

## *A Base Point Free Theorem of Reid Type*

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**Abstract.** Assume that  $X$  is a normal projective variety over  $\mathbb{C}$ , of dimension  $\leq 3$ , and that  $(X, \Delta)$  is a log variety that is weakly Kawamata log terminal. Let  $L$  be a nef Cartier divisor such that  $aL - (K_X + \Delta)$  is nef and log big on  $(X, \Delta)$  for some  $a \in \mathbb{N}$ . Then  $\text{Bs}|mL| = \emptyset$  for every  $m \gg 0$ .

### Introduction

We generally use the notation and terminology of [Utah].

Let  $X$  be a normal projective variety over  $\mathbb{C}$ . Let  $(X, \Delta)$  be a log variety which is log canonical. We assume that  $K_X + \Delta$  is a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor.

Let  $\Theta = \sum_{i=1}^s \Theta_i$  be a reduced divisor with only simple normal crossings on an  $n$ -dimensional non-singular complete variety over  $\mathbb{C}$ . We denote  $\mathbf{Strata}(\Theta) := \{\Gamma \mid 1 \leq k \leq n, 1 \leq i_1 < i_2 < \cdots < i_k \leq s, \Gamma \text{ is an irreducible component of } \Theta_{i_1} \cap \Theta_{i_2} \cap \cdots \cap \Theta_{i_k} \neq \emptyset\}$ .

First we recall the famous base point free theorem of Kawamata and Shokurov.

**THEOREM 0.** (Theorem 3-1-1 of [KMM]) *Assume that  $(X, \Delta)$  is weakly Kawamata log terminal (wklt). Let  $L$  be a nef Cartier divisor such that  $aL - (K_X + \Delta)$  is ample for some  $a \in \mathbb{N}$ . Then  $\text{Bs}|mL| = \emptyset$  for every  $m \gg 0$ .*

We note that, if  $aL - (K_X + \Delta)$  is nef and big but not ample, there exists a counterexample (cf. Remark 3-1-2 of [KMM]).

In [Rd], Reid introduced the notion of “log big” and suggested that, if  $aL - (K_X + \Delta)$  is nef and log big on  $(X, \Delta)$ , then the theorem holds.

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DEFINITION. (due to Reid [Rd]) Let  $f: Y \rightarrow X$  be a log resolution such that  $K_Y = f^*(K_X + \Delta) + \sum a_j E_j$  (where  $a_j \geq -1$ ). Let  $L$  be a Cartier divisor on  $X$ .  $L$  is called *nef and log big* on  $(X, \Delta)$  if  $L$  is nef and big and  $(L|_{f(\Gamma)})^{\dim f(\Gamma)} > 0$  for any member  $\Gamma$  of **Strata** $(\sum_{a_j=-1} E_j)$ .

REMARK. The definition of the notion of “nef and log big” does not depend on the choice of the log resolution  $f$  (Claim of [Fu]).

Now we state the main theorem of this paper.

MAIN THEOREM. Assume that  $\dim X \leq 3$  and that  $(X, \Delta)$  is weakly Kawamata log terminal (wklt). Let  $L$  be a nef Cartier divisor such that  $aL - (K_X + \Delta)$  is nef and log big on  $(X, \Delta)$  for some  $a \in \mathbb{N}$ . Then  $\text{Bs}|mL| = \emptyset$  for every  $m \gg 0$ .

REMARK. In [Fu0], the author proved the theorem under the conditions that  $X$  is non-singular,  $\Delta$  is a simple normal crossing divisor and  $\dim X$  is arbitrary.

### 1. Preliminaries

We collect some results needed for the proof of Main Theorem.

PROPOSITION 1. (Reid Type Vanishing, cf. [Fu]) Assume that  $(X, \Delta)$  is wklt. Let  $D$  be a  $\mathbb{Q}$ -Cartier integral Weil divisor. If  $D - (K_X + \Delta)$  is nef and log big on  $(X, \Delta)$ , then  $H^i(X, \mathcal{O}_X(D)) = 0$  for every  $i > 0$ .

PROOF. The proof is analogous to that of the main theorem of [Fu]. But, when we read [Fu], we must take notice of the facts below.

FACT 1. For real numbers  $a$  and  $b$ ,  $\lceil a + b \rceil \geq \lceil a \rceil + \lceil b \rceil$ . Hence  $\lceil f^*D + E \rceil \geq \lceil f^*D \rceil + \lceil E \rceil$ . Thus  $f_*\mathcal{O}_Y(\lceil f^*D + E \rceil) = \mathcal{O}_X(D)$ .

FACT 2.  $\lceil f^*D + E \rceil - (K_Y + f_*^{-1}[\Delta] + \{-f^*D - E\})$  is nef and log big on  $(Y, f_*^{-1}[\Delta] + \{-f^*D - E\})$ .  $\square$

PROPOSITION 2. (Kawamata, cf. the proofs of Lemma 3 of [Ka] and Theorem 3-1-1 of [KMM]) Assume that  $(X, \Delta)$  is wklt. Let  $L$  be a nef Cartier divisor such that  $aL - (K_X + \Delta)$  is nef and big for some  $a \in \mathbb{N}$ . If  $\text{Bs}|mL| \cap [\Delta] = \emptyset$  for every  $m \gg 0$ , then  $\text{Bs}|mL| = \emptyset$  for every  $m \gg 0$ .

## 2. Proof of Main Theorem

We prove the main theorem by induction on  $\dim X$ .

*Step 0.* Let  $f : Y \rightarrow X$  be a log resolution of  $(X, \Delta)$  such that the following conditions are satisfied:

- (1)  $Y$  is projective (from the assumption that  $X$  is projective and the definition of the notion of wklt),
- (2)  $\text{Exc}(f)$  consists of divisors,
- (3)  $K_Y + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E$ ,
- (4)  $E$  and  $F$  are  $f$ -exceptional effective  $\mathbb{Q}$ -divisors such that  $\text{Supp}(E)$  and  $\text{Supp}(F)$  do not have common irreducible components,
- (5)  $\lfloor F \rfloor = 0$ .

Note that, for any member  $G \in \mathbf{Strata}(f_*^{-1}\lfloor \Delta \rfloor)$ ,  $\text{Exc}(f)$  does not include  $G$  (from an argument in p. 99 of [Sh], cf. the proof of Proposition of [Fu]).

*Step 1.* We shall show that we may assume that  $X$  is  $\mathbb{Q}$ -factorial.

Apply the relative log minimal model program to  $f : (Y, f_*^{-1}\Delta + F) \rightarrow X$ .

We end up with a  $\mathbb{Q}$ -factorial weakly Kawamata log terminal variety (over  $X$ )  $\mu : (Z, \mu_*^{-1}\Delta + (F)_Z) \rightarrow X$  such that  $K_Z + \mu_*^{-1}\Delta + (F)_Z$  is  $\mu$ -nef.

Because  $K_Z + \mu_*^{-1}\Delta + (F)_Z = \mu^*(K_X + \Delta) + (E)_Z$ ,  $(E)_Z$  is  $\mu$ -nef. Thus  $(E)_Z$  is a  $\mu$ -exceptional  $\mu$ -nef effective divisor. Applying 1.1 of [Sh] to  $-(E)_Z$ , we obtain the fact that  $(E)_Z = 0$ . Hence  $\mu^*(K_X + \Delta) = K_Z + \mu_*^{-1}\Delta + (F)_Z$ .

We put  $M := aL - (K_X + \Delta)$ . Here we prove that  $\mu^*M$  is nef and log big on  $(Z, \mu_*^{-1}\Delta + (F)_Z)$ . Let  $G_0$  be a divisor of  $\mathbb{C}(X)$  whose discrepancy with respect to  $(Z, \mu_*^{-1}\Delta + (F)_Z)$  is  $-1$ . Let  $W$  be a non-singular projective variety and  $f_1 : W \rightarrow Y$  and  $\mu_1 : W \rightarrow Z$  birational morphisms, such that  $G_0$  is a non-singular prime divisor on  $W$  and that  $ff_1 = \mu\mu_1$ . Then  $f_1(G_0) \in \mathbf{Strata}(f_*^{-1}\lfloor \Delta \rfloor)$  (cf. the proof of Claim of [Fu]). So  $\text{Exc}(f)$  does not include  $f_1(G_0)$ .

Put  $B := f(\text{Exc}(f))$ . By Zariski's Main Theorem, the set-theoretical inverse image  $f^{-1}(B) = \text{Exc}(f)$ . We note that  $Y \setminus \text{Exc}(f) \cong Z \setminus \mu^{-1}(B)$ .

Because  $\mu_1(G_0) \cap (Z \setminus \mu^{-1}(B)) \cong f_1(G_0) \cap (Y \setminus \text{Exc}(f)) \neq \emptyset$ ,  $\mu_1(G_0)$  and  $f_1(G_0)$  are birationally equivalent.

Thus  $\mu_1(G_0)$  and  $f(f_1(G_0))$  are birationally equivalent. Hence  $\mu^*M$  is nef and big on  $\mu_1(G_0)$ . Therefore  $\mu^*M$  is nef and log big on  $(Z, \mu_*^{-1}\Delta + (F)_Z)$ .

Thus we may assume that  $X$  is  $\mathbb{Q}$ -factorial.

*Step 2.* From 3.8 of [Sh], every irreducible component of  $[\Delta]$  is normal.

Let  $S$  be an irreducible component of  $[\Delta]$ .

We put  $S_0 := f_*^{-1}S$ ,  $\text{Diff}(0) := (f|_{S_0})_*(f^*(K_X + S)|_{S_0} - (K_Y + S_0)|_{S_0})$  and  $\text{Diff}(\Delta - S) := \text{Diff}(0) + (\Delta - S)|_S$ . We note that  $\text{Diff}(0) \geq 0$  (the Subadjunction Lemma, 3.2.2 of [Sh], cf. 5-1-9 of [KMM]). Then  $K_{S_0} + (f_*^{-1}\Delta - S_0)|_{S_0} + F|_{S_0} = f^*(K_S + \text{Diff}(\Delta - S)) + E|_{S_0}$ .

Because  $(f|_{S_0})_*((f_*^{-1}\Delta - S_0)|_{S_0} + F|_{S_0} - E|_{S_0}) = \text{Diff}(\Delta - S)$  and  $\text{Diff}(\Delta - S) = \text{Diff}(0) + (\Delta - S)|_S \geq 0$ ,  $[\text{Diff}(\Delta - S)]$  is reduced. Here  $M|_S$  is nef and big. And, for any member  $\Gamma$  of **Strata** $((f_*^{-1}[\Delta] - S_0)|_{S_0})$ ,  $M|_{f(\Gamma)}$  is nef and big. Hence  $(S, \text{Diff}(\Delta - S))$  is log terminal and  $M|_S$  is nef and log big on  $(S, \text{Diff}(\Delta - S))$ .

We note that  $\text{Exc}(f|_{S_0})$  may not consist of divisors. But, because  $\text{Exc}(f) \cap S_0$  includes  $\text{Exc}(f|_{S_0})$ ,  $\text{Exc}(f|_{S_0})$  does not include any member of **Strata** $((f_*^{-1}[\Delta] - S_0)|_{S_0})$ . Hence, by a finite composition of blowing ups with centers included in  $\text{Exc}(f|_{S_0})$ , we get a log resolution of  $(S, \text{Diff}(\Delta - S))$  such that the exceptional locus consists of divisors with discrepancies  $> -1$  (because, for every divisor  $\nu \in \mathbb{C}(S)$  whose discrepancy with respect to  $(S, \text{Diff}(\Delta - S))$  is  $-1$ ,  $\text{center}_{S_0}(\nu) \in \text{Strata}((f_*^{-1}[\Delta] - S_0)|_{S_0})$ ). Therefore  $(S, \text{Diff}(\Delta - S))$  is wklt (Lemma 4 of [Ka], [Sz]).

Thus  $|mL|_S|$  is base point free for  $m \gg 0$ , by induction hypothesis.

*Step 3.* We consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-S) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_S \rightarrow 0.$$

Tensoring with  $\mathcal{O}_X(mL)$  for  $m \geq a$ , we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(mL - S) \rightarrow \mathcal{O}_X(mL) \rightarrow \mathcal{O}_S(mL) \rightarrow 0$$

(Since the sheaf  $\mathcal{O}_X(mL)$  is invertible,  $\mathcal{O}_X(mL) \otimes \mathcal{O}_X(-S) = \mathcal{O}_X(mL - S)$ ).

Because  $mL - S - (K_X + \Delta - S)$  is nef and log big on  $(X, \Delta - S)$ ,  $H^1(X, \mathcal{O}_X(mL - S)) = 0$  from Proposition 1. Hence the linear system

$|mL|$  on  $X$  cuts out a complete linear system  $|mL|_S|$  on  $S$ . Therefore  $\text{Bs}|mL| \cap [\Delta] = \emptyset$  for  $m \gg 0$ .

Thus Proposition 2 implies the assertion.  $\square$

REMARK. The proof above implies that the main theorem holds also under the conditions that  $\dim X = 4$  and that  $X$  is  $\mathbb{Q}$ -factorial.

More generally our proof shows the following implication:

If the log minimal model program works in all dimensions, then the main theorem holds without the condition that  $\dim X \leq 3$ .

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