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# *On Quadratic Differential Metrics on a Closed Riemann Surface*

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**Abstract.** We study properties of the space of quadratic differential metrics on a closed Riemann surface of genus $g \geq 2$. First, we introduce a natural metric on this space defined via length distortions which is proper and compact. Second, we study topological properties of this space and show equivalence of various convergences. Besides, we relate the preceding metric to another metric which is obtained via global Lipschitz constants.

## 1. Introduction

In this paper, we study properties of the space of quadratic differential metrics on a closed Riemann surface of genus $g \geq 2$. Let $X$ be a closed Riemann surface of genus $g \geq 2$. Let $Q(X)$ be the Banach space of holomorphic quadratic differentials on $X$ endowed with the $L_1$-norm $||\cdot||_1$, and $Q_1(X)$ the unit sphere in $Q(X)$. It is well-known that ([6, 8]) $Q(X)$ can be viewed as the cotangent space to the Teichmüller space $T(X)$ at $X$. Each non-zero $q \in Q(X)$ induces ([19]) a metric $|q|$ on $X$ known as a quadratic differential metric or $q$-metric. These metrics are complete geodesic metrics and have been studied extensively. The importance of such metrics lies in many aspects, e.g., they serve as ([7, 19]) extremal metrics in the definition of extremal lengths. Let $|Q(X)| = \{|q| : q \in Q(X)\}$ be the space of quadratic differential metrics on $X$. For a non-zero $q$, to each homotopy class of a closed curve $\alpha$ one associates its $q$-length $l_q(\alpha)$ (see Section 2 for details). In this paper, we will study properties of $|Q(X)|$ in terms of the length functions $l_q(\alpha)$.

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Throughout the paper, all closed curves are essential, and there is no distinction between a closed curve and its homotopy class. Denote by $\mathcal{S}$ (resp., $\mathcal{C}$) the set of homotopy classes of simple closed curves (resp., closed curves) on $X$. The following interesting result is shown by Marden and Strebel in [14].

Theorem A ([14], Theorem 4.3). *Let $\phi, \psi \in Q(X)$. If $l_\phi(\alpha) = l_\psi(\alpha)$ for every $\alpha \in \mathcal{S}$, then $\phi = e^{i\theta}\psi$ for some constant angle $\theta$.*

Remark 1. The original statement of this result in [14] is more general, while for our purpose it is sufficient to restrict to the above case. Note that Theorem A is actually a result on $|Q(X)|$. It implies that a $q$-metric is uniquely determined by its $q$-lengths over $\mathcal{S}$.

It is of interest to know further properties of the space $|Q(X)|$ with the aid of $q$-lengths. For this purpose, we consider the normalized space $|Q_1(X)|$. Define, for $|q_1|$ and $|q_2|$ in $|Q_1(X)|$,

$$d(|q_1|, |q_2|) = \log D(|q_1|, |q_2|),$$

where

$$D(|q_1|, |q_2|) = \sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_1}(\alpha)}{l_{q_2}(\alpha)}\} \sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_2}(\alpha)}{l_{q_1}(\alpha)}\}.$$

Then in Theorem 1 we will show

Theorem 1. *$d$ is a metric on $|Q_1(X)|$. $d(|q_1|, |q_2|)$ is finite for any $|q_1|$ and $|q_2|$ in $|Q_1(X)|$.*

From its definition, the metric $d$ is described by maximizing the length distortions under quadratic differential metrics.

Remark 2. In the Teichmüller space $T(X)$, we have the following interesting metrics:

$$d_{P_1}(X_1, X_2) = \log \sup_{\alpha \in \mathcal{S}} \{\frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)}\},$$

$$d_{P_2}(X_1, X_2) = \log \sup_{\alpha \in \mathcal{S}} \{\frac{l_{X_1}(\alpha)}{l_{X_2}(\alpha)}\}$$

and

$$d_L(X_1, X_2) = \log \max\{\sup_{\alpha \in \mathcal{S}} \frac{l_{X_1}(\alpha)}{l_{X_2}(\alpha)}, \sup_{\alpha \in \mathcal{S}} \frac{l_{X_2}(\alpha)}{l_{X_1}(\alpha)}\},$$

where $X_1, X_2 \in T(X)$, $l_{X_i}(\cdot)$ is the hyperbolic length with respect to $X_i$, $i = 1, 2$. $d_{P_1}$ and $d_{P_2}$ are introduced by Thurston ([20]) and called Thurston's asymmetric metrics. They are real metrics except for symmetry. $d_L$ is called the length spectrum metric, it is a real metric. Clearly, $d_{P_i} \leq d_L$, $i = 1, 2$. These metrics and their relations to the Teichmüller metric have been studied by many authors (see, e.g., [10, 11, 12, 13] [15, 16, 17, 18]).

It is also possible to define similar metrics $d_{\mathcal{P}_1}$, $d_{\mathcal{P}_2}$ and $d_{\mathcal{L}}$ on $|Q_1(X)|$ as $d_{P_1}$, $d_{P_2}$ and $d_L$ on the Teichmüller space, respectively. However, we will not do this. Instead, we will be concerned with the above metric $d$. $d$ is analogous to $d_{P_i}$, $i = 1, 2$ and $d_L$. It is symmetric. Moreover, $d$ is the biggest (see Lemma 3) one among $d_{\mathcal{P}_1}$, $d_{\mathcal{P}_2}$ and $d_{\mathcal{L}}$, which is also complete (see Corollary 3).

Natural topologies can be put on $|Q_1(X)|$, e.g., the $\mathcal{S}$-topology and the $\mathcal{C}$-topology (see Section 2 for details). In Theorem 2, we will study topological properties of $|Q_1(X)|$. We will show the equivalence of various convergences, including convergences in the metric topology induced by $d$, the $\mathcal{S}$-topology and the $\mathcal{C}$-topology, etc.

THEOREM 2. *Let $\{|q_n|\}_{n=1}^{\infty}$ be a sequence in $|Q_1(X)|$. Then the following are equivalent:*

*(i) $|q_n| \to |q|$ in the $\mathcal{S}$-topology in $|Q_1(X)|$, $n \to \infty$,*

*(ii) $|q_n| \to |q|$ in the $\mathcal{C}$-topology in $|Q_1(X)|$, $n \to \infty$,*

*(iii) for any convergent subsequence $q_{n_k}$ of $q_n$, there exists a constant angle $\theta$ such that $q_{n_k} \to e^{i\theta} q$ in the $L_1$-norm topology in $Q_1(X)$, $k \to \infty$,*

*(iv) $d(|q_n|, |q|) \to 0$, $n \to \infty$.*

As direct consequences of Theorem 2, we have the following corollaries which also indicate the naturality of the metric $d$.

COROLLARY 1. *The metric topology induced by $d$ is compatible with the $\mathcal{S}$-topology and also the $\mathcal{C}$-topology on $|Q_1(X)|$.*

COROLLARY 2. *The metric space $(|Q_1(X)|, d)$ is compact and proper.*

Corollary 3. *The metric space* $(|Q_1(X)|, d)$ *is complete and totally bounded.*

Finally, we will relate $d$ to another natural metric $\tau$ on $|Q_1(X)|$ which is defined by minimizing the global Lipschitz constants. More precisely, for $|q_1|, |q_2| \in |Q_1(X)|$ we define

$$\tau(|q_1|, |q_2|) = \inf_{f \sim id} \{\log(L(f)L(f^{-1}))\},$$

where the infimum is taken over all the Lipschitz homeomorphisms $f$ homotopic to $id : X \to X$, and $L(f)$ is the Lipschitz constant of $f$ given by

$$L(f) = \sup_{x,y \in X,\ x \neq y} \frac{\delta_2(f(x), f(y))}{\delta_1(x,y)}$$

in which $\delta_i$ is the distance function induced by $|q_i|$, $i = 1, 2$, respectively. In Proposition 1, we note the following relation between the metrics $d$ and $\tau$.

Proposition 1. *For any* $|q_1|$, $|q_2| \in |Q_1(X)|$,

$$d(|q_1|, |q_2|) \leq \tau(|q_1|, |q_2|).$$

## 2. Preliminaries

In this section, we briefly recall some necessary backgrounds. For references, see [5, 6, 8, 9, 19].

### 2.1. Quadratic differential

Let $X = \mathbb{H}/\Gamma$ be the Fuchsian uniformization of $X$, where $\mathbb{H}$ is the upper half plane. A holomorphic quadratic differential $q$ on $X$ is a holomorphic function on $\mathbb{H}$ such that

$$q(g(z))(g'(z))^2 = q(z),\ z \in \mathbb{H},\ g \in \Gamma.$$

Let $Q(X)$ be the space of holomorphic quadratic differentials on $X$. Endowed with the $L_1$-norm $|| \cdot ||_1$, $Q(X)$ is a Banach space ([19]). The theory of quadratic differentials and their connections with Teichmüller theory have been studied extensively (see, e.g., [6, 19]). The following result describes the equivalence of various convergences in $Q(X)$. In view of this, we may switch freely in these convergences.

Lemma 1 ([14], P. 181). *As $n \to \infty$, the following conditions are equivalent in $Q(X)$ :*

*$(a)$ $q_n$ converges locally uniformly to $q$,*

*$(b)$ $q_n$ converges uniformly to $q$,*

*$(c)$ $q_n$ converges to $q$ in the $L_1$-norm topology.*

## 2.2. Quadratic differential metric

A non-zero $q \in Q(X)$ induces a metric $|q|$ on $X$ which is locally given by $|q(z)|^{\frac{1}{2}}|dz|$. Such a metric is called a quadratic differential metric or a $q$-metric. For a closed curve $\alpha \subset X$, denote ([19])

$$l_q(\alpha) = \inf_{\gamma \sim \alpha} \Big\{ \int_\gamma |q(z)|^{\frac{1}{2}}|dz| \Big\},$$

where the infimum is taken over all curves $\gamma$ in the homotopy class of $\alpha$. $l_q(\alpha)$ is called the $q$-length or quadratic differential length of $\alpha$.

Let $|Q(X)| = \{|q| : q \in Q(X)\}$ be the space of quadratic differential metrics on $X$. Given a closed curve $\alpha \in \mathcal{C}$, the quadratic differential lengths $l_q(\alpha)$ define a length function on $|Q(X)|$. We endow $|Q(X)|$ with the following two topologies: the $\mathcal{S}$-topology (resp., $\mathcal{C}$-topology) which is the weakest topology that makes all the length functions corresponding to simple closed curves (resp., closed curves) continuous. Note that these two topologies are actually equivalent (see "$(i) \Leftrightarrow (ii)$" in Theorem 2).

## 2.3. Measured foliation

A measured foliation $\mathcal{F}$ on a topological surface $S$ of genus $g \geq 2$ is a singular foliation on $S$, where the singularities are isolated and $k$-pronged ($k \geq 3$), equipped with a measure $\mu$ on transverse arcs which is invariant under translation along leaves (see [5] for more details). The space $\mathcal{MF}$ of equivalence classes of measured foliations is defined where two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if $i(\alpha, \mathcal{F}_1) = i(\alpha, \mathcal{F}_2)$ for every $\alpha \in \mathcal{S}$, where $i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is the classical intersection number. Two classes $[\mathcal{F}_1]$ and $[\mathcal{F}_2]$ are projectively equivalent if there is a constant $r > 0$ so that $i(\alpha, \mathcal{F}_1) = ri(\alpha, \mathcal{F}_2)$ for every $\alpha \in \mathcal{S}$. The space of projective equivalence classes of measured foliations is denoted by $\mathcal{PMF}$.

We have the following (cf. [9]) descriptions of $\mathcal{MF}$ and $\mathcal{PMF}$. In particular, $\mathcal{PMF}$ is compact.

THEOREM B (Thurston). *$\mathcal{MF}$ is homeomorphic to a $6g-6$ dimensional ball. $\mathcal{PMF}$ is homeomorphic to a $6g-7$ dimensional sphere.*

The space $\mathcal{MF}$ may be viewed as a certain completion of positive real multiples of simple closed curves.

THEOREM C (Thurston). *There is an embedding of $\mathcal{S}\times\mathbb{R}_+$ into $\mathcal{MF}$. The image of this embedding is dense in $\mathcal{MF}$. The image of $\mathcal{S}$ in $\mathcal{PMF}$ is also dense.*

### 2.4. Continuity of the length function

The length function $l_q(\alpha):|Q(X)|\times\mathcal{S}\to\mathbb{R}$ extends to a length function $l:|Q(X)|\times\mathcal{MF}\to\mathbb{R}$. In this subsection, we state the continuity of $l$. For this purpose, we recall the following general results.

Let $C(X)$ be the space of geodesic currents on $X$ endowed with the weak$^*$ topology, where a geodesic current is a $\pi_1(X)$-invariant Randon measure on the space of un-oriented geodesics on the universal covering of $X$ ([3]).

THEOREM D ([3]). *There is an embedding of $\mathcal{C}$ into $C(X)$.*

([2], §4.2)([3], Proposition 3). *There is a continuous, symmetric, bilinear extension $i:C(X)\times C(X)\to\mathbb{R}$ of $i:\mathcal{C}\times\mathcal{C}\to\mathbb{R}$.*

There is ([3]) an embedding $\mathcal{MF}\to C(X)$ which is the unique continuous linear extension of the inclusion of $\mathcal{S}$ into $\mathcal{C}$. We will insist on the same symbol $\alpha$ for a closed curve (or, $\mathcal{F}$ for a measured foliation) and its image under the above embedding.

There is ([4]) also an embedding of the flat space of all quadratic differential metrics modeled on a closed surface into $C(X)$. In particular, in the special case for $|Q_1(X)|$, we have an embedding $L:|Q_1(X)|\to C(X)$ sending $|q|$ to the geodesic current $L_q$. The topology on $|Q_1(X)|$ involved in this embedding is the one such that $|q_n|\to|q|$ if $l_{q_n}(\alpha)\to l_q(\alpha)$ for every $\alpha\in\mathcal{C}$. The naturality of this embedding lies in the following

THEOREM E ([4], Theorem 4). *In the embedding $L:|Q_1(X)|\to C(X)$,*

$$l_q(\alpha)=i(L_q,\alpha)$$

*holds for every $\alpha\in\mathcal{C}$.*

From Theorems D and E, we conclude a particular consequence that

Lemma 2. *The length function $l_q(\alpha) : |Q(X)| \times \mathcal{S} \to \mathbb{R}$ extends to a continuous function from $|Q(X)| \times \mathcal{MF}$ to $\mathbb{R}$, which we also denote by $l$.*

## 3. Main Results

### 3.1. The metric $d$

In this subsection, we introduce a natural metric on $|Q_1(X)|$ as follows. For $|q_1|$ and $|q_2|$ in $|Q_1(X)|$, define

$$d(|q_1|, |q_2|) = \log D(|q_1|, |q_2|),$$

where

$$D(|q_1|, |q_2|) = \sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_1}(\alpha)}{l_{q_2}(\alpha)}\} \sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_2}(\alpha)}{l_{q_1}(\alpha)}\}.$$

Remark 3. From the density of weighted simple closed curves in $\mathcal{PMF}$ as in Theorem C, one observes that the quantity $D(|q_1|, |q_2|)$ can also be expressed as

$$D(|q_1|, |q_2|) = \sup_{\mathcal{F} \in \mathcal{PMF}} \{\frac{l_{q_1}(\mathcal{F})}{l_{q_2}(\mathcal{F})}\} \sup_{\mathcal{F} \in \mathcal{PMF}} \{\frac{l_{q_2}(\mathcal{F})}{l_{q_1}(\mathcal{F})}\}.$$

Since $\mathcal{PMF}$ is compact, the two suprema in the above expression can be attained.

Here, we present a natural question as follows.

Question 1. Give descriptions of these measured foliations that realize the suprema in $D(|q_1|, |q_2|)$.

To show that the quantity $d$ gives a real metric, we need the following

Lemma 3. *Let $|q_1|$, $|q_2| \in |Q_1(X)|$. Then*

$$\sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_1}(\alpha)}{l_{q_2}(\alpha)}\} \geq 1.$$

$$\sup_{\alpha\in\mathcal{S}}\{\frac{l_{q_2}(\alpha)}{l_{q_1}(\alpha)}\}\geq 1.$$
$$D(|q_1|,|q_2|)\geq 1.$$

PROOF. Recall that ([7]) (see also [9]) there is a homeomorphism $\mathfrak{q}$ which maps a measured foliation $\mathcal{F}$ and a Riemann surface $X$ to a holomorphic quadratic differential $\mathfrak{q}(\mathcal{F},X)\in Q(X)$ whose horizontal foliation is $\mathcal{F}$. The metric induced by such a quadratic differential is extremal (see, e.g., p. 270 in [4]) in the extremal length $ext_X(\mathcal{F})$ of $\mathcal{F}$. Normalize $\mathfrak{q}(\mathcal{F},X)$ to have unit $L_1$-norm and denote the resulting quadratic differential by $q$ for short. Then from the extremality,

$$l_q(\mathcal{F})=\sqrt{ext_X(\mathcal{F})}\geq l_{q'}(\mathcal{F})$$

for any $|q'|\in|Q_1(X)|$. Therefore, for any $|q_1|$ and $|q_2|$ in $|Q_1(X)|$,

$$\sup_{\mathcal{F}\in\mathcal{PMF}}\{\frac{l_{q_1}(\mathcal{F})}{l_{q_2}(\mathcal{F})}\}\geq\frac{l_{q_1}(\mathcal{F}_1)}{l_{q_2}(\mathcal{F}_1)}\geq 1,$$

where $\mathcal{F}_1$ is the horizontal foliation of $q_1$. Similarly, we get

$$\sup_{\mathcal{F}\in\mathcal{PMF}}\{\frac{l_{q_2}(\mathcal{F})}{l_{q_1}(\mathcal{F})}\}\geq 1.$$

Consequently, the lemma follows from Remark 3. □

THEOREM 1. *$d$ is a metric on $|Q_1(X)|$. $d(|q_1|,|q_2|)$ is finite for any $|q_1|$ and $|q_2|$ in $|Q_1(X)|$.*

PROOF. The finiteness of $d(|q_1|,|q_2|)$ follows from Remark 3. It is obvious that $d$ is symmetric, it satisfies the triangle inequality, and $d(|q_1|,|q_2|)=0$ if $|q_1|=|q_2|$.

The non-negativity of $d(|q_1|,|q_2|)$ is guaranteed by Lemma 3. We verify that $d(|q_1|,|q_2|)=0$ implies $|q_1|=|q_2|$. If $d(|q_1|,|q_2|)=0$, then from Lemma 3 we have

$$\sup_{\alpha\in\mathcal{S}}\{\frac{l_{q_1}(\alpha)}{l_{q_2}(\alpha)}\}=\sup_{\alpha\in\mathcal{S}}\{\frac{l_{q_2}(\alpha)}{l_{q_1}(\alpha)}\}=1.$$

Consequently $l_{q_1}(\alpha)=l_{q_2}(\alpha)$ for every $\alpha\in\mathcal{S}$. Therefore we conclude from Theorem A that $|q_1|=|q_2|$. □

### 3.2. Convergences in $|Q_1(X)|$

In this subsection, we compare various convergences in $|Q_1(X)|$, including the convergence in the metric topology induced by $d$. We show the equivalence of these convergences.

THEOREM 2. *Let $\{|q_n|\}_{n=1}^{\infty}$ be a sequence in $|Q_1(X)|$. Then the following are equivalent:*

*(i)* $|q_n| \to |q|$ *in the $\mathcal{S}$-topology in* $|Q_1(X)|$, $n \to \infty$,

*(ii)* $|q_n| \to |q|$ *in the $\mathcal{C}$-topology in* $|Q_1(X)|$, $n \to \infty$,

*(iii) for any convergent subsequence $q_{n_k}$ of $q_n$, there exists a constant angle $\theta$ such that $q_{n_k} \to e^{i\theta} q$ in the $L_1$-norm topology in $Q_1(X)$, $k \to \infty$,*

*(iv)* $d(|q_n|, |q|) \to 0$, $n \to \infty$.

PROOF. We will show the relations $(i) \implies (iii) \implies (ii) \implies (i)$ and $(i) \iff (iv)$.

$(i) \implies (iii)$: Suppose $l_{q_n}(\alpha) \to l_q(\alpha)$ for every $\alpha \in \mathcal{S}$, $n \to \infty$. Consider any convergent subsequence $q_{n_k}$ of $q_n$ which tends to $q'$ in $L_1$-norm as $k \to \infty$, for some $q' \in Q_1(X)$. By the continuity of $l_\phi(\alpha) : Q(X) \to \mathbb{R}$ for each fixed $\alpha \in \mathcal{C}$ (Lemma on page 78 in [7], or [14]), $l_{q_{n_k}}(\alpha) \to l_{q'}(\alpha)$ for each $\alpha \in \mathcal{S}$, $k \to \infty$. Consequently $l_q(\alpha) = l_{q'}(\alpha)$ for every $\alpha \in \mathcal{S}$. Therefore, from Theorem A we conclude $q' = e^{i\theta} q$ for some $\theta$. Hence $q_{n_k} \to e^{i\theta} q$ in $L_1$-norm as $k \to \infty$.

$(iii) \implies (ii)$ : Let $q_{n_k}$ be a convergent subsequence of $q_n$ such that $q_{n_k} \to e^{i\theta} q$ in $L_1$-norm for some angle $\theta$ as $k \to \infty$. Then it follows from the continuity of $l_\phi(\alpha) : Q(X) \to \mathbb{R}$ that $l_{q_{n_k}}(\alpha) \to l_q(\alpha)$ for every $\alpha \in \mathcal{C}$, $k \to \infty$. This implies that $|q_{n_k}| \to |q|$ in the $\mathcal{C}$-topology in $|Q_1(X)|$, $k \to \infty$. Applying the above reasoning to any convergent subsequence of $q_n$, we conclude that $|q_n| \to |q|$ in the $\mathcal{C}$-topology in $|Q_1(X)|$ as $n \to \infty$.

$(ii) \implies (i)$ : This is clear.

$(iv) \implies (i)$: If $d(|q_n|, |q|) \to 0$, then

$$\sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_n}(\alpha)}{l_q(\alpha)}\} \sup_{\alpha \in \mathcal{S}} \{\frac{l_q(\alpha)}{l_{q_n}(\alpha)}\} = \frac{\sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_n}(\alpha)}{l_q(\alpha)}\}}{\inf_{\alpha \in \mathcal{S}} \{\frac{l_{q_n}(\alpha)}{l_q(\alpha)}\}} \to 1, \ n \to \infty. \tag{1}$$

To simplify the notations, we denote

$$\sup_{\alpha \in \mathcal{S}} \{\frac{l_{q_n}(\alpha)}{l_q(\alpha)}\}$$

and

$$\sup_{\alpha\in\mathcal{S}}\{\frac{l_q(\alpha)}{l_{q_n}(\alpha)}\}$$

by $a_n$ and $b_n$ respectively, $n=1,2,\cdots$. We claim that

$$a_n\to 1,\ b_n\to 1,\ n\to\infty. \tag{2}$$

To see this, it is sufficient to show that each convergent subsequence of $a_n$ (resp., $b_n$) must converge to 1. Let $a_{n_k}$ be any convergent subsequence of $a_n$ with $a_{n_k}\to a$, $k\to\infty$. From Lemma 3, $a\geq 1$. Assume $a>1$. Since $a_nb_n\to 1$ $(n\to\infty)$, the subsequence $a_{n_k}b_{n_k}$ also tends to 1 $(k\to\infty)$. Then $b_{n_k}\to 1/a$ $(k\to\infty)$ with $1/a<1$. But by Lemma 3 the limit of $b_{n_k}$ should be at least 1, a contradiction. This shows $a_n\to 1$, $n\to\infty$. Similarly, $b_n\to 1$ as $n\to\infty$. Thus we have verified the claim in (2).

On the other hand, for every $\alpha\in\mathcal{S}$, we always have

$$1\leq \frac{\frac{l_{q_n}(\alpha)}{l_q(\alpha)}}{\inf_{\alpha\in\mathcal{S}}\frac{l_{q_n}(\alpha)}{l_q(\alpha)}}\leq\frac{\sup_{\alpha\in\mathcal{S}}\frac{l_{q_n}(\alpha)}{l_q(\alpha)}}{\inf_{\alpha\in\mathcal{S}}\frac{l_{q_n}(\alpha)}{l_q(\alpha)}},\ n=1,2,\cdots. \tag{3}$$

By (1) and (3) we obtain

$$l_{k_nq_n}(\alpha)\to l_q(\alpha),\ n\to\infty, \tag{4}$$

where

$$k_n=\Big(\inf_{\alpha\in\mathcal{S}}\frac{l_{q_n}(\alpha)}{l_q(\alpha)}\Big)^{-2}=b_n^2.$$

From (2), $k_n\to 1$ as $n\to\infty$. Thus it follows from (4) that for each $\alpha\in\mathcal{S}$, $l_{q_n}(\alpha)$ is bounded. Consequently, for each $\alpha\in\mathcal{S}$,

$$l_{k_nq_n}(\alpha)-l_{q_n}(\alpha)\to 0,\ n\to\infty. \tag{5}$$

Therefore, (4) and (5) yield

$$l_{q_n}(\alpha)\to l_q(\alpha),\ n\to\infty.$$

$(i)\Longrightarrow(iv)$: It follows from Remark 3 that there exist two sequences $\mu_n$ and $\nu_n$ in $\mathcal{MF}$ such that

$$D(|q_n|,|q|)=\frac{l_{q_n}(\mu_n)}{l_q(\mu_n)}\frac{l_q(\nu_n)}{l_{q_n}(\nu_n)},\ n=1,2,\cdots.$$

From the compactness of $\mathcal{PMF}$ and the homogeneity of $D(|q_n|, |q|)$ with respect to $\mu_n$ and $\nu_n$, we may assume in the sequence $D(|q_n|, |q|)$ that $\mu_n \to \mu$ and $\nu_n \to \nu$ in $\mathcal{MF}$, $n \to \infty$. By the hypothesis $(i)$ and the implication $(i) \Longrightarrow (ii)$, $l_{q_n}(\alpha) \to l_q(\alpha)$ for every $\alpha \in \mathcal{C}$ as $n \to \infty$. Consequently from the continuity of the length function $l$ as in Lemma 2, we get $D(|q_n|, |q|) \to 1$ and $d(|q_n|, |q|) \to 0$, $n \to \infty$. $\Box$

In particular, Theorem 2 implies the following corollaries.

Corollary 1. *The metric topology induced by $d$ is compatible with the $\mathcal{S}$-topology and also the $\mathcal{C}$-topology on $|Q_1(X)|$.*

Proof. "$(i) \Longleftrightarrow (iv)$" and "$(ii) \Longleftrightarrow (iv)$" in Theorem 2. $\Box$

Corollary 2. *The metric space $(|Q_1(X)|, d)$ is compact and proper.*

Proof. Let $\{|q_n|\}_{n=1}^{\infty} \subset |Q_1(X)|$. By the compactness of $(Q_1(X), ||\cdot||_1)$, there exists a subsequence $q_{n_k}$ of $q_n$ such that $q_{n_k} \to q'$ $(k \to \infty)$ in $L_1$-norm, for some $q' \in Q_1(X)$. From the proof of the implication "$(iii) \Longrightarrow (ii)$" in Theorem 2, we know $|q_{n_k}| \to |q'|$ in the $\mathcal{C}$-topology in $|Q_1(X)|$. Applying the implication "$(ii) \Longrightarrow (iv)$" in Theorem 2 to the subsequence $q_{n_k}$, we conclude that $d(|q_{n_k}|, |q'|) \to 0$, $k \to \infty$. $\Box$

By the Heine–Borel theorem, Corollary 2 implies the following

Corollary 3. *The metric space $(|Q_1(X)|, d)$ is complete and totally bounded.*

### 3.3. The Lipschitz metric $\tau$

We may also consider another natural metric on $|Q_1(X)|$ defined as follows. Let $|q_1|, |q_2| \in |Q_1(X)|$, and $f : (X, |q_1|) \to (X, |q_2|)$ be a Lipschitz homeomorphism. Recall that the Lipschitz constant of $f$ is defined as

$$L(f; |q_1|, |q_2|) = \sup_{x,y \in X,\ x \neq y} \frac{\delta_2(f(x), f(y))}{\delta_1(x, y)},$$

where $\delta_i$ is the distance function induced by $|q_i|$, $i = 1, 2$, respectively. For simplicity, denote $L(f; |q_1|, |q_2|)$ by $L(f)$. Then the Lipschitz metric on $|Q_1(X)|$ is defined as

$$\tau(|q_1|, |q_2|) = \inf_{f \sim id} \{\log(L(f)L(f^{-1}))\},$$

where the infimum is taken over all the Lipschitz homeomorphisms $f$ homotopic to $id : X \to X$. Similar definitions have been studied in [20] and ([1], Section 12.4.2.3.1) in different contexts.

It is easy to see that

$$l_{q_2}(f(\alpha)) \leq L(f) l_{q_1}(\alpha)$$

and

$$l_{q_1}(f^{-1}(\beta)) \leq L(f^{-1}) l_{q_2}(\beta)$$

hold for any closed curves $\alpha$ and $\beta$. Thus by the definitions of $d$ and $\tau$, we obtain

Proposition 1. *For any* $|q_1|$, $|q_2| \in |Q_1(X)|$,

$$d(|q_1|, |q_2|) \leq \tau(|q_1|, |q_2|).$$

To end this paper, we raise the following question on the relation between $d$ and $\tau$.

Question 2. Is it true that $d = \tau$?

The metric $d$ is defined by maximizing the length distortions of simple closed curves, while the metric $\tau$ is defined by minimizing the global Lipschitz constants. Thus if the answer to this question is positive, then it unifies these two viewpoints.

We will study the questions listed above and some further properties of the metrics $d$ and $\tau$ in the near future.

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