博士論文

論文題目　Algebraic structure on the space of intertwining operators
(絡作用素の空間上の代数構造)

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1 Introduction

The aim of this thesis is to study branching laws of real reductive Lie groups by algebraic methods.

1.1 Irreducible decomposition

The main problem is to analyze the restriction of an irreducible representation of a Lie group (resp. Lie algebra) with respect to a closed subgroup (resp. Lie subalgebra). The problem is called the branching problem. If the given irreducible representation is unitary, the following fact assures us that the restriction has an irreducible decomposition.

Fact 1.1 (Mautner and Teleman)**.** *Let U be a unitary representation of a locally compact group G. Then U has an irreducible decomposition:*

$$U \simeq \int_{\widehat{G}}^{\oplus} m(\pi) V_\pi d\mu(\pi),$$

where $\widehat{G}$ is the unitary dual of G and V_π is a representation space of π.

The measurable function $m : \widehat{G'_{\mathbb{R}}} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is called the multiplicity function. For the case of Lie group representations, R. Goodman showed in the proof of [12, Lemma 3.1] that the direct integral decomposition is compatible with the Lie algebra action.

Fact 1.2 (Goodman)**.** *Let U be a unitary representation of a Lie group G with direct integral decomposition:*

$$U \simeq \int_{Z}^{\oplus} U_z d\mu(z).$$

Then for any vector $v \in U^\infty$ defined by a section $z \mapsto v(z)$, $v(z)$ belongs to U_z^∞ for μ-almost every z, where U^∞ is the space of smooth vectors with respect to the G-action. Furthermore, for any $X \in \mathcal{U}(\mathfrak{g})$, we have $(Xv)(z) = X(v(z))$ for μ-almost every z.

1.2 Branching problem

Our main concern is the branching problem of unitary representations of real reductive Lie groups. For the branching problem of real reductive Lie groups, we refer to [35, 40]. Let $G_{\mathbb{R}}$ be a real reductive Lie group and $G'_{\mathbb{R}}$ a reductive subgroup of $G_{\mathbb{R}}$. For any irreducible unitary representation V of $G_{\mathbb{R}}$, the restriction $V|_{G'_{\mathbb{R}}}$ to $G'_{\mathbb{R}}$ has an irreducible decomposition by the theorem of Mautner and Teleman:

$$V|_{G'_{\mathbb{R}}} \simeq \int_{\widehat{G'_{\mathbb{R}}}}^{\oplus} m(\pi) V_\pi d\mu(\pi). \tag{1.2.1}$$

Since $G'_{\mathbb{R}}$ is reductive, the irreducible decomposition is unique. The irreducible decomposition is called the branching law of V with respect to $G'_{\mathbb{R}}$.

The branching problem for compact Lie groups has been studied by many mathematicians, and explicit branching laws have been obtained such as the Clebsh–Gordan formula, the Pieri rule, the branching laws for $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{U}(n), \mathrm{U}(n-1)), (\mathrm{SO}(n), \mathrm{SO}(n-1))$ and $(\mathrm{Sp}(n), \mathrm{Sp}(n-1))$, the Littlewood–Richardson coefficient, Kostant's formula and the Littelmann path model. Conversely, the branching problem for non-compact reductive Lie groups is difficult in general, and branching laws were known at the end of 1980s only for specific unitary representations such as holomorphic discrete series representations [17], [20], [53], [68], the Segal-Shale-Weil representation [18, 19], [23] and K-type formulas.

In the late 1980s, T. Kobayashi discovered discretely decomposable branching laws of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$. Let θ be a Cartan involution of $G_{\mathbb{R}}$ preserving $G'_{\mathbb{R}}$. Set $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$ and $\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$. We denote by K and $\mathfrak{g}$ the complexification of $K_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$, respectively. We use a similar notation for $G'_{\mathbb{R}}$ such as $K'_{\mathbb{R}}$, K' and $\mathfrak{g}'$. For a representation V of $G_{\mathbb{R}}$, we write V_K for the subspace of all $K_{\mathbb{R}}$-finite vectors. In the series of papers [27, 28, 30, 31], Kobayashi initiated and developed the theory of discretely decomposable $(\mathfrak{g}, K)$-modules and gave examples of explicit branching laws for $A_{\mathfrak{q}}(\lambda)$.

Definition 1.3 (T. Kobayashi [30, Definition 1.1])**.** A $(\mathfrak{g}, K)$-module V is said to be *discretely decomposable* if V has a $(\mathfrak{g}, K)$-module filtration $V_0 \subset V_1 \subset \cdots$ such that $\bigcup_i V_i = V$ and each V_i is finite length.

He gave criteria for the discrete decomposability of a restriction of an irreducible $(\mathfrak{g}, K)$-module by the asymptotic K-support [30] and the associated variety [31], and gave necessary and sufficient conditions for the discrete decomposability of a restriction of $A_{\mathfrak{q}}(\lambda)$. An important property is that the discrete decomposability of a $(\mathfrak{g}, K)$-module implies the discrete decomposability of a unitary representation of $G_{\mathbb{R}}$ as follows.

Fact 1.4 (T. Kobayashi [32, Theorem 2.7])**.** *Let V be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that $V_K|_{(\mathfrak{g}', K')}$ is discretely decomposable. Then $V_K|_{(\mathfrak{g}', K')}$ is decomposed into the direct sum of irreducible* $(\mathfrak{g}', K')$*-*

modules:

$$V_K|_{(\mathfrak{g}',K')} \simeq \bigoplus_{\pi \in \widehat{G'_{\mathbb{R}}}} m(\pi)(V_\pi)_{K'},$$

and $V|_{G'_{\mathbb{R}}}$ *is decomposed into the direct sum of irreducible unitary representations with the same multiplicity function* $m(\pi)$*:*

$$V|_{G'_{\mathbb{R}}} \simeq {\sum_{\pi \in \widehat{G'_{\mathbb{R}}}}}^{\oplus} m(\pi) V_\pi.$$

In the framework, discretely decomposable restrictions, explicit branching laws were computed for some unitary representations [6], [13], [27, 28, 30, 31], [42], [56], [59], [63], [72]. The discretely decomposable restrictions of $A_{\mathfrak{q}}(\lambda)$ with respect to symmetric subgroups were classified by T. Kobayashi and Y. Oshima [44], and the branching laws were obtained by Y. Oshima in [62].

1.3 Intertwining operator

The space of all intertwining operators is important to study the branching problem. Let V be an irreducible $(\mathfrak{g}, K)$-module and V' an irreducible $(\mathfrak{g}', K')$-module. Consider the following two vector spaces:

$$\mathrm{Hom}_{\mathfrak{g}',K'}(V, V'),$$
$$\mathrm{Hom}_{\mathfrak{g}',K'}(V', V).$$

An element of $\mathrm{Hom}_{\mathfrak{g}',K'}(V, V')$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ is called an *intertwining operator*.

The two spaces have a natural $\mathcal{U}(\mathfrak{g})^{G'}$-module structure. For the case of compact G', it is well-known that the action on $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ is irreducible. In particular, when $G'_{\mathbb{R}}$ is equal to the maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$, the $\mathcal{U}(\mathfrak{g})^K$-module plays an important role in the theory of $(\mathfrak{g}, K)$-modules such as Harish-Chandra's subquotient theorem [14]. Remark that for non-compact $G'_{\mathbb{R}}$, the $\mathcal{U}(\mathfrak{g})^{G'}$-modules may be reducible.

The space $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ is deeply related to the discrete decomposability:

Fact 1.5 (T. Kobayashi [31, Lemma 1.5])**.** *Let* V *be an irreducible* $(\mathfrak{g}, K)$*-module.* $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ *is non-zero for some irreducible* $(\mathfrak{g}', K')$*-module if and only if* $V|_{(\mathfrak{g}',K')}$ *is discretely decomposable.*

In many cases, the restriction of an irreducible $(\mathfrak{g}, K)$-module to the subpair $(\mathfrak{g}', K')$ is not discretely decomposable, and $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ is zero for any irreducible $(\mathfrak{g}', K')$-module V'. In such cases, any general theories to deal with branching laws are not known. Nevertheless, some geometric and analytic methods work well for some irreducible unitary representations [7], [17], [42], [43], [47], [58], [60], [61], [80].

The space $\mathrm{Hom}_{\mathfrak{g}',K'}(V', V)$ may have many information about branching laws with continuous spectrum. T. Kobayashi proposed the program to construct symmetry breaking operators explicitly. Here a symmetric breaking operators means a continuous $G'_{\mathbb{R}}$-intertwining operator from a continuous irreducible (or finite length) representation of $G_{\mathbb{R}}$ to one of $G'_{\mathbb{R}}$. The explicit construction was obtained for principal series representations and $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{O}(n+1,1), \mathrm{O}(n,1))$ [47], and holomorphic discrete series representations [45, 46]. A relation between symmetry breaking operators and $(\mathfrak{g}, K)$-module intertwining operators was discussed in [39], and recent developments and open problems on this topic are in [40].

1.4 Multiplicity-free representation

The concept of a multiplicity-free representation is just as important as that of the discrete decomposability.

Definition 1.6. For a unitary representation V of $G_{\mathbb{R}}$, we denote by $\mathcal{M}_{G_{\mathbb{R}}}(V)$ the essential supremum of the multiplicity function of the irreducible decomposition. V is said to be *multiplicity-free* if $\mathcal{M}_{G_{\mathbb{R}}}(V) = 1$, and to have *uniformly bounded multiplicities* if $\mathcal{M}_{G_{\mathbb{R}}}(V) < \infty$.

We use the same terminology for completely reducible $(\mathfrak{g}, K)$-modules and algebraic representations.

The Fourier transform, the Fourier series expansion and spherical harmonics are classical and important examples of multiplicity-free representations. In the representation theory of Lie groups, many multiplicity-free representations are known such as the Clebsh–Gordan formula, the Pieri rule, the Peter–Weyl theorem, the branching laws for $(G_{\mathbb{R}}, G'_{\mathbb{R}}) = (\mathrm{U}(n), \mathrm{U}(n-1))$ and $(\mathrm{SO}(n), \mathrm{SO}(n-1))$, the Cartan–Helgason theorem, the Plancherel formulas for Riemannian symmetric spaces and group manifolds.

A multiplicity-free representation has a 'canonical' irreducible decomposition, and the representation yields some natural transform like the Fourier transform. Therefore, finding a multiplicity-free representation may be related to finding some good analysis and geometry. We refer the reader to [34] for the point of view.

A spherical variety is one of the geometric objects to produce multiplicity-free representations.

Definition 1.7. Let G be a complex reductive algebraic group with Borel subgroup B. A G-variety X is said to be *spherical* if B has an open orbit on X.

By [76, Theorem 2], an affine G-variety X is spherical if and only if the coordinate ring $\mathbb{C}[X]$ of X is a multiplicity-free G-module.

T. Kobayashi introduced the notion of visible action on a complex manifold in [33], and showed the propagation theorem of multiplicity-free property in [29, 38]. Many multiplicity-free representations can be explained in the machinery [34]. An advantage of the notion is that infinite-dimensional unitary representations of any Lie group such as non-reductive Lie groups can be treated in the framework. An example of applications of visible actions is the branching laws of unitary highest weight modules with respect to symmetric subgroups [36].

Let $G_{\mathbb{R}}$ be a connected simple Lie group of Hermitian type with Cartan involution θ and $G'_{\mathbb{R}}$ a symmetric subgroup of $G_{\mathbb{R}}$ preserved by θ. Put $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$ and $K'_{\mathbb{R}} := (G'_{\mathbb{R}})^{\theta}$. Let $\mathfrak{a}'_{\mathbb{R}}$ be a maximal abelian subspace in $\mathfrak{g}^{-\theta} \cap (\mathfrak{g}'_{\mathbb{R}})^{\perp}$. Set $M_{\mathbb{R}} := Z_{K'_{\mathbb{R}}}(\mathfrak{a}'_{\mathbb{R}})$.

Fact 1.8 (T. Kobayashi [34, Theorem 18, Theorem 34], [36, Theorem A])**.** *Let $\mathcal{H}$ be a unitary highest weight module of $G_{\mathbb{R}}$ embedded in $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ for some irreducible unitary representation F of $K_{\mathbb{R}}$. If $F|_{M_{\mathbb{R}}}$ is multiplicity-free, then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free. In particular, if $\mathcal{H}$ is of scalar type, then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free.*

Fact 1.9 (T. Kobayashi [29, Theorem B])**.** *Retain the notation in the above fact. If $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is a symmetric pair of holomorphic type (i.e. $\mathfrak{g}'$ contains the center of $\mathfrak{k}$), then $\mathcal{H}|_{G'_{\mathbb{R}}}$ has uniformly bounded multiplicities.*

The second fact asserts $\mathcal{M}_{G'_{\mathbb{R}}}(\mathcal{H}) < \infty$. In this case, T Kobayashi stated in [36, Remark 1.5] that using the Howe duality [19], we could relate the multiplicity function and the stable branching coefficients defined by F. Sato [70].

1.5 Stability of multiplicities

If a representation has uniformly bounded multiplicities, the multiplicity function may have a stability property. We state Sato's stability theorem [70] as follows.

Let G be a connected complex semisimple algebraic group and G' a connected reductive subgroup of G. Assume that (G, G') is a spherical pair, that is, there is a Borel subgroup B of G such that BG' is open dense in G. Put

$$L := \{g \in G : gBG' = BG'\} \cap G'.$$

Then L is a reductive subgroup of G' by a Theorem of Brion–Luna–Vust [4]. Note that the set of equivalence classes of irreducible representations of L can be parametrized by a set of characters of $B \cap G' \subset L$. We denote by Λ^+ the set of all dominant integral weights of B. For a weight $\lambda \in \Lambda^+$, we write $F^G(\lambda)$ for the finite-dimensional irreducible representation of G with highest weight λ. Set

$$\Lambda^+(G/G') := \left\{\lambda \in \Lambda^+ : F^G(\lambda)^{G'} \neq 0\right\}.$$

Under this setting, F. Sato proved the following theorem.

Fact 1.10 (F. Sato [70, Theorem 3])**.** *Let F be a finite-dimensional irreducible representation of G'. Then for any $\lambda \in \Lambda^+$, there exists a weight $\nu_0 \in \Lambda^+(G/G')$ such that*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{G'}(F^G(\lambda + \nu_0 + \nu), F)) = \dim_{\mathbb{C}}(\mathrm{Hom}_L(F^L(\lambda|_{B\cap G'}), F))$$

for any $\nu \in \Lambda^+(G/G')$.

The above fact asserts two things: the multiplicity function of $\mathrm{Ind}_{G'}^G(F)$ is invariant by the translation of $\Lambda^+(G/G')$ for enough large parameters; and for such parameters, the multiplicity function can be described by the multiplicity function of the fiber F. The first property is called stability by F. Sato in [70].

For the case of symmetric pairs $(G_{\mathbb{R}}, G'_{\mathbb{R}})$, the stability property appeared in Wallach's book [77, Cor. 8.5.15]. In the case, we can see the phenomena in some literatures [28, Lemma 3.4], [49]. Stable branching coefficients was computed for some concrete compact Lie groups [54], [74].

2 Main results

In this section, we state the main theorems in this thesis.

Let $G_{\mathbb{R}}$ be a reductive Lie group with Cartan involution θ and $G'_{\mathbb{R}}$ a reductive subgroup of $G_{\mathbb{R}}$ closed under θ. We put $K_{\mathbb{R}} := G_{\mathbb{R}}^{\theta}$ and $\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}})$ and denote by K and $\mathfrak{g}$ the complexifications of $K_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$, respectively. In a similar way, we define K' and $\mathfrak{g}'$ for $G'_{\mathbb{R}}$.

2.1 Direct integral and intertwining operators

For a representation V of $G_{\mathbb{R}}$, we write V_K for the subspace of $K_{\mathbb{R}}$-finite vectors.

Theorem 2.1. *Let V be an irreducible unitary representation of $G_{\mathbb{R}}$. Suppose that the irreducible decomposition of $V|_{G'_{\mathbb{R}}}$ is as in (1.2.1). Then for almost every $\pi \in \widehat{G'_{\mathbb{R}}}$, there exist a $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $\mathbb{C}^{m(\pi)}$ and a surjective $(\mathfrak{g}', K')$-module and $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:*

$$\phi_\pi : V_K \to (V_\pi)_{K'} \otimes \mathbb{C}^{m(\pi)}$$

such that the vector field $(\pi \mapsto \phi_\pi(v))$ is equal to v in V for any $v \in V_K$.

Remark 2.2. A similar result for the Plancherel formulas on homogeneous spaces is well-known [2].

For the proof of the theorem, we use the reduction theorem by A. E. Nussbaum [57], which is a generalization of von Neumann's reduction theorem for bounded operators to closed operators. Since V_K and $\mathcal{U}(\mathfrak{g})^{G'}$ are at most countable-dimensional, we can define ϕ_π for almost every π. The compatibility with the $\mathfrak{g}'$-action is proved by Fact 1.2.

Definition 2.3. For a $(\mathfrak{g}, K)$-module V and a $(\mathfrak{g}', K')$-module V', we define

$$H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$$

as the space of all coinvariants of $V \otimes (V')^*_{K'}$. Then $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ has a natural $\mathcal{U}(\mathfrak{g})^{G'}$-module structure.

Remark 2.4. In the context of the Howe duality [19], the space $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ appears as the full theta lift. Hence the $\mathcal{U}(\mathfrak{g})^{G'}$-module structure on $H_0(\mathfrak{g}', K'; V \otimes (V')^*_{K'})$ is used in the study of the Howe duality [50], [52].

By Theorem 2.1, there is a surjective $\mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism:

$$H_0(\mathfrak{g}', K'; V_K \otimes (V_\pi^*)_{K'}) \to \mathbb{C}^{m(\pi)}$$

for almost every π. This is one of the reasons to study $\mathcal{U}(\mathfrak{g})^{G'}$-modules.

2.2 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: generalized Verma modules

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ constructed from a semisimple element $H \in \mathfrak{g}'$. Define $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ in a similar way for $\mathfrak{g}'$. We fix a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{l}'$ and extend it to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$.

For a finite-dimensional irreducible $\mathfrak{l}$-module F, we define a generalized Verma module by

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F.$$

The following theorem is needed to consider the branching problem of generalized Verma modules.

Fact 2.5 (T. Kobayashi [37])**.** *Under the setting,* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ *is discretely decomposable and* $\mathfrak{g}'$*-admissible.*

Following Knapp–Vogan's book [26], we recall the notion of the good range. A finite-dimensional irreducible $\mathfrak{l}$-module F is said to be in the *good range* if the infinitesimal character λ of F satisfies

$$\mathrm{Re}(\lambda + \rho(\mathfrak{u}), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Under this setting, the following theorems hold.

Theorem 2.6. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module in the good range. Suppose that* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ *is completely reducible and* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')$ *is an irreducible direct summand. Then the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module* $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is irreducible.*

Theorem 2.7. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module in the good range. Then the length of the* $\mathcal{U}(\mathfrak{g})^{G'}$*-module* $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ *is less than or equal to the length of* $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}((F')^* \otimes \mathbb{C}_{-2\rho(\mathfrak{u})})$.

2.3 Outline of the proof of Theorem 2.6

To study the $\mathcal{U}(\mathfrak{g})^{G'}$-modules, we define a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module structure on the space of $\Delta(G')$-finite linear maps as follows.

Definition 2.8. Let V be a $(\mathfrak{g}, K)$-module and V' be a $(\mathfrak{g}', K')$-module. Then $\mathfrak{g}' \oplus \mathfrak{g}$ and $K' \times K$ act on $\mathrm{Hom}_{\mathbb{C}}(V', V)$. $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ is defined as the sum of finite-dimensional $(\Delta(\mathfrak{g}'), \Delta(K'))$-submodules which lift to a representation of $\Delta(G')$. Then $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G')}$ becomes a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. We define a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module $\mathrm{Hom}_{\mathbb{C}}(V, V')_{\Delta(G')}$ in the same way.

Remark 2.9. If $G'_{\mathbb{R}}$ is equal to $G_{\mathbb{R}}$, a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module is a Harish-Chandra module of a complex reductive Lie group. In this case, for objects V, V' of the BGG category $\mathcal{O}$, $\mathrm{Hom}_{\mathbb{C}}(V', V)_{\Delta(G)}$ was studied by many mathematicians because the module is related to primitive ideals of the universal enveloping algebra and principal series representations of complex semisimple Lie groups (e.g. [3], [5], [8, 9], [22]).

An important property of the module is that the $\Delta(G')$-invariant part of the module is equal to the space of all intertwining operators. Hence we can study the $\mathcal{U}(\mathfrak{g})^{G'}$-module through the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. More precisely, the following two propositions hold.

Proposition 2.10. *Retain the settings in the above. Put*

$$\mathcal{I} := \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')} \cap \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})\Delta(\mathfrak{g}').$$

Then there is an algebra isomorphism:

$$\begin{array}{rccc} \alpha : & \mathcal{U}(\mathfrak{g})^{G'} & \simeq & \mathcal{U}(\mathfrak{g}' \oplus \mathfrak{g})^{\Delta(G')}/\mathcal{I} \\ & \Cup & & \Cup \\ & X & \mapsto & I \otimes X + \mathcal{I}. \end{array}$$

Proposition 2.11. *Let W be a $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module. Then the length of the $\mathcal{U}(\mathfrak{g})^{G'}$-module on $W^{\Delta(G')}$ is bounded by the length of W. In particular if W is irreducible, then $W^{\Delta(G')}$ is irreducible or zero.*

To prove Theorem 2.7, 2.6 and 2.17, we construct $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules using the Zuckerman derived functor $R^i\Gamma$. Let L' be the analytic subgroup of G' with Lie algebra $\mathfrak{l}'$. For a finite-dimensional irreducible $\mathfrak{l}$-module F with infinitesimal character λ, let $\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}(\lambda)$ be the full subcategory of the relative BGG category $\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}$ whose object V satisfies that $V \otimes F$ can lifts to a representation of L'. We denote by $\mathcal{F}(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$ the category of $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-modules of finite length. The following theorem is a key result.

Theorem 2.12. *Let F be a finite-dimensional irreducible $\mathfrak{l}$-module with infinitesimal character λ in the good range. Set $S := \dim_{\mathbb{C}}(\mathfrak{u}')$. Then the following functor gives a category embedding:*

$$\mathcal{O}^{\mathfrak{g}'}_{\mathfrak{q}'}(\lambda) \ni M \mapsto R^S\Gamma^{\Delta(G')}_{\Delta(L')}(M \otimes \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(F)) \in \mathcal{F}(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G')),$$

that is, the functor is exact and fully faithful, and maps irreducible objects to irreducible objects.

Remark 2.13. If $\mathfrak{g} = \mathfrak{g}'$, the theorem was proved by T. J. Enright [8, Chapter 16] except for the full faithfulness.

Remark 2.14. For a non-symmetric pair $(\mathfrak{g}, \mathfrak{k})$, a $(\mathfrak{g}, \mathfrak{k})$-module with some finiteness conditions is called a generalized Harish-Chandra module by I. Penkov and G. Zuckerman [64, 65, 66].

The proof of the theorem follows the proofs in Knapp–Vogan's book [26], Wallach's book [79] and Penkov–Zuckerman's papers.

Under the settings of Theorem 2.6, using the functor, we can prove the following $\mathcal{U}(\mathfrak{g})^{G'}$-module isomorphism:

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)) \simeq R^S\Gamma_{\Delta(L')}^{\Delta(G')}(M \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))^{\Delta(G')},$$

where M is a unique irreducible quotient of $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}((F')^* \otimes \mathbb{C}_{-2\rho(\mathfrak{u}')})$. Thus the irreducibility of $\mathrm{Hom}_{\mathfrak{g}',K'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'), \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is reduced to the irreducibility of $R^S\Gamma_{\Delta(L')}^{\Delta(G')}(L \otimes \mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ by Proposition 2.11.

2.4 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: Zuckerman derived functor modules

To apply Theorem 2.6 to Zuckerman derived functor modules, we define a quasi-abelian parabolic subalgebra.

Definition 2.15 (quasi-abelian)**.** $\mathfrak{q}$ is said to be *quasi-abelian* with respect to $\mathfrak{g}'$ if $(\alpha, \beta) \geq 0$ holds for any $\alpha \in \Delta(\mathfrak{u}', \mathfrak{h}')$ and $\beta \in \Delta(\mathfrak{u}'', \mathfrak{h}')$.

Remark 2.16. In the case of $G'_{\mathbb{R}} = K_{\mathbb{R}}$, the notion of a quasi-abelian parabolic subalgebra was used by Enright–Parthasarathy–Wallach–Wolf [10] to study Zuckerman derived functor modules.

If $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{g}'$, the completely reducibility always holds as long as F is in the good range. We assume $H \in \mathfrak{k}'$, and set $K_L := Z_K(H)$ and $K'_L := Z_{K'}(H)$. Then the following theorem holds.

Theorem 2.17. *Let F be a finite-dimensional irreducible $(\mathfrak{l}, K_L)$-module in the good range. Suppose that there exists an ideal $\mathfrak{k}_1$ of $\mathfrak{k}$ such that $H \in \mathfrak{k}_1$ and $\mathfrak{u} \cap \mathfrak{k} \subset \mathfrak{k}_1 \subset \mathfrak{g}'$, and suppose that $\mathfrak{q}$ is quasi-abelian with respect to $\mathfrak{g}'$. Put $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. Then $R^S\Gamma_{K_L}^{K}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))|_{(\mathfrak{g}',K')}$ is completely reducible and each direct summand is of the form $R^S\Gamma_{K'_L}^{K'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F'))$ with F' in the good range. Moreover, $\mathrm{Hom}_{\mathfrak{g}',K'}(R^S\Gamma_{K'_L}^{K'}(\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}'}(F')), R^S\Gamma_{K_L}^{K}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)))$ is irreducible as a $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

Remark 2.18. As we mentioned in the introduction, T. Kobayashi gave a necessary and sufficient condition for the discrete decomposability of $A_{\mathfrak{q}}(\lambda)$ (including discrete series representations), and gave some examples of explicit branching laws in [27, 28, 30, 31].

Remark 2.19. One of important cases satisfying the assumptions is the case of discretely decomposable restrictions of discrete series representations with respect to symmetric subgroups. For small discrete series representations and its restrictions to symmetric subgroups, the branching law was computed by Gross–Wallach in [13]. For any discrete series representations and non-symmetric subgroups, Duflo–Vargas gave a formula of the multiplicities like Blattner's formula in [6]. The subgroup K_1 in our setting is the same as in [13] and [6].

Under the assumptions in the theorem, $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F)|_{\mathfrak{g}'}$ is completely reducible. Hence the proof is reduced to Theorem 2.6 by the following fact.

Fact 2.20 (Gross–Wallach [13, Lemma 7])**.** *Under the assumptions in Theorem 2.17, $R^S\Gamma^K_{K_L}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$ is isomorphic to $R^S\Gamma^{K_1}_{K_1\cap L}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(F))$.*

2.5 Irreducibility of $\mathcal{U}(\mathfrak{g})^{G'}$-module: Holomorphic discrete series representations

We study the $\mathcal{U}(\mathfrak{g})^{G'}$-module and $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module arising from the branching law with continuous spectrum.

Following Kanpp's book [25, Chapter VI], we recall holomorphic discrete series representations. Assume that $G_{\mathbb{R}}$ is a connected simple Lie group of Hermitian type. Fix an element $H \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}_{\mathbb{R}})$ such that $\mathrm{ad}_{\mathfrak{g}}(H)$ has eigenvalues $-1, 0$ and 1. Then $\mathfrak{g}$ is decomposed into the direct sum of eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(H)$:

$$\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$$

corresponding to eigenvalues $1, 0$ and -1, respectively. We put $\mathfrak{q} := \mathfrak{k} \oplus \mathfrak{p}_+$ and $\overline{\mathfrak{q}} = \mathfrak{k} \oplus \mathfrak{p}_-$. Then $G_{\mathbb{R}}/K_{\mathbb{R}}$ admits a $G_{\mathbb{R}}$-invariant complex structure such that the natural embedding $G_{\mathbb{R}}/K_{\mathbb{R}} \hookrightarrow G/\overline{Q}$ is holomorphic.

Fact 2.21 (Harish-Chandra [15])**.** *Let F be an irreducible unitary representation of $K_{\mathbb{R}}$. Then $\mathcal{O} \cap L^2(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ is non-zero if and only if F is in the good range with respect to $\mathfrak{q}$, where $\mathcal{O} \cap L^2$ means the space of all holomorphic and L^2 sections. Furthermore, if $\mathcal{O} \cap L^2(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)$ is non-zero, it is irreducible and unitary as a representation of $G_{\mathbb{R}}$.*

The irreducible unitary representation is called a *holomorphic discrete series representation.*

Assume that $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is a symmetric pair of anti-holomorphic type (i.e. $\mathfrak{g}'_{\mathbb{R}}$ does not contains the center of $\mathfrak{k}_{\mathbb{R}}$) and $G'_{\mathbb{R}}$ satisfies the following condition:

$$\mathrm{Ad}_{\mathfrak{g}}(G'_{\mathbb{R}}) = G' \cap \mathrm{Int}(\mathfrak{g}_{\mathbb{R}}),$$

where G' is the analytic subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}'$.

It is known that the branching law of a holomorphic discrete series representation with respect to $G'_{\mathbb{R}}$ is reduced to the Plancherel formula of the Riemannian symmetric space $G'_{\mathbb{R}}/K'_{\mathbb{R}}$. Hence the irreducible decomposition has a continuous spectrum.

Fact 2.22 (J. Repka [68], R. Howe [17], Ólafsson–Ørsted [58])**.** *For a holomorphic discrete series representation V of $G_{\mathbb{R}}$ realized in $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, \mathcal{V})$ for a holomorphic $G_{\mathbb{R}}$-equivariant vector bundle $\mathcal{V}$ on $G_{\mathbb{R}}/K_{\mathbb{R}}$, the following isomorphism holds:*

$$V|_{G'_{\mathbb{R}}} \simeq L^2(G'_{\mathbb{R}}/K'_{\mathbb{R}}, \mathcal{V}|_{G'_{\mathbb{R}}/K'_{\mathbb{R}}}).$$

Let $Q'_{\mathbb{R}} = M'_{\mathbb{R}}A'_{\mathbb{R}}N'_{\mathbb{R}}$ be a minimal parabolic subgroup of $G'_{\mathbb{R}}$. Take a Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{m}'$, and put $\mathfrak{h}' := \mathfrak{a}' \oplus \mathfrak{t}'$. Write $I(\delta, \nu)$ for the underlying Harish-Chandra module of the principal series representation induced from $(\delta, V_\delta) \in \widehat{M'_{\mathbb{R}}}$ and $\nu \in (\mathfrak{a}')^*$. We consider 'generic' principal series representations in the following sense.

Lemma 2.23. *Let μ be the infinitesimal character of δ. Assume*

$$\frac{2(-\nu - \mu + \rho(\mathfrak{n}'), \alpha)}{(\alpha, \alpha)} \notin \mathbb{Z} \text{ for any } \alpha \in \Delta(\mathfrak{n}', \mathfrak{h}').$$

Let W be an irreducible subquotient of $I(\delta, \nu)$. Then the following properties hold:

(a) $\mathrm{End}_{\mathbb{C}}(W)_{\Delta(G')}$ *is irreducible as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module;*

(b) *for any finite-dimensional representation F of G', $F \otimes W$ is completely reducible.*

Fix a maximal abelian subspace $\mathfrak{t}_{\mathbb{R}}$ of $(\mathfrak{g}'_{\mathbb{R}})^{\perp} \cap \mathfrak{k}_{\mathbb{R}}$. Then $\mathfrak{t}_{\mathbb{R}}$ is a maximal abelian subspace of $(\mathfrak{g}'_{\mathbb{R}})^{\perp}$. Choose a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ containing $\Delta(\mathfrak{p}_+, \mathfrak{t})$. Let γ_1 be the highest weight of $\Delta(\mathfrak{g}'_{\mathbb{R}}, \mathfrak{a}'_{\mathbb{R}})$ with respect to the parabolic subgroup $Q'_{\mathbb{R}}$, and β_1 be the highest weight of $\Delta(\mathfrak{p}_+, \mathfrak{t})$. Then the following theorem holds.

Theorem 2.24. *Retain the notation and the assumption in Lemma 2.23. Let $\mathbb{C}_\lambda$ be a one-dimensional representation of $K_\mathbb{R}$ with weight λ. Assume*

$$\pm\frac{(w(\nu - \rho(\mathfrak{n})) + \rho(\mathfrak{n}), \gamma_1)}{(\gamma_1, \gamma_1)} + \frac{(\lambda, \beta_1)}{(\beta_1, \beta_1)} \notin \mathbb{Z}$$

for any $w \in W_{\mathfrak{g}'_\mathbb{R}}$. Then the $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ is irreducible.

Remark 2.25. In the case of the trivial representation W (the theorem can not apply to this case), $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), \mathbf{1})_{\Delta(G')}$ is isomorphic to a degenerate principal series representation of some real form of G. The irreducibility of degenerate principal series representations can be determined from the data of the K-type decomposition and the $\mathfrak{p}$-action on each K-type. T. Hirai introduced this method to study degenerate principal series representations of Lorentz groups [16]. Many mathematicians computed the composition series of degenerate principal series representations by a similar way such as V. F. Molčanov [55], Klimyk–Gavrilik [24], Johnson–Wallach [21] and Kudla–Rallis [48].

The structure of $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ can be computed by a similar way. Under the assumptions of Theorem 2.24, $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$ is completely reducible and multiplicity-free as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module. Hence we can use the irreducible decomposition as a $(\mathfrak{g}' \oplus \mathfrak{g}', \Delta(G'))$-module instead of the K-type decomposition.

It follows from Harish-Chandra's classification of holomorphic discrete series representations (Fact 2.21) that if λ is in the good range, $\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda)$ is isomorphic to the underlying Harish-Chandra module of a holomorphic discrete series representation. We apply the Jantzen–Zuckerman translation functor to $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(\mathbb{C}_\lambda), W)_{\Delta(G')}$, and we obtain the following theorem.

Theorem 2.26. *Let F be an irreducible unitary representation of $K_\mathbb{R}$ in the good range, and let (δ, V_δ) be an irreducible subrepresentation of $F|_{M'_\mathbb{R}}$. Suppose that the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ acts on F by a character λ. Assume that λ, δ and $\nu \in (\mathfrak{a}')^*$ satisfy the conditions of Lemma 2.23 and Theorem 2.24. Let W be an irreducible subquotient of $I(\delta, \nu)$. Then $\mathrm{Hom}_\mathbb{C}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F), W)_{\Delta(G')}$ is an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module, and $\mathrm{Hom}_{\mathfrak{g}', K'}(\mathrm{ind}^\mathfrak{g}_\mathfrak{q}(F), W)$ is irreducible as a $\mathcal{U}(\mathfrak{g})^{G'}$-module.*

2.6 Stability of multiplicities

We generalize Sato's stability theorem (Fact 1.10) to the case of quasi-affine spherical varieties.

Let X be a quasi-affine spherical variety of a complex connected reductive algebraic group G with Borel subgroup $B = TN$. Then the coordinate ring $\mathbb{C}[X]$ is a multiplicity-free G-module. Fix an open orbit $Bx_0 \subset X$ and put $L := \{g \in G : gx_0 = x_0, gBx_0 \subset Bx_0\}$. Then L is a reductive subgroup of G by the theorem of Brion–Luna–Vust [4, Théorème 3.4]. Consider a finitely generated torsion-free $(\mathbb{C}[X], G)$-module M such as the space of all global sections of a G-equivariant vector bundle on X. Then the following theorem holds.

Theorem 2.27. *There exists a weight $\lambda_0 \in \Lambda^+(\mathbb{C}[X])$ such that*

$$m^G_M(\lambda + \lambda_0) = m^L_{M/\mathfrak{m}(x_0)M}(\lambda|_{B_{x_0}})$$

for any $\lambda \in \Lambda^+(M)$.

Here $m^G_M(\cdot)$ is the multiplicity function of the G-module M, and $\Lambda^+(M)$ is the set of weights of $B/N \simeq T$ in M^N. We denote by $\mathfrak{m}(x_0)$ the maximal ideal of $\mathbb{C}[X]$ corresponding to x_0.

Remark 2.28. If X is an affine homogeneous variety G/G' of G, the theorem is just Sato's stability theorem (Fact 1.10).

The proof is essentially the same as the proof of Sato's stability theorem [70]. The only difference is that we study some behavior of the evaluation map $M \to M/\mathfrak{m}(x_0)M$ instead of using the reductivity of G'.

As an application of Theorem 2.27, we obtain the following corollary. Recall the notation in Fact 1.9.

Corollary 2.29. *Let $\mathcal{H}$ be a holomorphic discrete series representation of $G_{\mathbb{R}}$. Suppose that $G'_{\mathbb{R}}$ is connected and $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ is a symmetric pair of holomorphic type. Then we have*

$$\mathcal{M}_{G'_{\mathbb{R}}}(\mathcal{H}) = \mathcal{M}_{M_{\mathbb{R}}}(\mathcal{H}^{\mathfrak{p}_+}_K),$$

where $\mathcal{M}_{G'_{\mathbb{R}}}(\mathcal{H})$ is the maximal value of the multiplicities. In particular, $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free if and only if $(\mathcal{H}^{\mathfrak{p}_+}_K)|_{M_{\mathbb{R}}}$ is multiplicity-free.

Remark 2.30. $\mathcal{M}_{G'_{\mathbb{R}}}(\mathcal{H}) < \infty$ and the 'if part' of the second assertion were proved by T. Kobayashi (Fact 1.8, 1.9).

We write σ for the involution defining the symmetric pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$. We can reduce the branching law of $\mathcal{H}|_{G'_{\mathbb{R}}}$ to the irreducible decomposition of $S(\mathfrak{p}^{-\sigma}_-) \otimes \mathcal{H}^{\mathfrak{p}_+}_K$ as a K'-module (see [20, Proposition 2.5] and [36, Lemma 8.8]). As in the proof of [36, Theorem 8.3], the irreducible decomposition of

$S(\mathfrak{p}_-^{-\sigma}) \otimes \mathcal{H}_K^{\mathfrak{p}_+}$ is considered as the K-type decomposition of some direct sum of holomorphic discrete series representations of the associated symmetric subgroup $G_{\mathbb{R}}^{\theta\sigma}$. Since $S(\mathfrak{p}_-^{-\sigma})$ is multiplicity-free by the Hua–Kostant–Schmid theorem [71], we can apply Theorem 2.27 to $S(\mathfrak{p}_-^{-\sigma}) \otimes \mathcal{H}_K^{\mathfrak{p}_+}$, and this shows the corollary.

2.7 Classification of multiplicity-free restrictions of holomorphic discrete series representations

We classify multiplicity-free restrictions of holomorphic discrete series representations with respect to symmetric subgroups.

Let $G_{\mathbb{R}}$ be a connected simple Lie group of Hermitian type with simply-connected complexification G, and σ be an involutive automorphism of $G_{\mathbb{R}}$. Put $G'_{\mathbb{R}} := G_{\mathbb{R}}^{\sigma}$. The following theorem is the classification result.

Theorem 2.31. *Let $\mathcal{H}$ be a holomorphic discrete series representation of $G_{\mathbb{R}}$. Put $F := \mathcal{H}_K^{\mathfrak{p}_+}$. Then $\mathcal{H}|_{G'_{\mathbb{R}}}$ is multiplicity-free if and only if F is one-dimensional or the highest weight of $F|_{[\mathfrak{k},\mathfrak{k}]}$ belongs to $\Lambda(\sigma)$ in Table 1.*

Remark 2.32. In the case of scalar type $\mathcal{H}$, the theorem was proved by T. Kobayashi [29, 34] (Fact 1.8). The multiplicity-freeness was shown for $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}}) = (\mathfrak{so}(2,n), \mathfrak{so}(2,n-1))$ by Jakobsen–Vergne [20, Corollary 3.1], and for $(\mathfrak{su}(p,q), \mathfrak{u}(p-1,q))$ by T. Kobayashi [36, Theorem 8.10].

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}_{\mathbb{R}}^{\sigma}$	$\pm$	st.	$\Lambda(\sigma)$
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p_1,q_1)+\mathfrak{su}(p_2,q_2)+\mathfrak{t}$	h	○	$(p_1+q_1=1, p+q-1)$ any
				$(p_1+q_1=2, p+q-2)$ $m\omega_i$
				ω_i, $m\omega_i(i=1,p\pm 1,p+q-1)$
	$\mathfrak{so}(p,q)$	a		ω_i
	$\mathfrak{sp}(p/2,q/2)$	a		$m\omega_i(i=1,p\pm 1,p+q-1)$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$	h		ω_i
	$\mathfrak{sp}(n,\mathbb{R})$	h		$(n=2)$ any
				$(n\le 4)$ $m\omega_i$
				$m\omega_i(i=1,n\pm 1,2n-1)$
	$\mathfrak{sl}(n,\mathbb{C})+\mathbb{R}$	a	○	$(n=2)$ $m\omega_i$
				ω_i, $m\omega_i(i=1,n\pm 1,2n-1)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2p)+\mathfrak{so}^*(2n-2p)$	h	○	$(\min(p,n-p)=1)$ $m\omega_i$
				ω_2,ω_n
	$\mathfrak{u}(p,n-p)$	h	○	$m\omega_i(i=2,n)$
				$(n=4, p\text{:odd})$ $m\omega_i(i=2,3,4)$
	$\mathfrak{so}(n,\mathbb{C})$	a		ω_2,ω_n

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}^{\sigma}_{\mathbb{R}}$	$\pm$	st.	$\Lambda(\sigma)$
	$\mathfrak{su}^*(n)+\mathbb{R}$	a	○	$m\omega_i (i=2,n)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,p)+\mathfrak{so}(n-p)$	h		$(p=n-1)$ any
				$(n\text{:odd})\ \omega_1$
			○	$(n\text{:even } p=0,n-2)\ m\omega_1, m\omega_2$
				$(n\text{:even})\ \omega_1,\omega_2$
	$\mathfrak{so}(1,p)+\mathfrak{so}(1,n-p)$	a		the same as above
	$\mathfrak{u}(1,n/2)$	h	○	$m\omega_i (i=1,2,n/2)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(p,\mathbb{R})+\mathfrak{sp}(n-p,\mathbb{R})$	h		$(\min(p,n-p)=1)\ m\omega_i$
				ω_2,ω_n
	$\mathfrak{u}(p,n-p)$	h	○	ω_i
	$\mathfrak{sp}(n/2,\mathbb{C})$	a		ω_2,ω_n
	$\mathfrak{sl}(n,\mathbb{R})+\mathbb{R}$	a	○	ω_i
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)+\mathfrak{t}$	h	○	$m\omega_6$
	$\mathfrak{so}^*(10)+\mathfrak{t}$	h	○	
	$\mathfrak{so}(2,8)+\mathfrak{t}$	h	○	
	$\mathfrak{su}(5,1)+\mathfrak{sl}(2,\mathbb{R})$	h		none
	$\mathfrak{su}(4,2)+\mathfrak{su}(2)$	h		none
	$\mathfrak{f}_{4(-20)}$	a		$m\omega_2,\ m\omega_3$
	$\mathfrak{sp}(2,2)$	a		ω_6
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)}+\mathfrak{t}$	h	○	none
	$\mathfrak{e}_{6(-14)}+\mathfrak{t}$	h	○	none
	$\mathfrak{so}(10,2)+\mathfrak{sl}(2,\mathbb{R})$	h		none
	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	h		none
	$\mathfrak{su}(6,2)$	h		none
	$\mathfrak{e}_{6(-26)}+\mathbb{R}$	a	○	none
	$\mathfrak{su}^*(8)$	a		none

Table 1: the classification of multiplicity-free restrictions of holomorphic discrete series representations

The circle of the column with title 'st' means that its classification can be reduced to the Stembridge classification, that is, $G'_{\mathbb{R}}$ has a one-dimensional center. The column with title '$\pm$' means that if the value is 'h', the symmetric pair is of holomorphic type, and if the value is 'a', the symmetric pair is of anti-holomorphic type. ω's are fundamental weights corresponding to simple roots given later. $m\omega_i$ means that $m\omega_i$ is in $\Lambda(\sigma)$ for any m.

To prove the classification result, the following theorem is useful. Fix a unitary character $(\zeta, \mathbb{C}_\zeta)$ of $K_{\mathbb{R}}$. For an irreducible unitary representation F

of $K_{\mathbb{R}}$ with infinitesimal character λ, we define

$$Z_{hol}(F) := \{z \in \mathbb{Z} : (\lambda + \rho(\mathfrak{p}_+), \alpha) < 0 \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h})\},$$
$$Z_{fin}(F) := \left\{ z \in \mathbb{Z} : \frac{2(\lambda + \rho(\mathfrak{p}_+), \alpha)}{(\alpha, \alpha)} \in \{1, 2, \ldots\} \text{ for any } \alpha \in \Delta(\mathfrak{p}_+, \mathfrak{h}) \right\},$$

and let $L(F)$ denote a unique irreducible submodule of $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} F)_{K_{\mathbb{R}}}$.

Theorem 2.33. *Let F be a unitary irreducible representation of $K_{\mathbb{R}}$. Then the following conditions are equivalent:*

(a) $\mathcal{M}_{G^\sigma_{\mathbb{R}}}(\overline{\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for any* $z \in Z_{hol}(F)$*;*

(b) $\mathcal{M}_{G^\sigma_{\mathbb{R}}}(\overline{\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(F \otimes \mathbb{C}_{z\zeta})}) = 1$ *for some* $z \in Z_{hol}(F)$*;*

(c) $\mathcal{M}_{G^\sigma}(L(F \otimes \mathbb{C}_{z\zeta})) = 1$ *for any* $z \in Z_{fin}(F)$*;*

(d) $\mathcal{M}_{M_{\mathbb{R}}}(F) = 1$.

where $M_{\mathbb{R}}$ is the subgroup of $K'_{\mathbb{R}}$ defined before Fact 1.8.

Remark 2.34. For the proof of the theorem, we use the method, analytic continuation of holomorphic discrete series representations [1], [69], [75], [78]. To prove the theorem, we need only that the family of the representations $\mathcal{O}(G_{\mathbb{R}}/K_{\mathbb{R}}, G_{\mathbb{R}} \times_{K_{\mathbb{R}}} (F \otimes \mathbb{C}_{z\zeta}))_{K_{\mathbb{R}}}$ depends on z polynomially, that is, any element of $\mathfrak{g}$ acts on the space by a differential operator with polynomial coefficient on z.

Remark 2.35. In the branching problem, the method of analytic continuation was used to study symmetry breaking operators [47].

Remark 2.36. The theorem asserts that the sufficient condition for the multiplicity-freeness given by T. Kobayashi (Fact 1.8) is a necessary condition for holomorphic discrete series representations.

By the theorem, the classification of multiplicity-free restrictions of holomorphic discrete series representations is reduced to that of finite-dimensional irreducible representations. In particular, in the case that $G'_{\mathbb{R}}$ has a one-dimensional center, J. R. Stembridge has classified multiplicity-free restrictions of finite-dimensional irreducible representations with respect to G' in [73]. Thus for such $G'_{\mathbb{R}}$, the classification is immediately done by Theorem 2.33 and the Stembridge classification.

As a consequence, we obtain the following proposition.

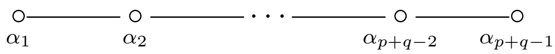
Corollary 2.37. *Let σ' be an involutive automorphism of $G_{\mathbb{R}}$. Assume that G^{σ} and $G^{\sigma'}$ are conjugate by an inner automorphism of G. Then we have $\Lambda(\sigma) = \Lambda(\sigma')$.*

Remark 2.38. The theorem asserts that the classification is not depend on a choice of real forms. The similarity of two groups with isomorphic complexifications can be found in many fields in the representation theory and the harmonic analysis. We give several examples:

- the Weyl unitary trick;
- the Flensted-Jensen duality [11];
- non-existence of compact Clifford–Klein forms [41];
- transfer of K-type [13];
- one to one correspondence of infinitesimal characters in the theory of the Howe duality [51], [67].

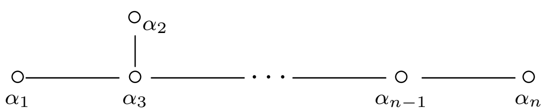
The simple roots of $\mathfrak{g}$ are numbered as follows.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$



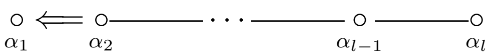
α_p is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$



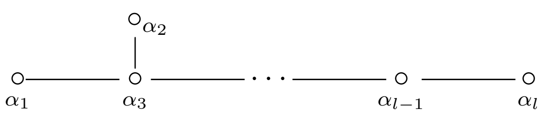
α_1 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2, n)$ If n is odd, the Dynkin diagram of $\mathfrak{g}$ is as follows:



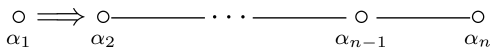
, where $l = (n+1)/2$. α_l is a unique non-compact simple root.

If n is even, the Dynkin diagram of $\mathfrak{g}$ is as follows:



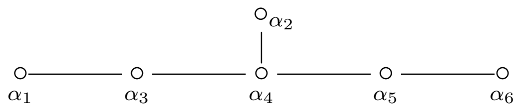
, where $l = n/2 + 1$. α_l is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$



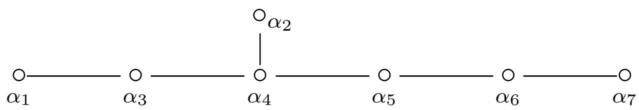
α_1 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{6(-14)}$



α_1 is a unique non-compact simple root.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{7(-25)}$



α_7 is a unique non-compact simple root.

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References

[1] F. A. Berezin. Quantization in complex symmetric spaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(2):363–402, 472, 1975.

[2] J. N. Bernstein. On the support of Plancherel measure. *J. Geom. Phys.*, 5(4):663–710 (1989), 1988.

[3] J. N. Bernstein and S. I. Gel′fand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.*, 41(2):245–285, 1980.

[4] M. Brion, D. Luna, and T. Vust. Espaces homogènes sphériques. *Invent. Math.*, 84(3):617–632, 1986.

[5] M. Duflo. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. *Ann. of Math. (2)*, 105(1):107–120, 1977.

[6] M. Duflo and J. A. Vargas. Branching laws for square integrable representations. *Proc. Japan Acad. Ser. A Math. Sci.*, 86(3):49–54, 2010.

[7] A. Dvorsky and S. Sahi. Tensor products of singular representations and an extension of the θ-correspondence. *Selecta Math. (N.S.)*, 4(1):11–29, 1998.

[8] T. J. Enright. *Lectures on representations of complex semisimple Lie groups*, volume 66 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin-New York, 1981. Lecture notes by Vyjayanthi Sundar.

[9] T. J. Enright. Unitary representations for two real forms of a semisimple Lie algebra: a theory of comparison. In *Lie group representations, I (College Park, Md., 1982/1983)*, volume 1024 of *Lecture Notes in Math.*, pages 1–29. Springer, Berlin, 1983.

[10] T. J. Enright, R. Parthasarathy, N. R. Wallach, and J. A. Wolf. Unitary derived functor modules with small spectrum. *Acta Math.*, 154(1-2):105–136, 1985.

[11] M. Flensted-Jensen. Discrete series for semisimple symmetric spaces. *Ann. of Math. (2)*, 111(2):253–311, 1980.

[12] R. Goodman. Complex Fourier analysis on a nilpotent Lie group. *Trans. Amer. Math. Soc.*, 160:373–391, 1971.

[13] B. Gross and N. Wallach. Restriction of small discrete series representations to symmetric subgroups. In *The mathematical legacy of*

Harish-Chandra (Baltimore, MD, 1998), volume 68 of *Proc. Sympos. Pure Math.*, pages 255–272. Amer. Math. Soc., Providence, RI, 2000.

[14] Harish-Chandra. Representations of semisimple Lie groups. II. *Trans. Amer. Math. Soc.*, 76:26–65, 1954.

[15] Harish-Chandra. Representations of semisimple Lie groups. VI. Integrable and square-integrable representations. *Amer. J. Math.*, 78:564–628, 1956.

[16] T. Hirai. On infinitesimal operators of irreducible representations of the Lorentz group of n-th order. *Proc. Japan Acad.*, 38:83–87, 1962.

[17] R. Howe. Reciprocity laws in the theory of dual pairs. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 159–175. Birkhäuser Boston, Boston, MA, 1983.

[18] R. Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.

[19] R. Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[20] H. P. Jakobsen and M. Vergne. Restrictions and expansions of holomorphic representations. *J. Funct. Anal.*, 34(1):29–53, 1979.

[21] K. D. Johnson and N. R. Wallach. Composition series and intertwining operators for the spherical principal series. I. *Trans. Amer. Math. Soc.*, 229:137–173, 1977.

[22] A. Joseph. Dixmier's problem for Verma and principal series submodules. *J. London Math. Soc. (2)*, 20(2):193–204, 1979.

[23] M. Kashiwara and M. Vergne. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44(1):1–47, 1978.

[24] A. U. Klimyk and A. M. Gavrilik. *The representations of the groups* $\mathrm{U}(n, 1)$ *and* $\mathrm{SO}_0(n, 1)$. Akad. Nauk Ukrain. SSR Inst. Teoret. Fiz., Kiev, 1976.

[25] A. W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[26] A. W. Knapp and D. A. Vogan, Jr. *Cohomological induction and unitary representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1995.

[27] T. Kobayashi. The restriction of $A_q(\lambda)$ to reductive subgroups. *Proc. Japan Acad. Ser. A Math. Sci.*, 69(7):262–267, 1993.

[28] T. Kobayashi. Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications. *Invent. Math.*, 117(2):181–205, 1994.

[29] T. Kobayashi. Multiplicity free theorem in branching problems of unitary highest weight modules. *Proceedings of Representation Theory held at Saga, Kyushu, 1997 (K. Mimachi, ed.)*, pages 9–17, 1997.

[30] T. Kobayashi. Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. II. Micro-local analysis and asymptotic K-support. *Ann. of Math. (2)*, 147(3):709–729, 1998.

[31] T. Kobayashi. Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties. *Invent. Math.*, 131(2):229–256, 1998.

[32] T. Kobayashi. Discretely decomposable restrictions of unitary representations of reductive Lie groups—examples and conjectures. In *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*, volume 26 of *Adv. Stud. Pure Math.*, pages 99–127. Math. Soc. Japan, Tokyo, 2000.

[33] T. Kobayashi. Geometry of multiplicity-free representations of $\mathrm{GL}(n)$, visible actions on flag varieties, and triunity. *Acta Appl. Math.*, 81(1-3):129–146, 2004.

[34] T. Kobayashi. Multiplicity-free representations and visible actions on complex manifolds. *Publ. Res. Inst. Math. Sci.*, 41(3):497–549, 2005.

[35] T. Kobayashi. Restrictions of unitary representations of real reductive groups. In *Lie theory*, volume 229 of *Progr. Math.*, pages 139–207. Birkhäuser Boston, Boston, MA, 2005.

[36] T. Kobayashi. Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs. In *Representation theory and automorphic forms*, volume 255 of *Progr. Math.*, pages 45–109. Birkhäuser Boston, Boston, MA, 2008.

[37] T. Kobayashi. Restrictions of generalized Verma modules to symmetric pairs. *Transform. Groups*, 17(2):523–546, 2012.

[38] T. Kobayashi. Propagation of the multiplicity-freeness property for holomorphic vector bundles. In *Lie Groups: Structure, Actions, and Representations*, volume 306 of *Progr. Math.* Birkhäuser Basel, 2013.

[39] T. Kobayashi. Shintani functions, real spherical manifolds, and symmetry breaking operators. In *Developments and retrospectives in Lie theory*, volume 37 of *Dev. Math.*, pages 127–159. Springer, Cham, 2014.

[40] T. Kobayashi. A program for branching problems in the representation theory of real reductive groups. In *Representations of Reductive Groups: In Honor of the 60th Birthday of David A. Vogan, Jr.*, volume 312 of *Progress in Mathematics*, pages 277–322. Birkhäuser/Springer, Cham, 2015.

[41] T. Kobayashi and K. Ono. Note on Hirzebruch's proportionality principle. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 37(1):71–87, 1990.

[42] T. Kobayashi and B. Ørsted. Analysis on the minimal representation of $\mathrm{O}(p,q)$. II. Branching laws. *Adv. Math.*, 180(2):513–550, 2003.

[43] T. Kobayashi, B. Ørsted, and M. Pevzner. Geometric analysis on small unitary representations of $\mathrm{GL}(N,\mathbb{R})$. *J. Funct. Anal.*, 260(6):1682–1720, 2011.

[44] T. Kobayashi and Y. Oshima. Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs. *Adv. Math.*, 231(3-4):2013–2047, 2012.

[45] T. Kobayashi and M. Pevzner. Differential symmetry breaking operators. I. General theory and F-method. *Selecta Math.*, 2015. Published OnLine.

[46] T. Kobayashi and M. Pevzner. Differential symmetry breaking operators. II. Rankin–Cohen operators for symmetric pairs. *Selecta Math.*, 2015. Published OnLine.

[47] T. Kobayashi and B. Speh. Symmetry breaking for representations of rank one orthogonal groups. *Mem. Amer. Math. Soc.*, 238(1126):v+110, 2015.

[48] S. S. Kudla and S. Rallis. Degenerate principal series and invariant distributions. *Israel J. Math.*, 69(1):25–45, 1990.

[49] J. Lepowsky. Multiplicity formulas for certain semisimple Lie groups. *Bull. Amer. Math. Soc.*, 77:601–605, 1971.

[50] J.-S. Li. Theta lifting for unitary representations with nonzero cohomology. *Duke Math. J.*, 61(3):913–937, 1990.

[51] J.-S. Li. The correspondences of infinitesimal characters for reductive dual pairs in simple Lie groups. *Duke Math. J.*, 97(2):347–377, 1999.

[52] J.-J. Ma. Derived functor modules, dual pairs and $\mathcal{U}(\mathfrak{g})^K$-actions. *J. Algebra*, 450:629–645, 2016.

[53] S. Martens. The characters of the holomorphic discrete series. *Proc. Nat. Acad. Sci. U.S.A.*, 72(9):3275–3276, 1975.

[54] K. Mashimo. On branching theorem of the pair $(G_2, \mathrm{SU}(3))$. *Nihonkai Math. J.*, 8(2):101–107, 1997.

[55] V. F. Molčanov. Representations of a pseudoorthogonal group that are connected with its cone. *Mat. Sb. (N.S.)*, 81 (123):358–375, 1970.

[56] J. Möllers and Y. Oshima. Discrete branching laws for minimal holomorphic representations. *J. Lie Theory*, 25(4):949–983, 2015.

[57] A. E. Nussbaum. Reduction theory for unbounded closed operators in Hilbert space. *Duke Math. J.*, 31:33–44, 1964.

[58] G. Ólafsson and B. Ørsted. Generalizations of the Bargmann transform. In *Lie theory and its applications in physics (Clausthal, 1995)*, pages 3–14. World Sci. Publ., River Edge, NJ, 1996.

[59] B. Ørsted and B. Speh. Branching laws for some unitary representations of $\mathrm{SL}(4, \mathbb{R})$. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 4:Paper 017, 19, 2008.

[60] B. Ørsted and J. Vargas. Restriction of square integrable representations: discrete spectrum. *Duke Math. J.*, 123(3):609–633, 2004.

[61] B. Ørsted and G. Zhang. L^2-versions of the Howe correspondence. I. *Math. Scand.*, 80(1):125–160, 1997.

[62] Y. Oshima. *Discrete branching laws of Zuckerman's derived functor modules.* PhD thesis, the University of Tokyo, mar 2013.

[63] Y. Oshima. On the restriction of Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups. *Amer. J. Math.*, 137(4):1099–1138, 2015.

[64] I. Penkov and G. Zuckerman. Generalized Harish-Chandra modules: a new direction in the structure theory of representations. *Acta Appl. Math.*, 81(1-3):311–326, 2004.

[65] I. Penkov and G. Zuckerman. Generalized Harish-Chandra modules with generic minimal $\mathfrak{k}$-type. *Asian J. Math.*, 8(4):795–811, 2004.

[66] I. Penkov and G. Zuckerman. On the structure of the fundamental series of generalized Harish-Chandra modules. *Asian J. Math.*, 16(3):489–514, 2012.

[67] T. Przebinda. The duality correspondence of infinitesimal characters. *Colloq. Math.*, 70(1):93–102, 1996.

[68] J. Repka. Tensor products of holomorphic discrete series representations. *Canad. J. Math.*, 31(4):836–844, 1979.

[69] P. J. Sally, Jr. *Analytic continuation of the irreducible unitary representations of the universal covering group of* $\mathrm{SL}(2, R)$. Memoirs of the American Mathematical Society, No. 69. American Mathematical Society, Providence, R. I., 1967.

[70] F. Sato. On the stability of branching coefficients of rational representations of reductive groups. *Comment. Math. Univ. St. Paul.*, 42(2):189–207, 1993.

[71] W. Schmid. Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen. *Invent. Math.*, 9:61–80, 1969/1970.

[72] B. Speh. Restriction of some representations of $U(p, q)$ to a symmetric subgroup. In *Representation theory and mathematical physics*, volume 557 of *Contemp. Math.*, pages 371–388. Amer. Math. Soc., Providence, RI, 2011.

[73] J. R. Stembridge. Multiplicity-free products and restrictions of Weyl characters. *Represent. Theory*, 7:404–439 (electronic), 2003.

[74] C. Tsukamoto. Stability of $\mathrm{SO}(n+3)/\mathrm{SO}(3)\times\mathrm{SO}(n)$ branching. *Tsukuba J. Math.*, 33(2):239–251, 2009.

[75] M. Vergne and H. Rossi. Analytic continuation of the holomorphic discrete series of a semi-simple Lie group. *Acta Math.*, 136(1-2):1–59, 1976.

[76] É. B. Vinberg and B. N. Kimel'fel'd. Homogeneous domains on flag manifolds and spherical subgroups of semisimple lie groups. *Funct. Anal. Appl.*, 12(3):168–174, 1978.

[77] N. R. Wallach. *Harmonic analysis on homogeneous spaces.* Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 19.

[78] N. R. Wallach. The analytic continuation of the discrete series. I, II. *Trans. Amer. Math. Soc.*, 251:1–17, 19–37, 1979.

[79] N. R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[80] G. Zhang. Tensor products of minimal holomorphic representations. *Represent. Theory*, 5:164–190 (electronic), 2001.