blow-up of $\boldsymbol{u}^n$ implies that $\lim_{t_n \to T(h)} \|\boldsymbol{u}^n\| = \lim_{n \to \infty} \|\boldsymbol{u}^n\| = \infty$.

We are now in a position to state numerical blow-up results.

Theorem 1.2.8 (Approximation of the blow-up time)**.** *Let $\boldsymbol{u}^n$ be the solution of the explicit scheme* (1.19) *or the implicit scheme* (1.20) *with the time increment control* (1.30) *and* (1.31)*, where the initial value is defined as* (1.25) *or* (1.27)*, respectively. In addition to the basic assumptions* (1.4) *and* (1.5) *on initial values, assume that $u_1(x)$ is so large that*

$$u_1(x) + u_0'(x) \ge 0, \not\equiv 0 \quad (x \in S_L). \tag{1.32}$$

Then, we have the following:

(i) *$\boldsymbol{u}^n \ge 0$ and $\boldsymbol{\phi}^n \ge \mathbf{0}$ (or $\boldsymbol{\phi}^{n+\frac{1}{2}} \ge \mathbf{0}$) for all $n \ge 0$.*

(ii) *If* (1.8) *holds true, $\boldsymbol{u}^n$ blows up in finite time $T(h)$ and*

$$T_\infty \le \liminf_{h \to 0} T(h). \tag{1.33}$$

(iii) *In addition to* (1.4)*, we assume that*

$$\lim_{t \to T_\infty} K(u(t)) = \infty, \tag{1.34}$$

then we have

$$T_\infty = \lim_{h \to 0} T(h). \tag{1.35}$$

Remark 1.2.9. The assumption (1.35) is somewhat restrictive. Essentially the same assumption is considered in [10]. However, we are unable to remove it at present. To find the sufficient condition for (1.35) to hold is an interesting open question.

Remark 1.2.10. All results presented above remain valid for $f(u) = u|u|^2$, since it is a C^4 function on $\mathbb{R}$.

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2 Error analysis of splitting methods for semilinear evolution equations

We consider a Strang-type splitting method for an abstract semilinear evolution equation $u_t = Au + F(u)$. Roughly speaking, the splitting method is a time-discretization approximation based on the decomposition of operators A and F. Particularly, the Strang method is a popular splitting method and is known to be convergent at a second order rate for some particular ODEs and PDEs. In this chapter, we propose a generalization of the Strang method and prove that our proposed method is convergent at a second order rate. Some numerical examples that confirm our theoretical result are given.

2.1 Introduction and main results

Let X be a Hilbert space equipped with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$, A be an m-dissipative linear operator in X with dence domain $D(A) \subset X$.

- For any $u \in D(A)$, $(Au, u) \leq 0$;
- For any $f \in X$ and $\lambda > 0$, there exists $u \in D(A)$ such that $u - \lambda Au = f$.

As is well-known, the operator A generates a contraction semigroup $\Phi_A(t) = e^{tA}$ if and only if A is m-dissipative with dense domain. We consider the following Cauchy problem for semilinear evolution equation:

$$\begin{cases} u_t = Au + F(u), \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $F : D(A) \to D(A)$ is a nonlinear operator. Typical examples of (2.1) are nonlinear Schrödinger equations in $\Omega \in \mathbb{R}^d$

$$u_t = i\Delta u + \alpha u|u|^2, \tag{2.2}$$

$$u_t = i\Delta u + \alpha u|u|^2 + \beta u|u|^4, \tag{2.3}$$

where α and β are complex constants. Setting $D(A) = \{v \in H_0^1(\Omega) \mid \Delta v \in L^2(\Omega)\}$, $Av = i\Delta v$, and $F(v) = \alpha v|v|^2$ in (2.2), we obtain (2.1).

The main purpose of this chapter is to study the so called splitting method, which is a semi-discrete approximation of (2.1) with respect to time variable t. The idea behind the splitting method is as follows. We denote the (nonlinear) solution operator (2.1) by $S(t)$. That is, the solution of (2.1) is given as $u(t) = S(t)u_0$; see (2.9) below. Then, we consider the time-discrete approximation to (2.1) at $t = n\Delta t$ as

$$u_n = \Psi(n\Delta t)u_0,$$

where $\Delta t > 0$ denotes a time increment and n a positive integer. Typical choices of Ψ are, for example,

$$\Psi(t) = \Phi_A(t)\Phi_F(t), \tag{2.4}$$

$$\Psi(t) = \Phi_F(t)\Phi_A(t), \tag{2.5}$$

$$\Psi(t) = \Phi_A(t/2)\Phi_F(t)\Phi_A(t/2) \tag{2.6}$$

where $\Phi_F(t)$ denotes the solution operator of $w_t = F(w)$. Particularly, (2.6) is called the Strang method.

Splitting methods are useful when $S(t)u_0$ is difficult to compute, while $\Phi_A(t)u_0$ and $\Phi_F(t)u_0$ are easy to compute. In addition, if (2.8) has conservation properties, then splitting methods basically preserve its discrete version. Splitting methods are widely used numerical methods for solving ODEs and PDEs.

Analysis of splitting methods for ODEs has been presented in many studies. For example, see Hairer *et al.*[20]. Some results on error analysis are also presented for PDEs. For example, results of error analysis for nonlinear Schrödinger equations can be found in e.g., Besse *et al.* [4] and Lubich [25].

However, to our best knowledge, little is known for abstract Cauchy problem of the form (2.1). Decombes and Thalhammer[14] and Jahnke and Lubich [21] presented an error analysis for the case in which F is a linear operator. For nonlinear abstract Cauchy problems, Borgna *et al.*[5] demonstrated that various splitting methods involving Strang method have first order accuracy. Namely, if Δt is sufficiently small, we have

$$\|S(n\Delta t)u_0 - \Psi(\Delta t)^n u_0\| \leq C\Delta t.$$

However, they did not demonstrate that Strang-type splitting method is a second order scheme:

$$\|S(n\Delta t)u_0 - \Psi(\Delta t)^n u_0\| \leq C\Delta t^2. \tag{2.7}$$

It should be kept in mind that (2.7) is established for the Strang method applied to particular PDEs; see Besse *et al.*[4] and Lubich[25]. Therefore, it is worth studying the Strang method for abstract Cauchy problem of the form (2.1) and deriving the second order error estimate.

On the other hand, the majority of previous studies have considered schemes that are split into two parts; $v_t = Av$ and $w_t = F(w)$. As a matter of fact, such two-parts splitting is applied tp (2.2), then the explicit solution formula for the ordinary differential equation $w_t = \alpha w|w|^2$ is available. However, the two-parts splitting is applied to (2.3), then we have to solve the ordinary differential equation

$w_t = \alpha w|w|^2 + \beta w|w|^4$ by numerical method since the exact solution is not available in the case.

Therefore, some researchers have proposed schemes that are split into more than two parts. However, the convergence properties of such schemes are not guaranteed in the case of PDEs.

In this paper, we propose a Strang-type splitting method that is split into three parts for (2.8). Moreover, we show that it is actually convergent at a second order rate.

Let us formulate our problem. For given nonlinear operators $F_1, F_2 : D(A) \to D(A)$, we set

$$F(v) = F_1(v) + F_2(v) \qquad (v \in D(A)).$$

For $u_0 \in D(A)$, we consider the Cauchy problem

$$\begin{cases} u_t = Au + F_1(u) + F_2(u), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.8}$$

and the corresponding integral equation:

$$u(t) = \Phi_A(t)u_0 + \int_0^t \Phi_A(t-s)F(u(s))ds, \quad t \in [0,T]. \tag{2.9}$$

We consider $D(A)$ and $D(A^2)$ as Hilbert spaces with

$$\begin{aligned} \|v\|_{D(A)} &= \|v\|_X + \|Av\|_X \quad \text{for} \quad v \in D(A), \\ \|v\|_{D(A^2)} &= \|v\|_{D(A)} + \|A^2 v\|_X \quad \text{for} \quad v \in D(A^2). \end{aligned}$$

For $i = 1, 2$, we assume that $F_i : D(A) \to D(A)$ satisfies the following conditions:

(F0) $F_i(0) = 0$,

(F1) $\|F_i'(v)w\|_{D(A)} \le L(\|v\|_{D(A)})\|w\|_{D(A)}$ for $v, w \in D(A)$,

(F2) $F_i(v) \in D(A^2)$ and $\|F_i(v)\|_{D(A^2)} \le L_2(\|v\|_{D(A)})\|v\|_{D(A^2)}$ for $v, w \in D(A^2)$,

(F3) $F_i(v) \in D(A^2)$ and $\|F_i(v) - F_i(w)\|_{D(A^2)} \le L_3(\max\{\|v\|_{D(A^2)}, \|w\|_{D(A^2)}\})\|v - w\|_{D(A^2)}$
for $v, w \in D(A^2)$,

(F4) $\|F_i'(v)w\|_X \le L_4(\|v\|_{D(A)})\|w\|_X$ for $v, w \in D(A)$,

(F5) $\|F_i''(v)(w,w)\|_X \le L_5(\|v\|_{D(A)})\|w\|_X\|w\|_{D(A)}$ for $v, w \in D(A)$.

Herein, F_i' and F_i'' denote the first and second Fréchet derivatives, $L, L_2, \cdots, L_5 : [0,\infty) \to [0,\infty)$ are decreasing functions.

We note that it follows from (F1) and (F0) that

(F6) $\|F_i(v) - F_i(w)\|_{D(A)} \le L(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v - w\|_{D(A)}$
for $v, w \in D(A)$,

(F7) $\|F_i(v)\|_{D(A)} \le L(\|v\|_{D(A)})\|v\|_{D(A)}$ for $v \in D(A)$.

Moreover, it follows from (F4) that

(F8) $\|F_i(v) - F_i(w)\|_X \le L_4(\max\{\|v\|_{D(A)}, \|w\|_{D(A)}\})\|v - w\|_X$
for $v, w \in D(A)$.

For simplicity, we write $F''(v)(w,w) = F''(v)w^2$ for $v, w \in D(A)$. Before stating the schemes and main results, we recall a general result for (2.9):

Proposition 2.1.1. *Assume* (F0)–(F1). *Then, for any* $u_0 \in D(A)$*, there exist* $T_{\max}(u_0) \in (0, \infty]$ *and a unique solution*

$$u \in C([0, T_{\max}(u_0)), D(A)) \cap C^1([0, T_{\max}(u_0), X)$$

of (2.9) *such that either the following* (i) *or* (ii) *holds:*

(i) $T_{\max}(u_0) = \infty$,

(ii) $T_{\max}(u_0) < \infty$ *and* $\lim_{t \uparrow T_{\max}(u_0)} \|u(t)\|_{D(A)} = \infty$.

Moreover, if $u_0 \in D(A^2)$*, then*

$$u \in C([0, T_{\max}(u_0)), D(A^2)) \cap C^1([0, T_{\max}(u_0)), D(A)).$$

For the proof of Proposition 2.1.1, see e.g., Section 4.3 of [8].

In order to state our scheme, for $i = 1, 2$, we consider the following Cauchy problem:

$$\begin{cases} w_{i,t} = F_i(w_i), & t \in [0, T], \\ w_i(0) = w_{i,0}, \end{cases} \tag{2.10}$$

and the corresponding integral equation:

$$w_i(t) = w_{i,0} + \int_0^t F_i(w_i(s))ds, \quad t \in [0, T]. \tag{2.11}$$

We denote the solution of (2.12) by $w_i(t) = \Phi_{F_i}(t)w_{i,0}$. That is,

$$\Phi_{F_i}(t)w_{i,0} = w_{i,0} + \int_0^t F_i(w_i(s))ds, \quad t \in [0, T]. \tag{2.12}$$

Then, our scheme to find $\Psi(t)u_0 \approx S(t)u_0$, reads as

$$\Psi(t)u_0 = \Phi_A(t/2)\Phi_{F_1}(t/2)\Phi_{F_2}(t)\Phi_{F_1}(t/2)\Phi_A(t/2)u_0. \tag{2.13}$$

Our scheme includes the Strang method by setting $F_1 = 0$.

We are now in a position to state the main results.

Theorem 2.1.2. *Assume* (F0)–(F5). *Let* $u_0 \in D(A^2)$, $T \in (0, T_{\max}(u_0))$ *and set*

$$m_0 = 8 \max_{t\in[0,T]} \|S(t)u_0\|_{D(A)}.$$

Then, there exists a positive constant h_0*, which depends only on* T, m_0 *and* $\|u_0\|_{D(A^2)}$*, such that*

$$\|(\Psi(h))^n u_0\|_{D(A)} \le m_0, \quad \|(\Psi(h))^n u_0\|_{D(A^2)} \le e^{\gamma_1 nh}\|u_0\|_{D(A^2)}, \tag{2.14}$$

$$\|S(nh)u_0 - (\Psi(h))^n u_0\|_{D(A)} \le \kappa_1 h\|u_0\|_{D(A^2)}, \tag{2.15}$$

$$\|S(nh)u_0 - (\Psi(h))^n u_0\|_{X} \le \kappa_2 h^2\|u_0\|_{D(A^2)}, \tag{2.16}$$

for all $h \in (0, h_0]$ *and* $n \in \mathbb{N}$ *satisfying* $nh \le T$*, where* γ_1 *is a positive constant depending only on* m_0*, and* κ_1, κ_2 *are positive constants depending only on* T *and* m_0.

3 Regularity and singularity of blow-up curve for $u_{tt} - u_{xx} = |u_t|^p$

We study a blow-up curve for the one dimensional wave equation $u_{tt} - u_{xx} = |u_t|^p$ with $p > 1$. The purpose of this paper is to show that the blow-up curve is a C^1 curve if the initial values are large and smooth enough. To prove the result, we convert the equation into a first order system, and then apply a modification of the method of Caffarelli and Friedman [7]. Moreover, we present some numerical investigations of the blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

3.1 Introduction and main results

In this paper, we consider the nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = |u_t|^p, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where

$$p > 1 \text{ is a constant such that the function } |s|^p \text{ is of class } C^4. \tag{3.2}$$

Here, u is an unknown real-valued function.

Let T^* and R^* be any positive constants, and set

$$B_{R^*} = \{x \mid |x| < R^*\}, \tag{3.3}$$

$$K_-(t_0, x_0) = \{(t,x) \mid |x - x_0| < t_0 - t,\ t > 0\}, \tag{3.4}$$

$$K_{T^*,R^*} = \bigcup_{x \in B_{R^*}} K_-(T^*, x). \tag{3.5}$$

We then consider the following function

$$T(x) = \sup\{t \in (0, T^*) \mid \ |u_t(t,x)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

In this paper, we call the set $\Gamma = \{(T(x), x) \mid x \in B_{R^*}\}$ the blow-up curve. Below, we identify Γ with T itself. There are two purposes of this paper. First, we demonstrate that T is continuously differentiable for the suitable initial values. Second, we present some numerical examples of the various blow-up curves. From the numerical results, we were able to confirm that the blow-up curves are smooth if the initial values are large and smooth enough. Moreover, we can predict that the blow-up curves have singular points if the initial values are not large enough even they are smooth enough.

We will state some analytical results from previous studies on the blow-up curves for nonlinear wave equations. The majority of previous studies have considered the following nonlinear wave equation:

$$u_{tt} - u_{xx} = F(u), \qquad t > 0,\ x \in \mathbb{R},$$

and corresponding blow-up curve

$$\tilde{T}(x) = \sup\{t \in (0, T^*) \mid \ |u(t,x)| < \infty\} \quad \text{for } x \in B_{R^*}.$$

We note that the definition of the blow-up curve is different from ours. The pioneering study on this topic was done by Caffarelli and Friedman [6], [7]. They investigated the case with $F(u) = |u|^p$. They demonstrated that $\tilde{T}$ in that case is continuously differentiable under suitable initial conditions. Moreover, Godin [18] showed that the blow-up curve with $F(u) = e^u$ is also continuously differentiable under appropriate initial conditions. It was also shown that the blow-up curve can be C^∞, in the case of $F(u) = e^u$ (see Godin [19]). Furthermore, Uesaka [33] considered the blow-up curve for the system of nonlinear wave equations.

On the other hand, Merle and Zagg [27] showed that there are cases where the blow-up curve has singular points, while the above results concern the smoothness of the blow-up curve.

As mentioned above, several results have been established on the blow-up curve when there are no nonlinear terms involving the derivative of the solution. On the other hand, to the best of our knowledge only one result has been found concerning the blow-up curve with nonlinear terms involving the derivative of solution. Ohta and Takamura [30] considered the nonlinear wave equation

$$u_{tt} - u_{xx} = (u_t)^2 - (u_x)^2, \qquad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{3.6}$$

This equation can be transformed into the wave equation $\partial_t^2 v - \partial_x^2 v = 0$ by

$$v(t,x) = \exp\{-u(t,x)\}, \quad u(t,x) = -\log\{v(t,x)\}.$$

Thanks to the linearization of (3.6), we can study the blow-up curve of (3.6).

However, we cannot apply this linearization to (3.1). Therefore, we employ an alternative method, which is to rewrite to (3.1) as a system that does not include the derivative of the solution in nonlinear terms. We basically apply the method introduced by Caffarelli and Friedman [7] to this system. However, we offer an alternative proof of [7] for showing that the blow-up curve of the blow-up limits is an affine function. Consequently, our proof is more elementary and easy to read. Our method would be applied to the original equation of [7].

We define ϕ and ψ as

$$\phi = u_t + u_x, \qquad \psi = u_t - u_x.$$

Then, we see that (3.1) is rewritten as

$$\begin{cases} D_-\phi = 2^{-p}|\phi+\psi|^p, & t>0,\ x\in\mathbb{R}, \\ D_+\psi = 2^{-p}|\phi+\psi|^p, & t>0,\ x\in\mathbb{R}, \\ \phi(0,x) = f(x), \quad \psi(0,x) = g(x), & x\in\mathbb{R}, \end{cases} \tag{3.7}$$

where $D_-v = v_t - v_x$, $D_+v = v_t + v_x$ and $f = u_1 + \partial_x u_0$, $g = u_1 - \partial_x u_0$. (The equivalency of between (3.1) and (3.7) will be described in Remark 3.1.2.)

Let $(\tilde{\phi}, \tilde{\psi})$ be the solution of

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = 2^{-p}|\tilde{\phi}+\tilde{\psi}|^p, & t>0, \\ \dfrac{d\tilde{\psi}}{dt} = 2^{-p}|\tilde{\phi}+\tilde{\psi}|^p, & t>0, \\ \tilde{\phi}(0) = \gamma_1, \quad \tilde{\psi}(0) = \gamma_2, \end{cases} \tag{3.8}$$

where γ_1 and γ_2 are some positive constants which will be fixed later. Then, we see that there exists a positive constant T_1 such that

$$\tilde{\phi}(t) + \tilde{\psi}(t) \to \infty \quad \text{as } t \to T_1.$$

We make the following assumptions.

(A1) $f \geq \gamma_1, \quad g \geq \gamma_2 \quad \text{in } B_{T^*+R^*}$.

(A2) $f, g \in C^4(B_{T^*+R^*})$.

(A3) There exists a constant $\varepsilon_0 > 0$ such that

$$2^{-p}(\gamma_1+\gamma_2)^p \geq (2+\varepsilon_0)\cdot \max_{x\in B_{T^*+R^*}} \{|f_x(x)| + |g_x(x)|\}.$$

(A4) $T_1 < T^*$.

(A5.1) There exists a constant $\varepsilon_1 > \dfrac{2}{2p-3}$ such that

$$2^{-p}(\gamma_1+\gamma_2)^p \geq (2+\varepsilon_1)\cdot \max_{x\in B_{T^*+R^*}} \{|f_x(x)| + |g_x(x)|\}.$$

(We notice that it follows from (3.2) that $p > 3/2$.)

(A5.2) There exists a constant $C^{(2)} > 0$ such that

$$(f+g)^{2p-1} \geq C^{(2)}\cdot \max_{x\in B_{T^*+R^*}} \{|f_{xx}(x)| + |g_{xx}(x)|\}.$$

(A5.3) There exists a constant $C^{(3)} > 0$ such that

$$(f+g)^{3p-2} \geq C^{(3)} \cdot \max_{x \in B_{T^*+R^*}} \{|\partial_x^3 f(x)| + |\partial_x^3 g(x)|\}.$$

We now state the main results of this paper.

Theorem 3.1.1. *Let T^* and R^* be arbitrary positive numbers. Assume that (A1)-(A5.3) hold true. Then, there exists a unique $C^1(B_{R^*})$ function T such that $0 < T(x) < T^*$ $(x \in B_{R^*})$ and a unique $(C^{3,1}(\Omega))^2$ solution (ϕ, ψ) of (3.7) satisfying*

$$\phi(t,x),\ \psi(t,x) \to \infty \quad \text{as} \quad t \to T(x) \tag{3.9}$$

for any $x \in B_{R^*}$, where $\Omega = \left\{(t,x) \in \mathbb{R}^2 \mid x \in B_{R^*},\ 0 < t < T(x)\right\}$.

Remark 3.1.2. The equation (3.1) is equivalent to (3.7). We set

$$u(t,x) = u_0(x) + \frac{1}{2}\int_0^t (\phi + \psi)(s,x)ds.$$

Then, u satisfies (3.1).

Remark 3.1.3. The assertion (3.9) implies that $u_t(t,x) \to \infty$ as $t \to T(x)$ $(x \in B_{R^*})$.

Next, we will mention numerical analysis of blow-up of nonlinear partial differential equations. There are many previous works of computation of blow-up solutions of various partial differential equations; See, for example, [28], [13], [10], [34], [31], [11] and [12].

We computed blow-up curve using the method of Cho [12] and obtained the various numerical results of blow-up curves.

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